

PLANE AND SPHERICAL TRIGONOMETRY

TEACHERS' EDITION



WENTWORTH - SMITH



Digitized by the Internet Archive
in 2023 with funding from
No Sponsor

PLANE AND SPHERICAL TRIGONOMETRY

TEACHERS' EDITION

BY

GEORGE WENTWORTH

AND

DAVID EUGENE SMITH

GINN AND COMPANY

BOSTON • NEW YORK • CHICAGO • LONDON
ATLANTA • DALLAS • COLUMBUS • SAN FRANCISCO

© COPYRIGHT, 1916, BY
GEORGE WENTWORTH AND DAVID EUGENE SMITH

ALL RIGHTS RESERVED

PRINTED IN THE UNITED STATES OF AMERICA

457.5

I

PREFACE

This book is intended for teachers, and *for teachers only*. The publishers will sell it to no person except to a teacher who is using the Wentworth-Smith Trigonometry as the regular textbook of his class. Every teacher who purchases a copy must consider himself in honor bound not to leave his copy where pupils can have access to it, and not to sell his copy except to the publishers, Messrs. Ginn and Company.

It is hoped that young teachers will derive considerable help from the systematic solutions of the exercises and that every teacher who is pressed for time will find great relief by not being obliged to work out every problem for himself.

Criticisms and corrections will be thankfully received.

GEORGE WENTWORTH
DAVID EUGENE SMITH

NOTE. The numbers in brackets quoted as reasons in cases like those on pages 197 and 343 of the Teachers' Edition refer to the numbered formulas on pages 185-186 of the text.

PLANE TRIGONOMETRY

TEACHERS' EDITION

Exercise 1. The Six Functions (Page 5)

1. In the figure of § 7, $\sin B = \frac{b}{c}$. Write the other five functions of the angle B .

$$\cos B = \frac{a}{c}.$$

$$\sec B = \frac{c}{a}.$$

$$\tan B = \frac{b}{a}.$$

$$\csc B = \frac{c}{b}.$$

$$\cot B = \frac{a}{b}.$$

2. Show that in the right triangle ACB (§ 7) the following relations exist:

$\sin A = \cos B,$	$\cos A = \sin B,$	$\tan A = \cot B,$
$\cot A = \tan B,$	$\sec A = \csc B,$	$\sec A = \sec B.$

In the triangle ACB (§ 7),

$$\sin A = \frac{a}{c}.$$

$$\cos A = \frac{b}{c}.$$

$$\tan A = \frac{a}{b}.$$

$$\cos B = \frac{a}{c}.$$

$$\sin B = \frac{b}{c}.$$

$$\cot B = \frac{a}{b}.$$

$$\therefore \sin A = \cos B.$$

$$\therefore \cos A = \sin B.$$

$$\therefore \tan A = \cot B.$$

$$\cot A = \frac{b}{a}.$$

$$\sec A = \frac{c}{b}.$$

$$\csc A = \frac{c}{a}.$$

$$\tan B = \frac{b}{a}.$$

$$\csc B = \frac{c}{b}.$$

$$\sec B = \frac{c}{a}.$$

$$\therefore \cot A = \tan B.$$

$$\therefore \sec A = \csc B.$$

$$\therefore \csc A = \sec B.$$

State which of the following is the greater :

3. $\sin A$ or $\tan A$.

$$\sin A = \frac{a}{c}.$$

$$\tan A = \frac{a}{b}.$$

But $c > b$.

$$\therefore \frac{a}{b} > \frac{a}{c},$$

and $\tan A > \sin A$.

4. $\cos A$ or $\cot A$.

$$\cos A = \frac{b}{c}.$$

$$\cot A = \frac{b}{a}.$$

But $c > a$.

$$\therefore \frac{b}{a} > \frac{b}{c},$$

and $\cot A > \cos A$.

5. $\sec A$ or $\tan A$.

$$\sec A = \frac{c}{b}.$$

$$\tan A = \frac{a}{b}.$$

But

$$c > a.$$

$$\therefore \frac{c}{b} > \frac{a}{b},$$

and

$$\sec A > \tan A.$$

6. $\csc A$ or $\cot A$.

$$\csc A = \frac{c}{a}.$$

$$\cot A = \frac{b}{a}.$$

But

$$c > b.$$

$$\therefore \frac{c}{a} > \frac{b}{a},$$

and

$$\csc A > \cot A.$$

Find the values of the six functions of A , if a , b , c respectively have the following values:

7. 3, 4, 5.

$$\sin A = \frac{a}{c} = \frac{3}{5}; \quad \cot A = \frac{b}{a} = \frac{4}{3};$$

$$\cos A = \frac{b}{c} = \frac{4}{5}; \quad \sec A = \frac{c}{b} = \frac{5}{4};$$

$$\tan A = \frac{a}{b} = \frac{3}{4}; \quad \csc A = \frac{c}{a} = \frac{5}{3}.$$

8. 5, 12, 13.

$$\sin A = \frac{a}{c} = \frac{5}{13}; \quad \cot A = \frac{b}{a} = \frac{12}{5};$$

$$\cos A = \frac{b}{c} = \frac{12}{13}; \quad \sec A = \frac{c}{b} = \frac{13}{12};$$

$$\tan A = \frac{a}{b} = \frac{5}{12}; \quad \csc A = \frac{c}{a} = \frac{13}{5}.$$

11. 3.9, 8, 8.9.

$$\sin A = \frac{a}{c} = \frac{3.9}{8.9} = \frac{39}{89};$$

$$\cos A = \frac{b}{c} = \frac{8}{8.9} = \frac{80}{89};$$

$$\tan A = \frac{a}{b} = \frac{3.9}{8} = \frac{39}{80};$$

$$\cot A = \frac{b}{a} = \frac{8}{3.9} = \frac{80}{39};$$

$$\sec A = \frac{c}{b} = \frac{8.9}{8} = \frac{89}{80};$$

$$\csc A = \frac{c}{a} = \frac{8.9}{3.9} = \frac{89}{39}.$$

12. 1.19, 1.20, 1.69.

$$\sin A = \frac{a}{c} = \frac{1.19}{1.69} = \frac{119}{169}; \quad \cot A = \frac{b}{a} = \frac{1.20}{1.19} = \frac{120}{119};$$

$$\cos A = \frac{b}{c} = \frac{1.20}{1.69} = \frac{120}{169}; \quad \sec A = \frac{c}{b} = \frac{1.69}{1.20} = \frac{169}{120};$$

$$\tan A = \frac{a}{b} = \frac{1.19}{1.20} = \frac{119}{120}; \quad \csc A = \frac{c}{a} = \frac{1.69}{1.19} = \frac{169}{119}.$$

9. 8, 15, 17.

$$\sin A = \frac{a}{c} = \frac{8}{17}; \quad \cot A = \frac{b}{a} = \frac{15}{8};$$

$$\cos A = \frac{b}{c} = \frac{15}{17}; \quad \sec A = \frac{c}{b} = \frac{17}{15};$$

$$\tan A = \frac{a}{b} = \frac{8}{15}; \quad \csc A = \frac{c}{a} = \frac{17}{8}.$$

10. 9, 40, 41.

$$\sin A = \frac{a}{c} = \frac{9}{41}; \quad \cot A = \frac{b}{a} = \frac{40}{9};$$

$$\cos A = \frac{b}{c} = \frac{40}{41}; \quad \sec A = \frac{c}{b} = \frac{41}{40};$$

$$\tan A = \frac{a}{b} = \frac{9}{40}; \quad \csc A = \frac{c}{a} = \frac{41}{9}.$$

13. What condition must be fulfilled by the lengths of the three lines a, b, c (§ 7) to make them the sides of a right triangle? Show that this condition is fulfilled in Exs. 7-12.

In order that the lines a, b, c may be the sides of a right triangle, their lengths must be such that $a^2 + b^2 = c^2$.

In Ex. 7, $3^2 + 4^2 = 9 + 16 = 25$; $5^2 = 25$.

In Ex. 8, $5^2 + 12^2 = 25 + 144 = 169$; $13^2 = 169$.

In Ex. 9, $8^2 + 15^2 = 64 + 225 = 289$; $17^2 = 289$.

In Ex. 10, $9^2 + 40^2 = 81 + 1600 = 1681$; $41^2 = 1681$.

In Ex. 11, $3.9^2 + 8^2 = 15.21 + 64 = 79.21$; $8.9^2 = 79.21$.

In Ex. 12, $1.19^2 + 1.20^2 = 1.4161 + 1.4400 = 2.8561$; $1.69^2 = 2.8561$.

Find the values of the six functions of A , if a, b, c respectively have the following values:

14. $2n, n^2 - 1, n^2 + 1$.

$$\sin A = \frac{a}{c} = \frac{2n}{n^2 + 1}; \quad \cot A = \frac{b}{a} = \frac{n^2 - 1}{2n};$$

$$\cos A = \frac{b}{c} = \frac{n^2 - 1}{n^2 + 1}; \quad \sec A = \frac{c}{b} = \frac{n^2 + 1}{n^2 - 1};$$

$$\tan A = \frac{a}{b} = \frac{2n}{n^2 - 1}; \quad \csc A = \frac{c}{a} = \frac{n^2 + 1}{2n}.$$

15. $n, \frac{n^2 - 1}{2}, \frac{n^2 + 1}{2}$.

$$\sin A = \frac{a}{c} = \frac{n}{\frac{n^2 + 1}{2}} = \frac{2n}{n^2 + 1}; \quad \cot A = \frac{b}{a} = \frac{\frac{n^2 - 1}{2}}{n} = \frac{n^2 - 1}{2n};$$

$$\cos A = \frac{b}{c} = \frac{\frac{n^2 - 1}{2}}{\frac{n^2 + 1}{2}} = \frac{n^2 - 1}{n^2 + 1}; \quad \sec A = \frac{c}{b} = \frac{\frac{n^2 + 1}{2}}{\frac{n^2 - 1}{2}} = \frac{n^2 + 1}{n^2 - 1};$$

$$\tan A = \frac{a}{b} = \frac{n}{\frac{n^2 - 1}{2}} = \frac{2n}{n^2 - 1}; \quad \csc A = \frac{c}{a} = \frac{\frac{n^2 + 1}{2}}{n} = \frac{n^2 + 1}{2n}.$$

16. $2mn, m^2 - n^2, m^2 + n^2$.

$$\sin A = \frac{a}{c} = \frac{2mn}{m^2 + n^2}; \quad \cot A = \frac{b}{a} = \frac{m^2 - n^2}{2mn};$$

$$\cos A = \frac{b}{c} = \frac{m^2 - n^2}{m^2 + n^2}; \quad \sec A = \frac{c}{b} = \frac{m^2 + n^2}{m^2 - n^2};$$

$$\tan A = \frac{a}{b} = \frac{2mn}{m^2 - n^2}; \quad \csc A = \frac{c}{a} = \frac{m^2 + n^2}{2mn}.$$

$$17. \frac{2mn}{m-n}, \quad m+n, \quad \frac{m^2+n^2}{m-n}.$$

$$\sin A = \frac{a}{c} = \frac{\frac{2mn}{m-n}}{\frac{m^2+n^2}{m-n}} = \frac{2mn}{m^2+n^2}; \quad \cot A = \frac{b}{a} = \frac{m+n}{\frac{2mn}{m-n}} = \frac{m^2-n^2}{2mn};$$

$$\cos A = \frac{b}{c} = \frac{m+n}{\frac{m^2+n^2}{m-n}} = \frac{m^2-n^2}{m^2+n^2}; \quad \sec A = \frac{c}{b} = \frac{\frac{m^2+n^2}{m-n}}{m+n} = \frac{m^2+n^2}{m^2-n^2};$$

$$\tan A = \frac{a}{b} = \frac{\frac{2mn}{m-n}}{m+n} = \frac{2mn}{m^2-n^2}; \quad \csc A = \frac{c}{a} = \frac{\frac{m^2+n^2}{m-n}}{\frac{2mn}{m-n}} = \frac{m^2+n^2}{2mn}.$$

18. As in Ex. 13, show that the condition for a right triangle is fulfilled in Exs. 14-17.

$$\text{In Ex. 14,} \quad (2n)^2 + (n^2-1)^2 = 4n^2 + n^4 - 2n^2 + 1 = n^4 + 2n^2 + 1;$$

$$(n^2+1)^2 = n^4 + 2n^2 + 1.$$

$$\text{In Ex. 15,} \quad (n)^2 + \left(\frac{n^2-1}{2}\right)^2 = n^2 + \frac{n^4-2n^2+1}{4} = \frac{n^4+2n^2+1}{4};$$

$$\left(\frac{n^2+1}{2}\right)^2 = \frac{n^4+2n^2+1}{4}.$$

In Ex. 16,

$$(2mn)^2 + (m^2-n^2)^2 = 4m^2n^2 + m^4 - 2m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4;$$

$$(m^2+n^2)^2 = m^4 + 2m^2n^2 + n^4.$$

$$\text{In Ex. 17,} \quad \left(\frac{2mn}{m-n}\right)^2 + (m+n)^2 = \frac{4m^2n^2}{m^2-2mn+n^2} + m^2 + 2mn + n^2$$

$$= \frac{m^4 + 2m^2n^2 + n^4}{m^2-2mn+n^2};$$

$$\left(\frac{m^2+n^2}{m-n}\right)^2 = \frac{m^4 + 2m^2n^2 + n^4}{m^2-2mn+n^2}.$$

Given $a^2 + b^2 = c^2$, find the six functions of A when:

$$19. a = b.$$

$$a^2 + b^2 = c^2, \quad \sin A = \frac{a}{c} = \frac{b}{b\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}, \quad \cot A = \frac{b}{a} = \frac{a}{b} = 1.$$

$$b^2 + b^2 = c^2,$$

$$2b^2 = c^2, \quad \cos A = \frac{b}{c} = \frac{b}{b\sqrt{2}} = \frac{1}{2}\sqrt{2}, \quad \sec A = \frac{c}{b} = \frac{b\sqrt{2}}{b} = \sqrt{2}.$$

$$c = b\sqrt{2}.$$

$$\tan A = \frac{a}{b} = \frac{b}{b} = 1.$$

$$\csc A = \frac{c}{a} = \frac{b\sqrt{2}}{b} = \sqrt{2}.$$

$$20. a = 2b.$$

$$a^2 + b^2 = c^2.$$

$$4b^2 + b^2 = c^2.$$

$$5b^2 = c^2.$$

$$c = b\sqrt{5}.$$

$$\sin A = \frac{a}{c} = \frac{2b}{b\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2}{5}\sqrt{5}.$$

$$\cot A = \frac{b}{a} = \frac{b}{2b} = \frac{1}{2}.$$

$$\cos A = \frac{b}{c} = \frac{b}{b\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{1}{5}\sqrt{5}.$$

$$\sec A = \frac{c}{b} = \frac{b\sqrt{5}}{b} = \sqrt{5}.$$

$$\tan A = \frac{a}{b} = \frac{2b}{b} = 2.$$

$$\csc A = \frac{c}{a} = \frac{b\sqrt{5}}{2b} = \frac{1}{2}\sqrt{5}.$$

$$21. a = \frac{2}{3}c.$$

$$a^2 + b^2 = c^2.$$

$$\frac{4}{9}c^2 + b^2 = c^2.$$

$$b^2 = \frac{5}{9}c^2.$$

$$b = \frac{c}{3}\sqrt{5}.$$

$$\sin A = \frac{a}{c} = \frac{\frac{2}{3}c}{c} = \frac{2}{3}.$$

$$\cot A = \frac{b}{a} = \frac{\frac{c}{3}\sqrt{5}}{\frac{2}{3}c} = \frac{1}{2}\sqrt{5}.$$

$$\cos A = \frac{b}{c} = \frac{\frac{c}{3}\sqrt{5}}{c} = \frac{1}{3}\sqrt{5}.$$

$$\sec A = \frac{c}{b} = \frac{c}{\frac{c}{3}\sqrt{5}} = \frac{3}{\sqrt{5}} = \frac{3}{5}\sqrt{5}.$$

$$\tan A = \frac{a}{b} = \frac{\frac{2}{3}c}{\frac{c}{3}\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2}{5}\sqrt{5}.$$

$$\csc A = \frac{c}{a} = \frac{c}{\frac{2}{3}c} = \frac{3}{2}.$$

Given $a^2 + b^2 = c^2$, find the six functions of B when :

$$22. a = 24, b = 143.$$

$$a^2 + b^2 = c^2.$$

$$\sin B = \frac{b}{c} = \frac{143}{145};$$

$$\cot B = \frac{a}{b} = \frac{24}{143};$$

$$24^2 + 143^2 = c^2.$$

$$c^2 = 21,025.$$

$$\cos B = \frac{a}{c} = \frac{24}{145};$$

$$\sec B = \frac{c}{a} = \frac{145}{24};$$

$$c = 145.$$

$$\tan B = \frac{b}{a} = \frac{143}{24};$$

$$\csc B = \frac{c}{b} = \frac{145}{143}.$$

$$23. b = 9.5, c = 19.3.$$

$$a^2 + c^2 = b^2$$

$$= 19.3^2 - 9.5^2$$

$$= 282.24.$$

$$a = 16.8.$$

$$\sin B = \frac{b}{c} = \frac{9.5}{19.3} = \frac{95}{193};$$

$$\cot B = \frac{a}{b} = \frac{16.8}{9.5} = \frac{168}{95};$$

$$\cos B = \frac{a}{c} = \frac{16.8}{19.3} = \frac{168}{193};$$

$$\sec B = \frac{c}{a} = \frac{19.3}{16.8} = \frac{193}{168};$$

$$\tan B = \frac{b}{a} = \frac{9.5}{16.8} = \frac{95}{168};$$

$$\csc B = \frac{c}{b} = \frac{19.3}{9.5} = \frac{193}{95}.$$

24. $a = 0.264$, $c = 0.265$.

$$b^2 = c^2 - a^2 = 0.265^2 - 0.264^2 = 0.000529.$$

$$b = 0.023.$$

$$\sin B = \frac{b}{c} = \frac{0.023}{0.265} = \frac{23}{265};$$

$$\cot B = \frac{a}{b} = \frac{0.264}{0.023} = \frac{264}{23};$$

$$\cos B = \frac{a}{c} = \frac{0.264}{0.265} = \frac{264}{265};$$

$$\sec B = \frac{c}{a} = \frac{0.265}{0.264} = \frac{265}{264};$$

$$\tan B = \frac{b}{a} = \frac{0.023}{0.264} = \frac{23}{264};$$

$$\csc B = \frac{c}{b} = \frac{0.265}{0.023} = \frac{265}{23}.$$

25. $b = 2\sqrt{pq}$, $c = p + q$.

$$a^2 = c^2 - b^2 = (p + q)^2 - (2\sqrt{pq})^2 = p^2 + 2pq + q^2 - 4pq = (p - q)^2$$

$$a = p - q.$$

$$\sin B = \frac{b}{c} = \frac{2\sqrt{pq}}{p + q};$$

$$\cot B = \frac{a}{b} = \frac{p - q}{2\sqrt{pq}} = \frac{p - q}{2pq} \sqrt{pq};$$

$$\cos B = \frac{a}{c} = \frac{p - q}{p + q};$$

$$\sec B = \frac{c}{a} = \frac{p + q}{p - q};$$

$$\tan B = \frac{b}{a} = \frac{2\sqrt{pq}}{p - q};$$

$$\csc B = \frac{c}{b} = \frac{p + q}{2\sqrt{pq}} = \frac{p + q}{2pq} \sqrt{pq}.$$

Given $a^2 + b^2 = c^2$, find the six functions of A and also the six functions of B when:

26. $a = \sqrt{p^2 + q^2}$, $b = \sqrt{2pq}$.

$$c^2 = a^2 + b^2 = p^2 + q^2 + 2pq.$$

$$c = p + q.$$

$$\sin A = \frac{a}{c} = \frac{\sqrt{p^2 + q^2}}{p + q};$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{2pq}}{\sqrt{p^2 + q^2}};$$

$$\cos A = \frac{b}{c} = \frac{\sqrt{2pq}}{p + q};$$

$$\sec A = \frac{c}{b} = \frac{p + q}{\sqrt{2pq}};$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{p^2 + q^2}}{\sqrt{2pq}};$$

$$\csc A = \frac{c}{a} = \frac{p + q}{\sqrt{p^2 + q^2}}.$$

$$\sin B = \frac{b}{c} = \frac{\sqrt{2pq}}{p + q};$$

$$\cot B = \frac{a}{b} = \frac{\sqrt{p^2 + q^2}}{\sqrt{2pq}};$$

$$\cos B = \frac{a}{c} = \frac{\sqrt{p^2 + q^2}}{p + q};$$

$$\sec B = \frac{c}{a} = \frac{p + q}{\sqrt{p^2 + q^2}};$$

$$\tan B = \frac{b}{a} = \frac{\sqrt{2pq}}{\sqrt{p^2 + q^2}};$$

$$\csc B = \frac{c}{b} = \frac{p + q}{\sqrt{2pq}}.$$

$$27. a = \sqrt{p^2 + p}, c = p + 1.$$

$$b^2 = c^2 - a^2 = p^2 + 2p + 1 - p^2 - p = p + 1.$$

$$b = \sqrt{p + 1}.$$

$$\sin A = \frac{a}{c} = \frac{\sqrt{p^2 + p}}{p + 1};$$

$$\cos A = \frac{b}{c} = \frac{\sqrt{p + 1}}{p + 1} = \frac{1}{\sqrt{p + 1}};$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{p^2 + p}}{\sqrt{p + 1}} = \sqrt{p};$$

$$\sin B = \frac{b}{c} = \frac{1}{\sqrt{p + 1}};$$

$$\cos B = \frac{a}{c} = \frac{\sqrt{p^2 + p}}{p + 1};$$

$$\tan B = \frac{b}{a} = \frac{\sqrt{p + 1}}{\sqrt{p^2 + p}} = \frac{\sqrt{p}}{p};$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{p + 1}}{\sqrt{p^2 + p}} = \frac{\sqrt{p}}{p};$$

$$\sec A = \frac{c}{b} = \frac{p + 1}{\sqrt{p + 1}} = \sqrt{p + 1};$$

$$\csc A = \frac{c}{a} = \frac{p + 1}{\sqrt{p^2 + p}} = \frac{\sqrt{p^2 + p}}{p}$$

$$\cot B = \frac{a}{b} = \frac{\sqrt{p^2 + p}}{\sqrt{p + 1}} = \sqrt{p};$$

$$\sec B = \frac{c}{a} = \frac{p + 1}{\sqrt{p^2 + p}} = \frac{\sqrt{p^2 + p}}{p};$$

$$\csc B = \frac{c}{b} = \frac{p + 1}{\sqrt{p + 1}} = \sqrt{p + 1}.$$

In the right triangle ACB , as shown in § 7:

28. Find the length of side a if
 $\sin A = \frac{3}{5}$, and $c = 20.5$.

$$\sin A = \frac{a}{c} = \frac{3}{5}.$$

$$\therefore \frac{a}{20.5} = \frac{3}{5}.$$

$$5a = 61.5.$$

$$a = 12.3.$$

29. Find the length of side b if
 $\cos A = 0.44$, and $c = 3.5$.

$$\cos A = \frac{b}{c} = 0.44.$$

$$\therefore \frac{b}{3.5} = 0.44.$$

$$b = 1.54.$$

30. Find the length of side a if
 $\tan A = 3\frac{2}{3}$, and $b = 2\frac{5}{11}$.

$$\tan A = \frac{a}{b} = \frac{11}{3}.$$

$$\therefore \frac{11a}{27} = \frac{11}{3}.$$

$$3a = 27.$$

$$a = 9.$$

31. Find the length of side b if
 $\cot A = 4$, and $a = 1700$.

$$\cot A = \frac{b}{a} = 4.$$

$$\therefore \frac{b}{1700} = 4.$$

$$b = 6800.$$

32. Find the length of the hypotenuse if $\sec A = 2$, and $b = 2000$.

$$\sec A = \frac{c}{b} = 2.$$

$$\therefore \frac{c}{2000} = 2.$$

$$c = 4000.$$

33. Find the length of the hypotenuse if $\csc A = 6.4$, and $a = 35.6$.

$$\csc A = \frac{c}{a} = 6.4.$$

$$\therefore \frac{c}{35.6} = 6.4.$$

$$c = 227.84.$$

Find the hypotenuse and other side of a right triangle, given:

34. $b = 6$, $\tan A = \frac{3}{2}$.

$$\tan A = \frac{a}{b} = \frac{3}{2}.$$

$$\therefore \frac{a}{6} = \frac{3}{2}.$$

$$a = 9.$$

$$a^2 + b^2 = c^2.$$

$$81 + 36 = c^2.$$

$$117 = c^2.$$

$$c = 3\sqrt{13}.$$

35. $a = 3.5$, $\cos A = 0.5$.

$$\cos A = \frac{b}{c} = 0.5.$$

$$b = 0.5c.$$

$$a^2 + b^2 = c^2.$$

$$12.25 + 0.25c^2 = c^2.$$

$$0.75c^2 = 12.25.$$

$$c^2 = 16\frac{1}{3}.$$

$$c = \frac{7}{3}\sqrt{3}, b = \frac{7}{6}\sqrt{3}.$$

36. $b = 4$, $\csc A = 1\frac{2}{3}$.

$$\csc A = \frac{c}{a} = \frac{5}{3}.$$

$$c = \frac{5}{3}a.$$

$$a^2 + b^2 = c^2.$$

$$a^2 + 16 = \frac{25a^2}{9}.$$

$$9a^2 + 144 = 25a^2.$$

$$16a^2 = 144.$$

$$a^2 = 9.$$

$$a = 3, c = 5.$$

37. $b = 2$, $\sin A = 0.6$.

$$\sin A = \frac{a}{c} = 0.6.$$

$$a = 0.6c.$$

$$a^2 + b^2 = c^2.$$

$$0.36c^2 + 4 = c^2.$$

$$0.64c^2 = 4.$$

$$c^2 = 6.25.$$

$$c = 2.5, a = 1.5.$$

38. The hypotenuse of a right triangle is 2.5 mi., $\sin A = 0.6$, and $\cos A = 0.8$. Compute the sides of the triangle.

$$\sin A = \frac{a}{c}.$$

$$\therefore a = c \sin A \\ = 0.6 \times 2.5 \text{ mi.} \\ = 1.5 \text{ mi.}$$

$$\cos A = \frac{b}{c}.$$

$$\therefore b = c \cos A \\ = 0.8 \times 2.5 \text{ mi.} \\ = 2 \text{ mi.}$$

Find, by means of the above table, the sides and hypotenuse of a right triangle, given:

40. $A = 20^\circ$, $c = 1$.

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A \\ = 0.342.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A \\ = 0.94.$$

41. $A = 20^\circ$, $c = 4$.

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A \\ = 4 \times 0.342 \\ = 1.368.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A \\ = 4 \times 0.940 \\ = 3.76.$$

42. $A = 20^\circ$, $c = 3.5$.

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A \\ = 3.5 \times 0.342 \\ = 1.197.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A \\ = 3.5 \times 0.940 \\ = 3.29.$$

43. $A = 20^\circ, c = 4.8.$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A$$

$$= 4.8 \times 0.342$$

$$= 1.6416.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A$$

$$= 4.8 \times 0.940$$

$$= 4.512.$$

44. $A = 20^\circ, c = 7\frac{1}{2}.$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A$$

$$= 7\frac{1}{2} \times 0.342$$

$$= 2.565.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A$$

$$= 7\frac{1}{2} \times 0.940$$

$$= 7.05.$$

45. $A = 40^\circ, c = 1.$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A$$

$$= 0.643.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A$$

$$= 0.766.$$

46. $A = 40^\circ, c = 3.$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A$$

$$= 3 \times 0.643$$

$$= 1.929.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A$$

$$= 3 \times 0.766$$

$$= 2.298.$$

47. $A = 40^\circ, c = 7.$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A$$

$$= 7 \times 0.643$$

$$= 4.501.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A$$

$$= 7 \times 0.766$$

$$= 5.362.$$

48. $A = 40^\circ, c = 10.7.$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A$$

$$= 10.7 \times 0.643$$

$$= 6.8801.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A$$

$$= 10.7 \times 0.766$$

$$= 8.1962.$$

49. $A = 40^\circ, c = 250.$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A$$

$$= 250 \times 0.643$$

$$= 160.75.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A$$

$$= 250 \times 0.766$$

$$= 191.5.$$

50. $A = 70^\circ, c = 2.$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A$$

$$= 2 \times 0.940$$

$$= 1.88.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A$$

$$= 2 \times 0.342$$

$$= 0.684.$$

51. $A = 70^\circ, a = 2.$

$$\csc A = \frac{a}{c}.$$

$$c = a \csc A$$

$$= 2 \times 1.064$$

$$= 2.128.$$

$$\cot A = \frac{b}{a}.$$

$$b = a \cot A$$

$$= 2 \times 0.364$$

$$= 0.728.$$

52. $A = 70^\circ, b = 2.$

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A$$

$$= 2 \times 2.747$$

$$= 5.494.$$

$$\sec A = \frac{c}{b}.$$

$$c = b \sec A$$

$$= 2 \times 2.924$$

$$= 5.848.$$

53. $A = 70^\circ, a = 25.$

$$\cot A = \frac{b}{a}.$$

$$b = a \cot A$$

$$= 25 \times 0.364$$

$$= 9.1.$$

$$\csc A = \frac{c}{a}.$$

$$c = a \csc A$$

$$= 25 \times 1.064$$

$$= 26.6.$$

54. $A = 70^\circ, b = 150.$

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A$$

$$= 150 \times 2.747$$

$$= 412.05.$$

$$\sec A = \frac{c}{b}.$$

$$c = b \sec A$$

$$= 150 \times 2.924$$

$$= 438.6.$$

55. By dividing the length of a vertical rod by the length of its horizontal shadow, the tangent of the angle of elevation of the sun at that time was found to be 0.82. How high is a tower, if the length of its horizontal shadow at the same time is 174.3 yd.?

$$\tan A = \frac{a}{b} = 0.82.$$

$$\begin{aligned}\therefore a &= 0.82 b \\ &= 0.82 \text{ of } 174.3 \text{ yd.} \\ &= 142.926 \text{ yd.}\end{aligned}$$

56. A pin is stuck upright on a table top and extends upward 1 in. above the surface. When its shadow is $\frac{7}{8}$ in. long, what is the tangent of the angle of elevation of the sun? How high is a telegraph pole whose horizontal shadow at that instant is 21 ft.?

$$\tan A = \frac{a}{b} = \frac{1}{\frac{7}{8}} = 1\frac{1}{7}.$$

$$\begin{aligned}\therefore a &= 1\frac{1}{7} b \\ &= 1\frac{1}{7} \times 21 \text{ ft.} \\ &= 24 \text{ ft.}\end{aligned}$$

Exercise 2. Functions of Complementary Angles (Page 7)

Express as functions of the complementary angle:

1. $\sin 30^\circ = \cos (90^\circ - 30^\circ) = \cos 60^\circ.$
2. $\cos 20^\circ = \sin (90^\circ - 20^\circ) = \sin 70^\circ.$
3. $\tan 40^\circ = \cot (90^\circ - 40^\circ) = \cot 50^\circ.$
4. $\sec 25^\circ = \csc (90^\circ - 25^\circ) = \csc 65^\circ.$
5. $\sin 50^\circ = \cos (90^\circ - 50^\circ) = \cos 40^\circ.$
6. $\tan 60^\circ = \cot (90^\circ - 60^\circ) = \cot 30^\circ.$
7. $\sec 75^\circ = \csc (90^\circ - 75^\circ) = \csc 15^\circ.$
8. $\csc 85^\circ = \sec (90^\circ - 85^\circ) = \sec 5^\circ.$
9. $\sin 60^\circ = \cos (90^\circ - 60^\circ) = \cos 30^\circ.$
10. $\cos 60^\circ = \sin (90^\circ - 60^\circ) = \sin 30^\circ.$
11. $\tan 45^\circ = \cot (90^\circ - 45^\circ) = \cot 45^\circ.$
12. $\sec 45^\circ = \csc (90^\circ - 45^\circ) = \csc 45^\circ.$
13. $\sin 75^\circ 30' = \cos (90^\circ - 75^\circ 30') = \cos 14^\circ 30'.$
14. $\tan 82^\circ 45' = \cot (90^\circ - 82^\circ 45') = \cot 7^\circ 15'.$
15. $\sec 68^\circ 15' = \csc (90^\circ - 68^\circ 15') = \csc 21^\circ 45'.$
16. $\cos 88^\circ 10' = \sin (90^\circ - 88^\circ 10') = \sin 1^\circ 50'.$

Express as functions of an angle less than 45° :

17. $\sin 65^\circ = \cos (90^\circ - 65^\circ) = \cos 25^\circ.$
18. $\tan 80^\circ = \cot (90^\circ - 80^\circ) = \cot 10^\circ.$
19. $\sec 77^\circ = \csc (90^\circ - 77^\circ) = \csc 13^\circ.$
20. $\cos 52^\circ = \sin (90^\circ - 52^\circ) = \sin 38^\circ.$
21. $\cot 61^\circ = \tan (90^\circ - 61^\circ) = \tan 29^\circ.$
22. $\csc 78^\circ = \sec (90^\circ - 78^\circ) = \sec 12^\circ.$
23. $\sin 89^\circ = \cos (90^\circ - 89^\circ) = \cos 1^\circ.$

24. $\cos 86^\circ = \sin (90^\circ - 86^\circ) = \sin 4^\circ$.
 25. $\sec 88^\circ = \csc (90^\circ - 88^\circ) = \csc 2^\circ$.
 26. $\sin 77\frac{1}{2}^\circ = \cos (90^\circ - 77\frac{1}{2}^\circ) = \cos 12\frac{1}{2}^\circ$.
 27. $\cos 82\frac{1}{2}^\circ = \sin (90^\circ - 82\frac{1}{2}^\circ) = \sin 7\frac{1}{2}^\circ$.
 28. $\tan 88.6^\circ = \cot (90^\circ - 88.6^\circ) = \cot 1.4^\circ$.

Find A , given the following relations:

- | | |
|----------------------------------|--------------------------------|
| 29. $90^\circ - A = A$. | 31. $90^\circ - A = 2A$. |
| $2A = 90^\circ$. | $90^\circ = 3A$. |
| $A = 45^\circ$. | $A = 30^\circ$. |
| 30. $\cos A = \sin A$. | 32. $\cos A = \sin 2A$. |
| $\cos A = \sin (90^\circ - A)$. | $\cos A = \sin (90^\circ - A)$ |
| $90^\circ - A = A$. | $90^\circ - A = 2A$. |
| $2A = 90^\circ$. | $3A = 90^\circ$. |
| $A = 45^\circ$. | $A = 30^\circ$. |

Exercise 3. Functions of 30° , 45° , and 60° (Page 9)

Given $\sqrt{3} = 1.7320$, express as decimal fractions the following:

- $\sin 30^\circ = \frac{1}{2} = 0.5$.
- $\cos 30^\circ = \frac{1}{2}\sqrt{3} = \frac{1}{2} \times 1.7320 = 0.8660$.
- $\tan 30^\circ = \frac{1}{3}\sqrt{3} = \frac{1}{3} \times 1.7320 = 0.5773$.
- $\cot 30^\circ = \sqrt{3} = 1.7320$.
- $\sec 30^\circ = \frac{2}{3}\sqrt{3} = \frac{2}{3} \times 1.7320 = 1.1547$.
- $\csc 30^\circ = 2$.
- $\sin 60^\circ = \frac{1}{2}\sqrt{3} = \frac{1}{2} \times 1.7320 = 0.8660$.
- $\cos 60^\circ = \frac{1}{2} = 0.5$.
- $\tan 60^\circ = \sqrt{3} = 1.7320$.
- $\cot 60^\circ = \frac{1}{3}\sqrt{3} = \frac{1}{3} \times 1.7320 = 0.5773$.
- $\sec 60^\circ = 2$.
- $\csc 60^\circ = \frac{2}{3}\sqrt{3} = \frac{2}{3} \times 1.7320 = 1.1547$.

Write the ratios of the following, simplifying the results:

- | | |
|---|--|
| 13. $\sin 45^\circ$ to $\sin 30^\circ$. | 15. $\tan 45^\circ$ to $\tan 30^\circ$. |
| $\frac{\sin 45^\circ}{\sin 30^\circ} = \frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}} = \sqrt{2}$. | $\frac{\tan 45^\circ}{\tan 30^\circ} = \frac{1}{\frac{1}{3}\sqrt{3}} = \sqrt{3}$. |
| 14. $\cos 45^\circ$ to $\cos 30^\circ$. | 16. $\cot 45^\circ$ to $\cot 30^\circ$. |
| $\frac{\cos 45^\circ}{\cos 30^\circ} = \frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{3}\sqrt{6}$. | $\frac{\cot 45^\circ}{\cot 30^\circ} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$. |

17. $\sec 45^\circ$ to $\sec 30^\circ$.

$$\frac{\sec 45^\circ}{\sec 30^\circ} = \frac{\sqrt{2}}{\frac{2}{3}\sqrt{3}} = \frac{1}{2}\sqrt{6}.$$

18. $\csc 45^\circ$ to $\csc 30^\circ$.

$$\frac{\csc 45^\circ}{\csc 30^\circ} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2}.$$

19. $\sin 30^\circ$ to $\sin 60^\circ$.

$$\frac{\sin 30^\circ}{\sin 60^\circ} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

20. $\cos 30^\circ$ to $\cos 60^\circ$.

$$\frac{\cos 30^\circ}{\cos 60^\circ} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}.$$

21. $\tan 30^\circ$ to $\tan 60^\circ$.

$$\frac{\tan 30^\circ}{\tan 60^\circ} = \frac{\frac{1}{3}\sqrt{3}}{\sqrt{3}} = \frac{1}{3}.$$

22. $\cot 30^\circ$ to $\cot 60^\circ$.

$$\frac{\cot 30^\circ}{\cot 60^\circ} = \frac{\sqrt{3}}{\frac{1}{3}\sqrt{3}} = 3.$$

23. $\sec 30^\circ$ to $\sec 60^\circ$.

$$\frac{\sec 30^\circ}{\sec 60^\circ} = \frac{2}{\frac{2}{3}\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

24. $\csc 30^\circ$ to $\csc 60^\circ$.

$$\frac{\csc 30^\circ}{\csc 60^\circ} = \frac{2}{\frac{2}{3}\sqrt{3}} = \sqrt{3}.$$

Express as functions of angles less than 45° :

25. $\sin 62^\circ 17' 40'' = \cos (90^\circ - 62^\circ 17' 40'') = \cos 27^\circ 42' 20''.$

26. $\tan 75^\circ 28' 35'' = \cot (90^\circ - 75^\circ 28' 35'') = \cot 14^\circ 31' 25''.$

27. $\sec 87^\circ 32' 51'' = \csc (90^\circ - 87^\circ 32' 51'') = \csc 2^\circ 27' 9''.$

28. $\cos 88^\circ 0' 27'' = \sin (90^\circ - 88^\circ 0' 27'') = \sin 1^\circ 59' 33''.$

29. $\sin 75.8^\circ = \cos (90^\circ - 75.8^\circ) = \cos 14.2^\circ.$

30. $\cos 82.75^\circ = \sin (90^\circ - 82.75^\circ) = \sin 7.25^\circ.$

31. $\tan 68.82^\circ = \cot (90^\circ - 68.82^\circ) = \cot 21.18^\circ.$

32. $\sec 85.95^\circ = \csc (90^\circ - 85.95^\circ) = \csc 4.05^\circ.$

Find A , given the following relations:

33. $90^\circ - A = 45^\circ - \frac{1}{2}A.$

$$-\frac{1}{2}A = -45^\circ.$$

$$A = 90^\circ.$$

34. $90^\circ - \frac{1}{2}A = A.$

$$-\frac{1}{2}A = -90^\circ.$$

$$A = 60^\circ.$$

35. $45^\circ + A = 90^\circ - A.$

$$2A = 45^\circ.$$

$$A = 22\frac{1}{2}^\circ, \text{ or } 22^\circ 30'.$$

36. $90^\circ - 4A = A.$

$$-5A = -90^\circ.$$

$$A = 18^\circ.$$

37. $90^\circ - A = nA.$

$$A(n+1) = 90^\circ.$$

$$A = \frac{90^\circ}{n+1}.$$

38. $\cos A = \sin (45^\circ - \frac{1}{2}A).$

$$\cos A = \sin (90^\circ - A).$$

$$90^\circ - A = 45^\circ - \frac{1}{2}A.$$

$$-\frac{1}{2}A = -45^\circ.$$

$$A = 90^\circ.$$

39. $\cot \frac{1}{2}A = \tan A.$

$$\cot \frac{1}{2}A = \tan (90^\circ - \frac{1}{2}A).$$

$$90^\circ - \frac{1}{2}A = A.$$

$$\frac{3}{2}A = 90^\circ.$$

$$A = 60^\circ.$$

40. $\tan (45^\circ + A) = \cot A.$

$$\cot A = \tan (90^\circ - A).$$

$$90^\circ - A = 45^\circ + A.$$

$$-2A = -45^\circ.$$

$$A = 22\frac{1}{2}^\circ, \text{ or } 22^\circ 30'.$$

41. $\cos 4A = \sin A.$

$$\cos 4A = \sin(90^\circ - 4A).$$

$$90^\circ - 4A = A.$$

$$-5A = -90^\circ.$$

$$A = 18^\circ.$$

42. $\cot A = \tan nA.$

$$\cot A = \tan(90^\circ - A).$$

$$90^\circ - A = nA.$$

$$A(n+1) = 90^\circ.$$

$$A = \frac{90^\circ}{n+1}.$$

43. By what must $\sin 45^\circ$ be multiplied to equal $\tan 30^\circ$?

$$x \sin 45^\circ = \tan 30^\circ.$$

$$x = \frac{\tan 30^\circ}{\sin 45^\circ} = \frac{\frac{1}{3}\sqrt{3}}{\frac{1}{2}\sqrt{2}} = \frac{1}{3}\sqrt{6}.$$

44. By what must $\sec 45^\circ$ be multiplied to equal $\csc 30^\circ$?

$$x \sec 45^\circ = \csc 30^\circ.$$

$$x = \frac{\csc 30^\circ}{\sec 45^\circ} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

49. What is the ratio of $\cos 45^\circ \csc 45^\circ$ to $\cos 30^\circ \csc 30^\circ$?

$$\frac{\cos 45^\circ \csc 45^\circ}{\cos 30^\circ \csc 30^\circ} = \frac{\frac{1}{2}\sqrt{2} \times \sqrt{2}}{\frac{1}{2}\sqrt{3} \times 2} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

50. What is the ratio of $\sin 45^\circ \sin 30^\circ$ to $\cos 45^\circ \cos 30^\circ$?

$$\frac{\sin 45^\circ \sin 30^\circ}{\cos 45^\circ \cos 30^\circ} = \frac{\frac{1}{2}\sqrt{2} \times \frac{1}{2}}{\frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

51. What is the ratio of $\tan 30^\circ \cot 30^\circ$ to $\tan 60^\circ \cot 60^\circ$?

$$\frac{\tan 30^\circ \cot 30^\circ}{\tan 60^\circ \cot 60^\circ} = \frac{\frac{1}{3}\sqrt{3} \times \sqrt{3}}{\sqrt{3} \times \frac{1}{3}\sqrt{3}} = 1.$$

52. From the statement $\tan 30^\circ = \frac{1}{3}\sqrt{3}$ find $\cot 60^\circ$.

$$\cot 60^\circ = \cot(90^\circ - 30^\circ) = \tan 30^\circ = \frac{1}{3}\sqrt{3}.$$

45. By what must $\cos 45^\circ$ be multiplied to equal $\tan 60^\circ$?

$$x \cos 45^\circ = \tan 60^\circ.$$

$$x = \frac{\tan 60^\circ}{\cos 45^\circ} = \frac{\sqrt{3}}{\frac{1}{2}\sqrt{2}} = \sqrt{6}.$$

46. By what must $\csc 60^\circ$ be divided to equal $\tan 45^\circ$?

$$\frac{\csc 60^\circ}{x} = \tan 45^\circ.$$

$$x = \frac{\csc 60^\circ}{\tan 45^\circ} = \frac{\frac{2}{3}\sqrt{3}}{1} = \frac{2}{3}\sqrt{3}.$$

47. By what must $\csc 30^\circ$ be divided to equal $\tan 30^\circ$?

$$\frac{\csc 30^\circ}{x} = \tan 30^\circ.$$

$$x = \frac{\csc 30^\circ}{\tan 30^\circ} = \frac{2}{\frac{1}{3}\sqrt{3}} = 2\sqrt{3}.$$

48. What is the ratio of $\sin 45^\circ \sec 45^\circ$ to $\cos 60^\circ$?

$$\frac{\sin 45^\circ \sec 45^\circ}{\cos 60^\circ} = \frac{\frac{1}{2}\sqrt{2} \times \sqrt{2}}{\frac{1}{2}} = 2.$$

Exercise 4. Use of the Table (Page 10)

From the table on page 11 find the values of the following :

- | | | |
|--------------------------------|---------------------------------|--------------------------------|
| 1. $\sin 5^\circ = 0.0872$. | 12. $\sin 76^\circ = 0.9703$. | 23. $\tan 45^\circ = 1.0000$. |
| 2. $\sin 14^\circ = 0.2419$. | 13. $\cos 24^\circ = 0.9135$. | 24. $\cot 45^\circ = 1.0000$. |
| 3. $\sin 21^\circ = 0.3584$. | 14. $\sin 66^\circ = 0.9135$. | 25. $\sec 0^\circ = 1.0000$. |
| 4. $\sin 30^\circ = 0.5000$. | 15. $\cos 35^\circ = 0.8192$. | 26. $\csc 90^\circ = 1.0000$. |
| 5. $\cos 85^\circ = 0.0872$. | 16. $\sin 55^\circ = 0.8192$. | 27. $\sec 15^\circ = 1.0353$. |
| 6. $\cos 76^\circ = 0.2419$. | 17. $\cot 5^\circ = 11.4301$. | 28. $\csc 75^\circ = 1.0353$. |
| 7. $\cos 69^\circ = 0.3584$. | 18. $\tan 85^\circ = 11.4301$. | 29. $\csc 12^\circ = 4.8097$. |
| 8. $\cos 60^\circ = 0.5000$. | 19. $\cot 11^\circ = 5.1446$. | 30. $\sec 78^\circ = 4.8097$. |
| 9. $\cos 6^\circ = 0.9945$. | 20. $\tan 79^\circ = 5.1446$. | 31. $\csc 44^\circ = 1.4396$. |
| 10. $\sin 84^\circ = 0.9945$. | 21. $\tan 21^\circ = 0.3839$. | 32. $\sec 46^\circ = 1.4396$. |
| 11. $\cos 14^\circ = 0.9703$. | 22. $\cot 69^\circ = 0.3839$. | |

33. Find the difference between $2 \sin 9^\circ$ and $\sin (2 \times 9^\circ)$.

$$\begin{aligned} 2 \sin 9^\circ &= 2 \times 0.1564 = 0.3128. \\ \sin (2 \times 9^\circ) &= \sin 18^\circ = 0.3090. \\ 0.3128 - 0.3090 &= 0.0038. \end{aligned}$$

34. Find the difference between $3 \tan 5^\circ$ and $\tan (3 \times 5^\circ)$.

$$\begin{aligned} 3 \tan 5^\circ &= 3 \times 0.0875 = 0.2625. \\ \tan (3 \times 5^\circ) &= \tan 15^\circ = 0.2679. \\ 0.2679 - 0.2625 &= 0.0054. \end{aligned}$$

35. Which is the larger, $2 \sec 10^\circ$ or $\sec (2 \times 10^\circ)$?

$$\begin{aligned} 2 \sec 10^\circ &= 2 \times 1.0154 = 2.0308. \\ \sec (2 \times 10^\circ) &= \sec 20^\circ = 1.0642. \\ 2 \sec 10^\circ &\text{ is the larger.} \end{aligned}$$

36. Which is the larger, $2 \csc 10^\circ$ or $\csc (2 \times 10^\circ)$?

$$\begin{aligned} 2 \csc 10^\circ &= 2 \times 5.7588 = 11.5176. \\ \csc (2 \times 10^\circ) &= \csc 20^\circ = 2.9238. \\ 2 \csc 10^\circ &\text{ is the larger.} \end{aligned}$$

37. Which is the larger, $2 \cos 15^\circ$ or $\cos (2 \times 15^\circ)$?

$$\begin{aligned} 2 \cos 15^\circ &= 2 \times 0.9659 = 1.9318. \\ \cos (2 \times 15^\circ) &= \cos 30^\circ = 0.8660. \\ 2 \cos 15^\circ &\text{ is the larger.} \end{aligned}$$

38. Compare $3 \sin 20^\circ$ with $\sin (3 \times 20^\circ)$; with $\sin (2 \times 20^\circ)$.

$$\begin{aligned} 3 \sin 20^\circ &= 3 \times 0.3420 = 1.0260. \\ \sin (3 \times 20^\circ) &= \sin 60^\circ = 0.8660. \\ \sin (2 \times 20^\circ) &= \sin 40^\circ = 0.6428. \end{aligned}$$

$3 \sin 20^\circ$ is larger than $\sin (3 \times 20^\circ)$ and larger than $\sin (2 \times 20^\circ)$.

39. Compare $3 \tan 10^\circ$ with $\tan(3 \times 10^\circ)$; with $\tan(2 \times 10^\circ)$.

$$3 \tan 10^\circ = 3 \times 0.1763 = 0.5289.$$

$$\tan(3 \times 10^\circ) = \tan 30^\circ = 0.5774.$$

$$\tan(2 \times 10^\circ) = \tan 20^\circ = 0.3640.$$

$3 \tan 10^\circ$ is smaller than $\tan(3 \times 10^\circ)$ and larger than $\tan(2 \times 10^\circ)$.

40. Compare $3 \cos 10^\circ$ with $\cos(3 \times 10^\circ)$; with $\cos(2 \times 10^\circ)$.

$$3 \cos 10^\circ = 3 \times 0.9848 = 2.9544.$$

$$\cos(3 \times 10^\circ) = \cos 30^\circ = 0.8660.$$

$$\cos(2 \times 10^\circ) = \cos 20^\circ = 0.9397.$$

$3 \cos 10^\circ$ is larger than $\cos(3 \times 10^\circ)$ and larger than $\cos(2 \times 10^\circ)$.

41. Is $\sin(10^\circ + 20^\circ)$ equal to $\sin 10^\circ + \sin 20^\circ$?

$$\sin(10^\circ + 20^\circ) = \sin 30^\circ = 0.5000.$$

$$\sin 10^\circ + \sin 20^\circ = 0.1736 + 0.3420 = 0.5156.$$

$\sin(10^\circ + 20^\circ)$ is not equal to $\sin 10^\circ + \sin 20^\circ$.

42. When the angle is increased from 0° to 90° which of the six functions are increased and which are decreased?

The sin, tan, and sec increase and the cos, cot, and csc decrease.

Exercise 5. Use of the Table (Page 12)

Using the values given in the table on page 11, show as above that the following are reciprocals:

1. $\sin 30^\circ, \csc 30^\circ.$

$$\sin 30^\circ \csc 30^\circ = 0.5 \times 2 = 1.$$

6. $\cos 10^\circ, \sec 10^\circ.$

$$\cos 10^\circ \sec 10^\circ = 0.9848 \times 1.0154$$

2. $\sin 25^\circ, \csc 25^\circ.$

$$= 0.99996592$$

$$\sin 25^\circ \csc 25^\circ = 0.4226 \times 2.3662$$

$$= 1.$$

$$= 0.99995612$$

$$= 1.0000.$$

7. $\sin 75^\circ, \csc 75^\circ.$

$$\sin 75^\circ \csc 75^\circ = 0.9659 \times 1.0353$$

3. $\cos 35^\circ, \sec 35^\circ.$

$$\cos 35^\circ \sec 35^\circ = 0.8192 \times 1.2208$$

$$= 0.99999627$$

$$= 1.00007936$$

$$= 1.$$

$$= 1.$$

8. $\cos 75^\circ, \sec 75^\circ.$

$$\cos 75^\circ \sec 75^\circ = 0.2588 \times 3.8637$$

4. $\sin 10^\circ, \csc 10^\circ.$

$$\sin 10^\circ \csc 10^\circ = 0.1736 \times 5.7588$$

$$= 0.99992556$$

$$= 0.99972768$$

$$= 1.$$

$$= 1.$$

5. $\tan 10^\circ, \cot 10^\circ.$

$$\tan 10^\circ \cot 10^\circ = 0.1763 \times 5.6713$$

$$= 0.99985019$$

$$= 1.$$

9. $\tan 75^\circ, \cot 75^\circ.$

$$\tan 75^\circ \cot 75^\circ = 3.7321 \times 0.2679$$

$$= 0.99982959$$

$$= 1.$$

10. From the table show that the ratio of $\sin 20^\circ \csc 20^\circ$ to $\tan 50^\circ \cot 50^\circ$ is 1.

$$\frac{\sin 20^\circ \csc 20^\circ}{\tan 50^\circ \cot 50^\circ} = \frac{0.3420 \times 2.9238}{1.1918 \times 0.8391} = \frac{0.99993960}{1.00003938} = \frac{1}{1} = 1.$$

11. Similarly, show that $\cos 40^\circ \sec 40^\circ : \tan 70^\circ \cot 70^\circ = 1$.

$$\frac{\cos 40^\circ \sec 40^\circ}{\tan 70^\circ \cot 70^\circ} = \frac{0.7660 \times 1.3054}{2.7475 \times 0.3640} = \frac{0.99993640}{1.00009000} = \frac{1}{1} = 1.$$

In the right triangle ACB , as shown in § 7:

12. Find the length of side a if $A = 30^\circ$, and $c = 75.2$.

$$\begin{aligned}\sin A &= \frac{a}{c} \\ a &= c \sin A \\ &= 75.2 \times 0.5000 \\ &= 37.6.\end{aligned}$$

13. Find the length of side a if $A = 45^\circ$, and $c = 1.414$.

$$\begin{aligned}\sin A &= \frac{a}{c} \\ a &= c \sin A \\ &= 1.414 \times 0.7071 \\ &= 0.9998394. \\ \therefore a &= 1.\end{aligned}$$

14. Find the length of side b if $A = 30^\circ$, and $c = 115.47$.

$$\begin{aligned}\cos A &= \frac{b}{c} \\ b &= c \cos A \\ &= 115.47 \times 0.8660 \\ &= 99.99702. \\ \therefore b &= 100.\end{aligned}$$

15. Find the length of side a if $A = 60^\circ$, and $b = 34.64$.

$$\begin{aligned}\tan A &= \frac{a}{b} \\ a &= b \tan A \\ &= 34.64 \times 1.7321 \\ &= 59.999944. \\ \therefore a &= 60.\end{aligned}$$

16. Find the length of side b if $A = 60^\circ$, and $c = 25.72$.

$$\begin{aligned}\cos A &= \frac{b}{c} \\ b &= c \cos A \\ &= 25.72 \times 0.5000 \\ &= 12.86.\end{aligned}$$

17. Find the length of side a if $A = 30^\circ$, and $c = 45.28$.

$$\begin{aligned}\sin A &= \frac{a}{c} \\ a &= c \sin A \\ &= 45.28 \times 0.5000 \\ &= 22.64.\end{aligned}$$

Exercise 6. Use of the Sine (Page 15)

Find a to four figures, given the following:

1. $c = 10$, $A = 10^\circ$.

$$a = c \sin A = 10 \times 0.1736 = 1.736.$$

2. $c = 15$, $A = 15^\circ$.

$$a = c \sin A = 15 \times 0.2588 = 3.882.$$

3. $c = 58$, $A = 45^\circ$.

$$a = c \sin A = 58 \times 0.7071 = 41.01.$$

4. $c = 75$, $A = 50^\circ$.

$$a = c \sin A = 75 \times 0.7660 = 57.45.$$

Find A , given the following:

5. $c = 10$, $a = 2.079$.

$$\sin A = \frac{a}{c} = \frac{2.079}{10} = 0.2079.$$

$$A = 12^\circ.$$

7. $c = 2$, $a = 1.2586$.

$$\sin A = \frac{a}{c} = \frac{1.2586}{2} = 0.6293$$

$$A = 39^\circ.$$

6. $c = 20$, $a = 6.840$.

$$\sin A = \frac{a}{c} = \frac{6.840}{20} = 0.3420.$$

$$A = 20^\circ.$$

8. $c = 50$, $a = 34.1$.

$$\sin A = \frac{a}{c} = \frac{34.1}{50} = 0.6820.$$

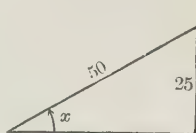
$$A = 43^\circ.$$

9. A 50-foot ladder resting against the side of a house reaches a point 25 ft. from the ground. What angle does it make with the ground?

$$\sin A = \frac{a}{c} = \frac{25}{50} = 0.5000.$$

$$\therefore A = 30^\circ.$$

The ladder makes an angle of 30° with the ground.

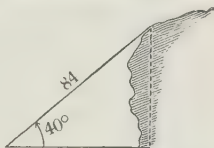


10. From the top of a rock a cord is stretched to a point on the ground, making an angle of 40° with the horizontal plane. The cord is 84 ft. long. Assuming the cord to be straight, how high is the rock?

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A = 84 \times 0.6428 = 53.9952.$$

The height of the rock is 54 ft.



11. Find the side of a regular decagon inscribed in a circle of radius 7 ft.

$$\text{Central angle} = 360^\circ \div 10 = 36^\circ.$$

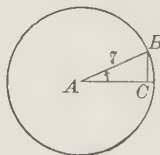
Half the central angle = $\frac{1}{2}$ of $36^\circ = 18^\circ$.

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A = 7 \times 0.3090 = 2.1630.$$

$$BC = 2.163 \text{ ft.}$$

The side of the decagon is 4.326 ft.

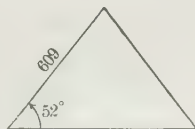


12. The edge of the Great Pyramid is 609 ft. and makes an angle of 52° with the horizontal plane. What is the height of the pyramid?

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A = 609 \times 0.7880 = 479.8920.$$

The height of the pyramid is 479.9 ft.

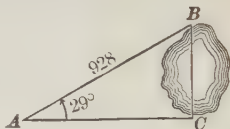


13. Wishing to measure BC , the length of a pond, a surveyor ran a line CA at right angles to BC . He measured AB and $\angle A$, finding that $AB = 928$ ft., and $A = 29^\circ$. Find the length of BC .

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A = 928 \times 0.4848 = 449.8944.$$

The length of BC is 449.9 ft.



Exercise 7. Use of the Cosine (Page 16)

Find b to four figures, given the following:

1. $c = 11, A = 10^\circ$.

6. $c = 2.8, A = 48^\circ$.

$$b = c \cos A = 11 \times 0.9848 = 10.83, \quad b = c \cos A = 2.8 \times 0.6691 = 1.873.$$

2. $c = 14, A = 10^\circ$.

7. $c = 9.7, A = 52^\circ$.

$$b = c \cos A = 14 \times 0.9613 = 13.46, \quad b = c \cos A = 9.7 \times 0.6157 = 5.972.$$

3. $c = 28, A = 24^\circ$.

8. $c = 11.2, A = 58^\circ$.

$$b = c \cos A = 28 \times 0.9135 = 25.58, \quad b = c \cos A = 11.2 \times 0.5299 = 5.935.$$

4. $c = 41, A = 39^\circ$.

9. $c = 12.5, A = 67^\circ$.

$$b = c \cos A = 41 \times 0.7771 = 31.86, \quad b = c \cos A = 12.5 \times 0.3907 = 4.884.$$

5. $c = 75, A = 42^\circ$.

10. $c = 28.25, A = 75^\circ$.

$$b = c \cos A = 75 \times 0.7431 = 55.73, \quad b = c \cos A = 28.25 \times 0.2588 = 7.311$$

Find A , given the following:

11. $c = 10, b = 9.848$.

14. $c = 17.6, b = 8.8$.

$$\cos A = \frac{b}{c} = \frac{9.848}{10} = 0.9848.$$

$$\cos A = \frac{b}{c} = \frac{8.8}{17.6} = 0.5000.$$

$$\therefore A = 10^\circ.$$

$$\therefore A = 60^\circ.$$

12. $c = 20, b = 19.126$.

15. $c = 500, b = 227$.

$$\cos A = \frac{b}{c} = \frac{19.126}{20} = 0.9563.$$

$$\cos A = \frac{b}{c} = \frac{227}{500} = 0.4540.$$

$$\therefore A = 17^\circ.$$

$$\therefore A = 63^\circ.$$

13. $c = 40, b = 35.952$.

16. $c = 600, b = 205.2$.

$$\cos A = \frac{b}{c} = \frac{35.952}{40} = 0.8988.$$

$$\cos A = \frac{b}{c} = \frac{205.2}{600} = 0.3420.$$

$$\therefore A = 26^\circ.$$

$$\therefore A = 70^\circ.$$

$$17. c = 200, b = 117.56.$$

$$\cos A = \frac{b}{c} = \frac{117.56}{200} = 0.5878.$$

$$\therefore A = 54^\circ.$$

$$19. c = 300, b = 102\frac{3}{5}.$$

$$\cos A = \frac{b}{c} = \frac{102.6}{300} = 0.3420.$$

$$\therefore A = 70^\circ.$$

$$18. c = 187, b = 93\frac{1}{2}.$$

$$\cos A = \frac{b}{c} = \frac{93.5}{187} = 0.5000.$$

$$\therefore A = 60^\circ.$$

$$20. c = 1000, b = 104\frac{1}{2}.$$

$$\cos A = \frac{b}{c} = \frac{104.5}{1000} = 0.1045.$$

$$\therefore A = 84^\circ.$$

21. A flagstaff breaks off 22 ft. from the top and, the parts still holding together, the top of the staff reaches the earth 11 ft. from the foot. What angle does it make with the ground?

$$\cos A = \frac{b}{c} = \frac{11}{22} = 0.5000.$$

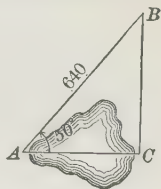
$$A = 60^\circ, \text{ the required angle.}$$

22. Wishing to measure the length of a pond, a class constructed a right triangle as shown in the figure. If $AB = 640$ ft. and $A = 50^\circ$, required the distance AC .

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 640 \times 0.6428 = 411.392.$$

The required distance is 411.4 ft.



23. In the same figure what is the length of AC when $AB = 500$ ft. and $A = 40^\circ$?

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 500 \times 0.7660 = 383.0000.$$

The required length is 383 ft.

24. In the same figure, if $AC = 731.4$ ft. and $AB = 1000$ ft., what is the size of angle A ?

$$\cos A = \frac{b}{c} = \frac{731.4}{1000} = 0.7314.$$

$$\therefore A = 43^\circ.$$

25. A regular hexagon is inscribed in a circle of radius 9 in. How far is it from the center to a side?

Half the angle at the center is $\frac{1}{2}$ of $\frac{360^\circ}{6} = 30^\circ$.

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 9 \times 0.8660 = 7.7940.$$

The distance from the center to a side is 7.794 in.

26. What is the area of a regular hexagon inscribed in a circle of radius 8 in.?

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 8 \times 0.8660 = 6.9280.$$

The apothem is 6.928 in.

$$\text{Area of hexagon} = 6 \times \frac{6.928 \times 8}{2} \text{ sq. in.} = 166.272 \text{ sq. in.}$$

27. A ship sails northeast 8 mi. It is then how many miles to the east of the starting point?

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 8 \times 0.7071 = 5.6568.$$

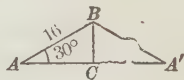
The ship is 5.657 miles east of the starting point.

28. Some 16-foot roof timbers make an angle of 30° with the horizontal in an A-shaped roof, as shown in the figure. Find AA' , the span of the roof.

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 16 \times 0.8660 = 13.8560.$$

$$AA' = 2 AC = 2b = 2 \times 13.856 = 27.712.$$



The span is 27.71 ft.

29. An equilateral triangle is inscribed in a circle of radius 12 in. How far is it from the center to a side?

Half the angle at the center is $\frac{1}{2}$ of $\frac{360^\circ}{3} = 60^\circ$.

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 12 \times 0.5000 = 6.$$

The distance from the center to a side is 6 in.

30. A crane AB , 30 ft. long, makes an angle of x degrees with the horizontal line AC . Find the distance AC when $x = 20$; when $x = 45$; when $x = 65$; when $x = 0$; when $x = 90$.

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 30 \cos 20^\circ = 30 \times 0.9397 = 28.1910.$$

$$AC = 28.19 \text{ ft. } Ans.$$

$$b = c \cos A = 30 \cos 45^\circ = 30 \times 0.7071 = 21.2130.$$

$$AC = 21.21 \text{ ft. } Ans.$$

$$b = c \cos A = 30 \cos 65^\circ = 30 \times 0.4226 = 12.6780.$$

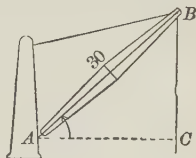
$$AC = 12.68 \text{ ft. } Ans.$$

$$b = c \cos A = 30 \cos 0^\circ = 30 \times 1 = 30.$$

$$AC = 30 \text{ ft. } Ans.$$

$$b = c \cos A = 30 \cos 90^\circ = 30 \times 0 = 0.$$

$$AC = 0 \text{ ft. } Ans.$$



31. In Ex. 30 what angle does the crane make with the horizontal when $AC = 15$ ft.? when $AC = 30$ ft.?

$$\cos A = \frac{b}{c} = \frac{15}{30} = 0.5000.$$

$$\therefore A = 60^\circ. \text{ } Ans.$$

$$\cos A = \frac{b}{c} = \frac{30}{30} = 1.0000.$$

$$\therefore A = 0^\circ. \text{ } Ans.$$

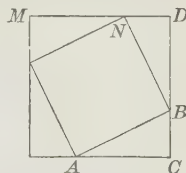
32. The square AN , of which the side is 200 ft., is inscribed in the square CM . AC is 181.26 ft. Required the angles that the sides of the small square make with the large one.

$$\cos A = \frac{b}{c} = \frac{181.26}{200} = 0.9063.$$

$$\therefore A = 25^\circ.$$

$$B = 90^\circ - 25^\circ = 65^\circ.$$

The required angles are 25° and 65° .



33. In Ex. 32 find the required angles when $AB = 15$ in. and $BC = 7\frac{1}{2}$ in.; when $AB = 20$ in. and $BC = 10.3$ in.

$$\cos B = \frac{a}{c} = \frac{7\frac{1}{2}}{15} = 0.5000.$$

$$\therefore B = 60^\circ.$$

$$A = 90^\circ - 60^\circ = 30^\circ.$$

$$\cos B = \frac{a}{c} = \frac{10.3}{20} = 0.5150.$$

$$\therefore B = 59^\circ.$$

$$A = 90^\circ - 59^\circ = 31^\circ.$$

The required angles are 30° and 60° ; 31° and 59° .

34. The edge of the Great Pyramid is 609 ft., and it makes an angle of 52° with the horizontal plane. What is the diagonal of the base?

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 609 \times 0.6157 = 374.9613.$$

$$2b = 749.9226.$$

The diagonal of the base is 749.9 ft.

Exercise 8. Use of the Tangent (Page 19)

Find a to four significant figures, given the following:

1. $b = 37, A = 18^\circ.$

$$\frac{a}{b} = \tan A.$$

$$a = b \tan A$$

$$= 37 \times 0.3249$$

$$= 12.0213.$$

$$a = 12.02.$$

2. $b = 26, A = 23^\circ.$

$$a = b \tan A$$

$$= 26 \times 0.4245$$

$$= 11.0370.$$

$$a = 11.04.$$

3. $b = 48, A = 31^\circ.$

$$a = b \tan A$$

$$= 48 \times 0.6009$$

$$= 28.8432.$$

$$a = 28.84.$$

4. $b = 62, A = 36^\circ.$

$$a = b \tan A$$

$$= 62 \times 0.7265$$

$$= 45.0430.$$

$$a = 45.04.$$

5. $b = 98, A = 45^\circ.$

$$a = b \tan A$$

$$= 98 \times 1$$

$$= 98.$$

6. $b = 4.8, A = 51^\circ.$

$$a = b \tan A$$

$$= 4.8 \times 1.2349$$

$$= 5.92752.$$

$$a = 5.928.$$

7. $b = 9.6, A = 57^\circ.$

$$a = b \tan A$$

$$= 9.6 \times 1.5399$$

$$= 14.78304.$$

$$a = 14.78.$$

8. $b = 23.4, A = 62^\circ.$

$$a = b \tan A$$

$$= 23.4 \times 1.8807$$

$$= 44.00838.$$

$$a = 44.01.$$

9. $b = 28.7, A = 75^\circ.$

$$a = b \tan A$$

$$= 28.7 \times 3.7321$$

$$= 107.11127.$$

$$a = 107.1.$$

10. $b = 39.7, A = 85^\circ.$

$$a = b \tan A$$

$$= 39.7 \times 11.4301$$

$$= 453.77497.$$

$$a = 453.8.$$

Find A , given the following:

11. $a = 6, b = 6.$

$$\tan A = \frac{a}{b} = \frac{6}{6} = 1.$$

$$\therefore A = 45^\circ.$$

12. $a = 0.281, b = 2.$

$$\tan A = \frac{a}{b} = \frac{0.281}{2} = 0.1405.$$

$$\therefore A = 8^\circ.$$

13. $a = 4.752, b = 30.$

$$\tan A = \frac{a}{b} = \frac{4.752}{30} = 0.1584.$$

$$\therefore A = 9^\circ.$$

14. $a = 13.772, b = 40.$

$$\tan A = \frac{a}{b} = \frac{13.772}{40} = 0.3443$$

$$\therefore A = 19^\circ.$$

15. $a = 2.424, b = 6.$

$$\tan A = \frac{a}{b} = \frac{2.424}{6} = 0.4040.$$

$$\therefore A = 22^\circ.$$

16. $a = 20.503, b = 10.$

$$\tan A = \frac{a}{b} = \frac{20.503}{10} = 2.0503.$$

$$\therefore A = 64^\circ.$$

17. A man standing 120 ft. from the foot of a church spire finds that the angle of elevation of the top is 50° . If his eye is 5 ft. 8 in. from the ground, what is the height of the spire?

$$a = b \tan A = 120 \times 1.1918 = 143.016.$$

$$a = 143.$$

The height of the spire is 143 ft. + 5 ft. 8 in. = 148 ft. 8 in.

18. When a flagstaff 55.43 ft. high casts a shadow 100 ft. long on a horizontal plane, what is the angle of elevation of the sun?

$$\tan A = \frac{a}{b} = \frac{55.43}{100} = 0.5543.$$

$$\therefore A = 29^\circ.$$

Angle of elevation of the sun is 29° .

19. A ship S is observed at the same instant from two lighthouses, L and L' , 3 mi. apart. $\angle L'LS$ is found to be 40° and $\angle LL'S$ is found to be 90° . What is the distance of the ship from L' ? What is its distance from L ?

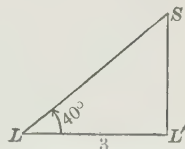
$$\tan A = \frac{a}{b}.$$

$$\begin{aligned} a &= b \tan A \\ &= 3 \times 0.8391 \\ &= 2.5173. \end{aligned}$$

$$\cos A = \frac{b}{c}.$$

$$0.7660 = \frac{3}{c}.$$

$$c = 3.9164.$$



The distance of the ship from L' is 2.517 mi. and from L it is 3.916 mi.

20. From the top of a rock which rises vertically, including the instrument, 134 ft. above a river bank the angle of depression of the opposite bank is found to be 40° . How wide is the river?

$$\tan A = \frac{a}{b}.$$

$$b = \frac{a}{\tan A} = \frac{134}{0.8391} = 159.69.$$

The river is 159.7 ft. wide.

21. An A-shaped roof has a span AA' of 24 ft. The ridgepole R is 12 ft. above the horizontal line AA' . What angle does AR make with AA' ? with RA' ? with the perpendicular from R on AA' ?

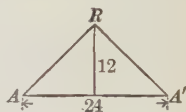
$$\tan A = \frac{a}{b}.$$

$$\frac{1}{2} \times 24 \text{ ft.} = 12 \text{ ft.}$$

$$\tan A = \frac{1}{1} = 1$$

$$\therefore A = 45^\circ.$$

$$\angle RAA' = 90^\circ.$$



AR makes with AA' an angle of 45° , with RA' an angle of 90° , and with the perpendicular from R on AA' an angle of 45° .

22. The foot of a ladder is 17 ft. 6 in. from a wall, and the ladder makes an angle of 42° with the horizontal when it leans against the wall. How far up the wall does it reach?

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A = 17\frac{1}{2} \times 0.9004 = 15.7570.$$

The ladder reaches 15.76 ft. up the wall.

23. A post subtends an angle of 7° from a point on the ground 50 ft. away. What is the height of the post?

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A = 50 \times 0.1228 = 6.1400.$$

The post is 6.14 ft. high.

24. The diameter of a one-cent piece is $\frac{3}{8}$ in. If the coin is held so that it subtends an angle of 40° at the eye, what is its distance from the eye?

$$\text{Angle } A = \frac{1}{2} \text{ of } 40^\circ = 20^\circ.$$

$$\text{Radius of coin} = \frac{3}{8} \text{ in.}$$

$$\tan A = \frac{a}{b}.$$

$$b = \frac{a}{\tan A} = \frac{\frac{3}{8}}{0.3640} = 1.0302$$

The coin is 1.03 in. distant from the eye.

Exercise 9. Use of the Cotangent (Page 20)

Find b to four significant figures, given the following :

1. $a = 29, A = 48^\circ.$

$$b = a \cot A = 29 \times 0.9004 = 26.11.$$

2. $a = 38, A = 72^\circ.$

$$b = a \cot A = 38 \times 0.3249 = 12.35.$$

3. $a = 56, A = 19^\circ.$

$$b = a \cot A = 56 \times 2.9042 = 162.6.$$

4. $a = 72, A = 40^\circ.$

$$b = a \cot A = 72 \times 1.1918 = 85.81.$$

5. $a = 425, A = 38^\circ.$

$$b = a \cot A = 425 \times 1.2799 = 544.0.$$

6. $a = 19\frac{1}{2}, A = 36^\circ.$

$$b = a \cot A = 19\frac{1}{2} \times 1.3764 = 26.84.$$

7. $a = 24.8, A = 43^\circ.$

$$b = a \cot A = 24.8 \times 1.0724 = 26.60.$$

8. $a = 256.8, A = 75^\circ.$

$$b = a \cot A = 256.8 \times 0.2679 = 68.80.$$

Find A , given the following :

9. $a = 72, b = 72.$

$$\cot A = \frac{b}{a} = \frac{72}{72} = 1.0000.$$

$$\therefore A = 45^\circ.$$

10. $a = 60, b = 128.67.$

$$\cot A = \frac{b}{a} = \frac{128.67}{60} = 2.1445.$$

$$\therefore A = 25^\circ.$$

11. How far from a tree 50 ft. high must a person lie in order to see the top at an angle of elevation of 60° ?

$$\cot A = \frac{b}{a}.$$

$$b = a \cot A = 50 \times 0.5774 = 28.8700.$$

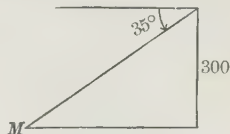
The distance from the tree is 28.87 ft.

12. From the top of a tower 300 ft. high, including the instrument, a point on the ground is observed to have an angle of depression of 35° . How far is the point from the tower ?

$$\cot A = \frac{b}{a}. \quad \text{Angle } A = 35^\circ = \text{angle } M.$$

$$b = a \cot A = 300 \times 1.4281 = 428.43.$$

The point is 428.4 ft. from the building.



13. From the extremity of the shadow cast by a church spire 150 ft. high the angle of elevation of the top is 53° . What is the length of the shadow ?

$$\cot A = \frac{b}{a}.$$

$$b = a \cot A = 150 \times 0.7536 = 113.0400.$$

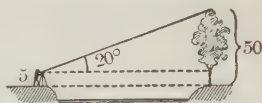
The length of the shadow is 113 ft.

14. A tree known to be 50 ft. high, standing on the bank of a stream, is observed from the opposite bank to have an angle of elevation of 20° . The angle is measured on a line 5 ft. above the foot of the tree. How wide is the stream?

$$\cot A = \frac{b}{a}.$$

$$b = a \cot A = (50 - 5) \times 2.7475 = 123.6375.$$

The width of the stream is 123.6 ft.



Exercise 10. Use of the Secant (Page 21)

Find c to four significant figures, given the following:

1. $b = 36$, $A = 27^\circ$.

4. $b = 22\frac{1}{2}$, $A = 48^\circ$.

$c = b \sec A = 36 \times 1.1223 = 40.40$. $c = b \sec A = 22\frac{1}{2} \times 1.4945 = 33.63$

2. $b = 48$, $A = 39^\circ$.

5. $b = 33.4$, $A = 53^\circ$.

$c = b \sec A = 48 \times 1.2868 = 61.77$. $c = b \sec A = 33.4 \times 1.6616 = 55.50$.

3. $b = 74$, $A = 43^\circ$.

6. $b = 148.8$, $A = 64^\circ$.

$c = b \sec A = 74 \times 1.3673 = 101.2$. $c = b \sec A = 148.8 \times 2.2812 = 339.4$.

Find A , given the following:

7. $b = 10$, $c = 13\frac{1}{4}$.

8. $b = 17.8$, $c = 35.6$.

$$\sec A = \frac{c}{b} = \frac{13\frac{1}{4}}{10} = 1.3250.$$

$$\sec A = \frac{c}{b} = \frac{35.6}{17.8} = 2.0000.$$

$$\therefore A = 41^\circ.$$

$$\therefore A = 60^\circ.$$

9. A ladder rests against the side of a building, and makes an angle of 28° with the ground. The foot of the ladder is 20 ft. from the building. How long is the ladder?

$$\sec A = \frac{c}{b}.$$

$$c = b \sec A = 20 \times 1.1326 = 22.6520.$$

The ladder is 22.65 ft. long.

10. From a point 50 ft. from a house a wire is stretched to a window so as to make an angle of 30° with the horizontal. Find the length of the wire, assuming it to be straight.

$$\sec A = \frac{c}{b}.$$

$$c = b \sec A = 50 \times 1.1547 = 57.7350.$$

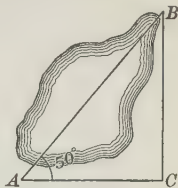
The wire is 57.74 ft. long.

11. In measuring the distance AB a surveyor ran the line AC , making an angle of 50° with AB , and the line BC perpendicular to AC . He measured AC and found that it was 880 ft. Required the distance AB .

$$\sec A = \frac{c}{b}.$$

$$c = b \sec A = 880 \times 1.5557 = 1369.0160.$$

The distance AB is 1369 ft.



12. From the extremity of the shadow cast by a tree the angle of elevation of the top is 47° . The shadow is 62 ft. 6 in. long. How far is it from the top of the tree to the extremity of the shadow?

$$\sec A = \frac{c}{b}.$$

$$c = b \sec A = 62.5 \times 1.4663 = 91.64375.$$

The distance from the top of the tree to the extremity of the shadow is 91.64 ft.

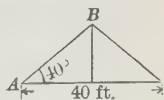
13. The span of this roof is 40 ft., and the roof timbers AB make an angle of 40° with the horizontal. Find the length of AB .

One half the span = 20 ft.

$$\sec A = \frac{c}{b}.$$

$$c = b \sec A = 20 \times 1.3054 = 26.1080.$$

The length AB is 26.11 ft.



Exercise 11. Use of the Cosecant (Page 22)

Find c to four significant figures, given the following :

1. $a = 24$, $A = 29^\circ$.

4. $a = 56\frac{1}{2}$, $A = 61^\circ$.

$$c = a \csc A = 24 \times 2.0627 = 49.50.$$

$$c = a \csc A = 56\frac{1}{2} \times 1.1434 = 64.60$$

2. $a = 36$, $A = 41^\circ$.

5. $a = 75.8$, $A = 69^\circ$.

$$c = a \csc A = 36 \times 1.5243 = 54.87.$$

$$c = a \csc A = 75.8 \times 1.0711 = 81.19.$$

3. $a = 56$, $A = 44^\circ$.

6. $a = 146.9$, $A = 74^\circ$.

$$c = a \csc A = 56 \times 1.4396 = 80.62.$$

$$c = a \csc A = 146.9 \times 1.0403 = 152.8$$

Find A , given the following :

7. $a = 10, c = 11.126.$

$$\csc A = \frac{c}{a} = \frac{11.126}{10} = 1.1126.$$

$$\therefore A = 64^\circ.$$

9. $a = 5\frac{1}{2}, c = 6.0687.$

$$\csc A = \frac{c}{a} = \frac{6.0687}{5\frac{1}{2}} = 1.1034.$$

$$\therefore A = 65^\circ.$$

8. $a = 13, c = 27.6913.$

$$\csc A = \frac{c}{a} = \frac{27.6913}{13} = 2.1301.$$

$$\therefore A = 28^\circ.$$

10. $a = 75, c = 106.065.$

$$\csc A = \frac{c}{a} = \frac{106.065}{75} = 1.4142.$$

$$\therefore A = 45^\circ.$$

11. Seen from a point on the ground the angle of elevation of an aeroplane is 64° . If the aeroplane is 1000 ft. above the ground, how far is it in a straight line from the observer?

$$\csc A = \frac{c}{a}.$$

$$c = a \csc A = 1000 \times 1.1126 = 1112.6000.$$

The aeroplane is 1113 ft. from the observer.

12. A ship sailing 47° east of north changes its latitude 28 mi. in 3 hr. What is its rate of sailing per hour?

$$A = 90^\circ - 47^\circ = 43^\circ.$$

$$\csc A = \frac{c}{a}.$$

$$c = a \csc A = 28 \times 1.4663 = 40.8564.$$

The rate of sailing is 13.69 mi. per hour.

13. A ship sailing 63° east of south changes its latitude 45 mi. in 5 hr. What is its rate of sailing per hour?

$$A = 90^\circ - 63^\circ = 27^\circ.$$

$$\csc A = \frac{c}{a}.$$

$$c = a \csc A = 45 \times 2.2027 = 99.1215.$$

The rate of sailing is 19.82 mi. per hour.

14. From the top of a lighthouse 100 ft., including the instrument above the level of the sea a boat is observed under an angle of depression of 22° . How far is the boat from the point of observation?

$$\csc A = \frac{c}{a}.$$

$$c = a \csc A = 100 \times 2.6695 = 266.95.$$

The point is 267.0 ft. from the point of observation.

15. Seen from a point on the ground the angle of elevation of the top of a telegraph pole 27 ft. high is 28° . How far is it from the point of observation to the top of the pole?

$$\csc A = \frac{c}{a}.$$

$$c = a \csc A = 27 \times 2.1301 = 57.5127.$$

The distance from the point of observation to the top of the pole is 57.51 ft.

16. What is the length of the hypotenuse of a right triangle of which one side is $11\frac{3}{4}$ in. and the opposite angle 43° ?

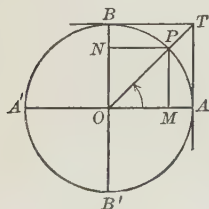
$$\csc A = \frac{c}{a}.$$

$$c = a \csc A = 11\frac{3}{4} \times 1.4663 = 17.229025.$$

The hypotenuse is 17.23 in.

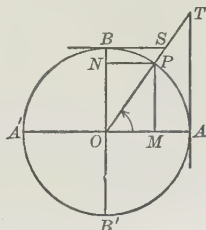
Exercise 12. Functions as Lines (Page 23)

1. Represent by lines the functions of 45° .



$$\begin{aligned}\sin 45^\circ &= MP; \cos 45^\circ = OM; \\ \tan 45^\circ &= AT; \cot 45^\circ = BT; \\ \sec 45^\circ &= OT; \csc 45^\circ = OT.\end{aligned}$$

2. Represent by lines the functions of an acute angle greater than 45° .



$$\begin{aligned}\sin x &= MP; \cos x = OM; \\ \tan x &= AT; \cot x = BS; \\ \sec x &= OT; \csc x = OS.\end{aligned}$$

Using the above figure, determine which is the greater :

3. $\sin x$ or $\tan x$.

$$MP = \sin x, AT = \tan x.$$

$$OM : MP = OA : AT.$$

$$\text{But } OM < OA.$$

$$\therefore MP < AT.$$

$$\therefore \sin x < \tan x,$$

$$\tan x > \sin x.$$

4. $\sin x$ or $\sec x$.

$$MP = \sin x, OT = \sec x.$$

$$OP < OT.$$

$$\text{In rt. } \triangle OMP, OP > MP.$$

$$\therefore MP < OT.$$

$$\therefore \sin x < \sec x,$$

$$\sec x > \sin x.$$

or

or

5. $\sec x$ or $\tan x$.

$$OT = \sec x, AT = \tan x.$$

In rt. $\triangle OAT$, $OT > AT$.

$$\therefore \sec x > \tan x.$$

6. $\csc x$ or $\cot x$.

$$OS = \csc x, BS = \cot x.$$

In rt. $\triangle BOS$, $OS > BS$.

$$\therefore \csc x > \cot x.$$

7. $\cos x$ or $\cot x$.

$$OM = \cos x, BS = \cot x.$$

$$OP : OS = OM : BS$$

$$OP < OS.$$

$$\therefore OM < BS.$$

$$\therefore \cos x < \cot x,$$

$$\cot x > \cos x.$$

or

8. $\cos x$ or $\csc x$.

$$OM = \cos x, OS = \csc x.$$

In rt. $\triangle OMP$, $OP > OM$.

$$\therefore OS > OM.$$

$$\therefore \csc x > \cos x.$$

Construct the angle x , given the following :

9. $\tan x = 3$.

Let BAB' be a unit circle with center O ; then construct a tangent to the circle at A . On this tangent take $AT = 3OA$; then AOT is the required angle.

10. $\csc x = 2$.

Let ABA' be a unit circle, with center O ; construct BN tangent to the circle at B . With center O and radius equal to $2OA$ describe an arc cutting BN at S ; draw OS .

Then AOS is the required angle.

11. $\cos x = \frac{1}{2}$.

Take $OM = \frac{1}{2}$ radius OA . At M erect a $\perp$ to meet the circumference at P . Draw OP .

Then POM is the angle required.

12. $\sin x = \cos x$.

Let $MP = \sin x$ and $OM = \cos x$.

But, by hypothesis, $MP = OM$.

Therefore, by geometry, $x = 45^\circ$.

Hence, construct an angle of 45° as the required angle.

13. $\sin x = 2 \cos x$.

Construct rt. $\angle PMO$, making $PM = 2OM$. Draw OP .

Then POM is the angle required.

14. $4 \sin x = \tan x$.

Take $\frac{1}{4}$ of radius OA to M . At M erect a $\perp$ to meet the circumference at P . Draw OP .

Then POM is the required angle.

15. Show that the sine of an angle is equal to one half the chord of twice the angle in a unit circle.

Given $\angle POA$.

Construct $POB = 2POA$. Draw chord PB . Then it is $\perp$ to OA ; and PM , its half, is the sine of POA .

$$\therefore \sin x = \frac{1}{2} \text{ chord } 2x.$$

16. Find x if $\sin x$ is equal to one half the side of a regular decagon inscribed in a unit circle.

Let AC be a side of a decagon.

Then $AOC = \frac{1}{10} \times 360^\circ = 36^\circ$. Draw OB bisecting AC . Then $\angle AOC$ is bisected, and $\angle AOB = 18^\circ$.

But the sine of $AOB = \frac{1}{2} AC$.

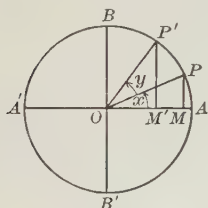
$$\therefore x \text{ or } AOB = 18^\circ.$$

Given x and y , $x + y$ being less than 90° , construct a line equal to:

17. $\sin(x + y) - \sin x$.

Let $P'M' = \sin(x + y)$ in a circle whose center is O , and $PM = \sin x$. Then, with a radius equal to PM , describe an arc from M' as center, cutting $P'M'$ in N . Then

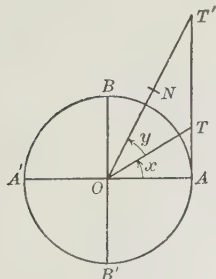
$$P'N = \sin(x + y) - \sin x.$$



18. $\tan(x + y) - \tan x$.

Let $AT' = \tan(x + y)$ in a circle whose center is O , and $AT = \tan x$. Then

$$AT' - AT = TT' = \tan(x + y) - \tan x.$$

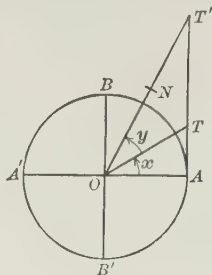


19. $\sec(x + y) - \sec x$.

Let $OT' = \sec(x + y)$ in a circle whose center is O , and $OT = \sec x$. Then, with a radius equal to OT ,

describe an arc from O as center, cutting OT' in N . Then

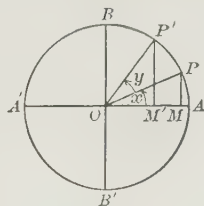
$$T'N = \sec(x + y) - \sec x.$$



20. $\cos x - \cos(x + y)$.

Let $OM' = \cos(x + y)$ in a circle whose center is O , and $OM = \cos x$.

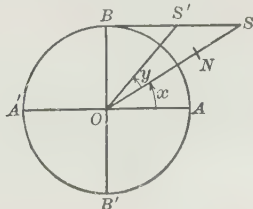
Then $OM - OM' = M'M$ and $M'M = \cos x - \cos(x + y)$.



21. $\cot x - \cot(x + y)$.

Let $BS' = \cot(x + y)$ in a circle whose center is O , and $BS = \cot x$.

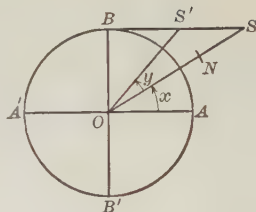
Then $BS - BS' = SS'$ and $SS' = \cot x - \cot(x + y)$.



22. $\csc x - \csc(x + y)$.

Let $OS' = \csc(x + y)$ in a circle whose center is O , and $OS = \csc x$. Then, with a radius equal to OS' , describe an arc from O as center, cutting OS in N . Then

$$NS = \csc x - \csc(x + y).$$



23. $\tan(x + y) - \sin(x + y) + \tan x - \sin x$.

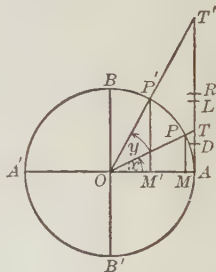
Let $AT' = \tan(x + y)$ in a circle whose center is O , and $P'M' = \sin(x + y)$, $AT = \tan x$, and $PM = \sin x$.

From A with a radius $= P'M'$ take AR .

From R with a radius $= AT$ take RD .

From D with a radius $= PM$ take DL in the opposite direction.

Then $T'L$ is the constructed value of $\tan(x + y) - \sin(x + y) + \tan x - \sin x$.



Given an angle x , construct an angle y such that:

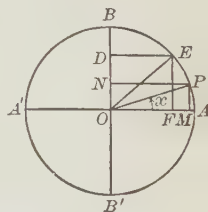
24. $\sin y = 2 \sin x$.

Let PM be the sine of the $\angle x$ in a circle whose center is O .

Draw PN perpendicular to the vertical diameter.

Then $NO = PM$.

Take ND on vertical diameter $= NO$. Draw DE perpendicular to vertical diameter, meeting circumference at E . Draw EF perpendicular to OM and draw OE .



$$OD = 2 MP \text{ by construction}$$

$$EF = OD, EF = 2 PM.$$

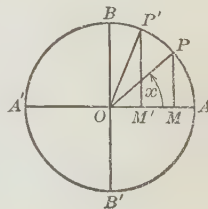
$$\therefore \angle EOM = \text{angle required.}$$

25. $\cos y = \frac{1}{2} \cos x$.

Let OM be the cosine of the $\angle x$ in a circle whose center is O .

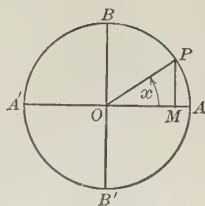
Take OM' on OM equal to $\frac{1}{2} OM$.

At M' erect a perpendicular $P'M'$ meeting the circumference in P' . Draw $P'O$. $\therefore \angle MOP' = \text{angle required}$.



26. $\sin y = \cos x$.

Let OM be the cosine of the $\angle x$ in a circle whose center is O . Then OM is the sine of the $\angle OPM$, the angle required.



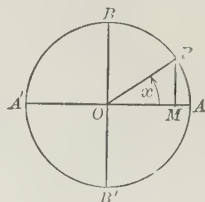
27. $\tan y = \cot x$.

Since $\tan y = \cot x$,

$$\frac{a}{b} = \frac{b}{a}.$$

$$\therefore a = b.$$

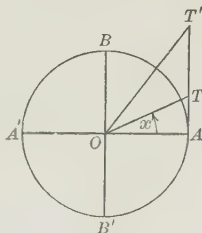
Hence, construct an isosceles right triangle. The required angle will be 45° .



28. $\tan y = 3 \tan x$.

Let AT be the tangent of the $\angle x$ in a circle whose center is O .

Produce AT to T' , making AT' equal to $3 AT$. Draw OT' . Then $\angle AOT'$ is the required angle.



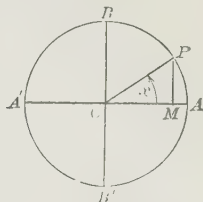
29. $\sec y = \csc x$.

Since $\sec y = \csc x$,

$$\frac{c}{b} = \frac{c}{a}.$$

$$\therefore a = b.$$

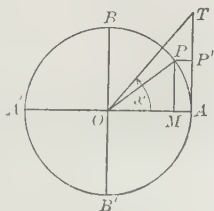
Hence, construct an isosceles right triangle. The required angle will be 45° .



30. $\sin y = \frac{1}{2} \tan x$.

Let AT be the tangent of the $\angle x$ in a circle whose center is O .

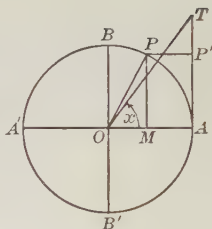
On AT take $AP' = \frac{1}{2} AT$ and draw through P' a line parallel to the horizontal axis cutting the circumference in P . Draw PM perpendicular to OA . Draw the line OP . Then $PM = P'A = \frac{1}{2} AT$. $\angle AOP$ is the required angle.



31. $\sin y = \frac{2}{3} \tan x$.

Let AT be the tangent of the $\angle x$ in a circle whose center is O .

On AT take AP' equal to $\frac{2}{3} AT$ and draw through P' a line parallel to the horizontal axis intersecting the circumference in P . Draw PM perpendicular to OA . Draw the line OP . Then $PM = P'A = \frac{2}{3} AT$. $\angle AOP$ is the required angle.



32. Show by construction that $2 \sin A > \sin 2A$, when $A < 45^\circ$.

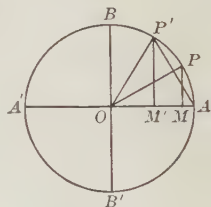
Construct $\angle AOP$ and $\angle POP'$ each equal to the given $\angle A$ and less than 45° . $PM = \sin A$, $P'M' = \sin 2A$.

$$P'A = 2 \sin A, \text{ and } P'M' = \sin 2A.$$

But

$$P'A > P'M'.$$

$$\therefore 2 \sin A > \sin 2A.$$



33. Show by construction that $\cos A < 2 \cos 2A$, when $A < 30^\circ$.

Construct $\angle AOP$ and $\angle POP'$ each equal to the given $\angle A$, less than 30° .

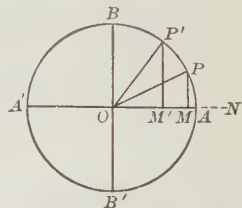
$$OM = \cos A, OM' = \cos 2A.$$

On OA lay off ON equal to $2 OM'$.

$$ON = 2 OM' > OM.$$

$$OM < ON.$$

$$\therefore \cos A < 2 \cos 2A.$$



34. Given two angles A and B , $A + B$ being less than 90° ; show that $\sin(A + B) < \sin A + \sin B$.

Construct $\angle AOP = \angle A$ and $\angle POP' = \angle B$.

Then $\sin(A + B) = P'M'$, $\sin A = PM$, and $\sin B = P'N$. Draw NR perpendicular to OA .

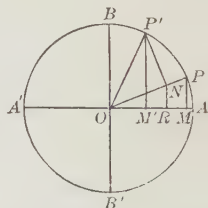
$$P'M' < P'N + NR,$$

and

$$PM > NR.$$

$$\therefore P'M' < P'N + PM.$$

$$\therefore \sin(A + B) < \sin A + \sin B.$$



35. Given $\sin x$ in a unit circle; find the length of a line in a circle of radius r corresponding in position to $\sin x$.

$1 : r = \sin x$: the required line.

Therefore the length of line required $= r \sin x$.

36. In a right triangle, given the hypotenuse c , and $\sin A = m$; find the two sides.

$$\sin A = \frac{a}{c} = m.$$

$$\therefore a = cm.$$

$$b^2 = c^2 - a^2 = c^2 - c^2 m^2.$$

$$b = c \sqrt{1 - m^2}.$$

37. In a right triangle, given the side b , and $\tan A = m$; find the other side and the hypotenuse.

$$\tan A = \frac{a}{b} = m.$$

$$\therefore a = bm.$$

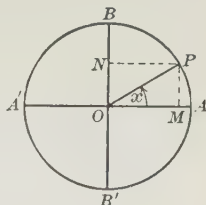
$$c^2 = a^2 + b^2 = b^2 m^2 + b^2.$$

$$c = b \sqrt{m^2 + 1}.$$

Construct, or show that it is impossible to construct, the angle x , given the following:

38. $\sin x = \frac{1}{2}$.

In a unit circle whose center is O , take ON on the vertical axis equal to $\frac{1}{2}$ of the radius. From N draw NP parallel to OA and meeting the circumference at P . From P draw PM perpendicular to OA and meeting the axis at M . Then $PM = NO = \frac{1}{2}$. Draw PO . Then $\angle POA$ is the angle required.



39. $\sin x = 1$.

In a unit circle, since $\sin x = 1$, a side of the triangle must equal 1, and hence the radius OB .

The angle whose sine is 1 is 90° .

The sine of the angle increases until it reaches the limit 1, as the angle increases to 90° .

40. $\sin x = \frac{5}{4}$.

It is impossible to construct the angle x , for a right triangle having a side greater than the hypotenuse does not exist

41. $\cos x = 0$.

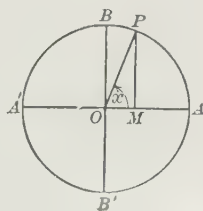
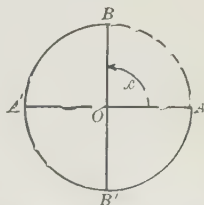
Since in a unit circle $\cos x = OM = 0$, M must coincide with O and angle $x = 90^\circ$.

42. $\cos x = \frac{4}{3}$.

It is impossible to construct the angle x , for a right triangle having a side greater than the hypotenuse does not exist.

43. $\cos x = \frac{1}{3}$.

Take $OM = \frac{1}{3}$ radius OA of a unit circle. At M erect a perpendicular intersecting the circumference at P . Draw OP . Then POM is the required angle.



44. $\tan x = \frac{4}{3}$.

In a unit circle whose center is O , draw AT the tangent at A , the extremity of the horizontal axis, and equal to $\frac{4}{3}$ of the radius OA . Draw OT . Then $\angle AOT$ is the required angle.

45. $\cot x = \frac{1}{2}$.

In a unit circle whose center is O , draw BS the tangent at B , the extremity of the vertical axis, and equal to $\frac{1}{2}$ of OB the radius. $BS = \frac{1}{2} = \cot x$. Then $\angle BOS$ is the required angle.

46. $\sec x = \frac{1}{2}$.

It is impossible to construct the angle x , for a right triangle having a side greater than the hypotenuse does not exist.

47. Using a protractor, draw the figure to show that

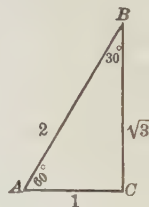
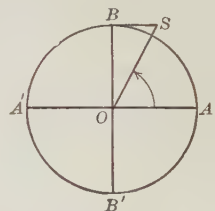
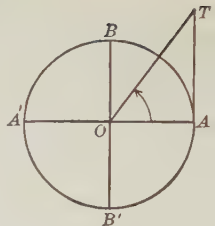
$$\sin 60^\circ = \cos \left(\frac{1}{2} \text{ of } 60^\circ \right),$$

and

$$\sin 30^\circ = \cos (2 \times 30^\circ).$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \cos 30^\circ = \frac{\sqrt{3}}{2}.$$

$$\sin 30^\circ = \frac{1}{2}, \cos 60^\circ = \frac{1}{2}.$$



Exercise 13. Functions as Lines. (Page 26)

1. Draw a figure to show that $\sin 90^\circ = 1$.

$$\sin x = MP.$$

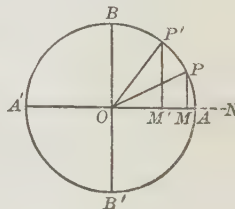
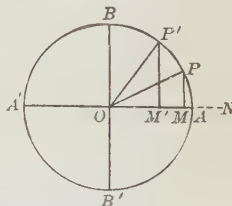
As the angle increases MP increases to $M'P'$, and so on, P' moving along the circumference of the circle until it reaches B , the limit. Then $x = 90^\circ$ and $OB = \sin 90^\circ = 1$, the radius of the circle.

2. What is the value of $\cos 90^\circ$? Draw a figure to show this.

$$\cos 90^\circ = 0; \cos x = OM.$$

As x increases M moves along OM towards O . At 90° , M coincides with O and hence

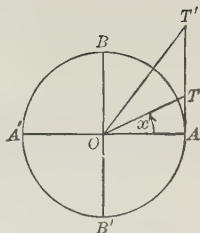
$$OM = \cos 90^\circ = 0.$$



3. What is the value of $\sec 0^\circ$? Draw a figure to show this.

$$\sec 0^\circ = 1; \sec x = OT'.$$

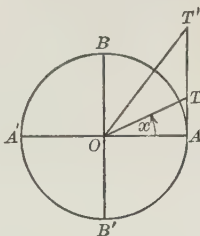
As x approaches 0° , OT' , OT , etc., approach OA . At 0° OT coincides with OA , and hence equals 1.



4. What is the value of $\tan 90^\circ$? Draw a figure to show this.

$$\tan 90^\circ = \infty; \tan x = AT.$$

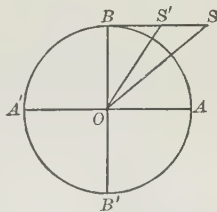
As x increases, AT increases to AT' , etc., until at 90° AT is infinitely long when OT is parallel to AT .



5. What is the value of $\cot 90^\circ$? Draw a figure to show this.

$$\cot 90^\circ = 0; \cot x = BS.$$

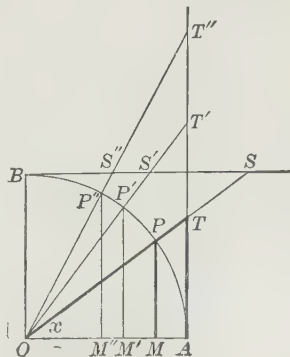
As x increases, BS decreases to BS' , etc., until S reaches B . Then $x = 90^\circ$ and $BS = 0$.



6. As the angle increases, which increases the more rapidly, the sine or the tangent? Show this by reference to the figure.

The tangent increases more rapidly.

In the figure, PM is the sine of angle x and AT the tangent. As x grows larger, the difference between $P'M'$ and $T'A'$ increases over the difference between PM and TA .



7. If you double an angle, does this double the sine? Show this by reference to the figure.

No.

$$\sin x = PM.$$

$$\sin 2x = P'M'.$$

On $M'P'$ lay off $M'N$ equal to MP . $M'N$ is not $\frac{1}{2}$ of $P'M'$.

$$2PM \text{ does not equal } P'M'.$$

8. If you bisect an angle, does this bisect the tangent? Prove it.

No.

$$\tan x = TA.$$

$$\tan \frac{x}{2} = T'A.$$

On AT lay off $T'D$ equal to AT' .

$$T'A \text{ does not equal } \frac{1}{2}TA.$$

9. State the angle for which these relations are true:

$$\sin x = \cos x, \tan x = \cot x, \sec x = \csc x.$$

Show this by reference to the figure.

$$\sin x = \cos x \text{ when } x = 45^\circ. \text{ Then } PM(\sin x) = OM(\cos x) = 1.$$

$$\tan x = \cot x \text{ when } x = 45^\circ. \text{ Then } TA(\tan x) = BS(\cot x) = 1.$$

$$\sec x = \csc x \text{ when } x = 45^\circ. \text{ Then } OT(\sec x) = OS(\csc x) = 1.$$

10. If you know that $\sin 40^\circ 15' = 0.6461$, and $\cos 40^\circ 15' = 0.7632$, and that the difference between each of these and the sine and cosine of $40^\circ 15' 30''$ is 0.0001, what is $\sin 40^\circ 15' 30''$? $\cos 40^\circ 15' 30''$?

$$\sin 40^\circ 15' 30'' = 0.6461 + 0.0001 = 0.6462.$$

$$\cos 40^\circ 15' 30'' = 0.7632 - 0.0001 = 0.7631.$$

11. If you know that $\tan 20^\circ 12' = 0.3679$, and that the difference between this and $\tan 20^\circ 12' 15''$ is 0.0001, what is $\tan 20^\circ 12' 15''$?

$$\tan 20^\circ 12' 15'' = 0.3679 + 0.0001 = 0.3680.$$

12. If you know that $\cot 20^\circ 12' = 2.7179$, and that the difference between this and $\cot 20^\circ 12' 15''$ is 0.0006, what is $\cot 20^\circ 12' 15''$?

$$\cot 20^\circ 12' 15'' = 2.7179 - 0.0006 = 2.7173.$$

13. If you know that $\tan 66.5^\circ = 2.2998$, and that the difference between this and $\tan 66.6^\circ$ is 0.0111, what is $\tan 66.6^\circ$?

$$\tan 66.6^\circ = 2.2998 + 0.0111 = 2.3109.$$

14. If you know that $\cos 57.4^\circ = 0.5388$, and that the difference between this and $\cos 57.5^\circ$ is 0.0015, what is $\cos 57.5^\circ$?

$$\cos 57.5^\circ = 0.5388 - 0.0015 = 0.5373.$$

Draw the angle x , for which the functions have the following values and state (page 11) to the nearest degree the value of the angle:

15. $\sin x = 0.1$.

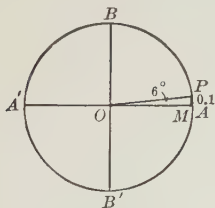
$$\sin x = 0.1.$$

$$x = 6^\circ.$$

Draw a unit circle with center O .

With the protractor lay off $\angle AOP = 6^\circ$.

Then $MP = \sin 6^\circ = 0.1$.

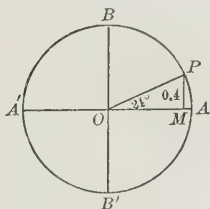


16. $\sin x = 0.4$.

$$\sin x = 0.4.$$

$$x = 24^\circ.$$

$$MP = \sin 24^\circ = 0.4.$$

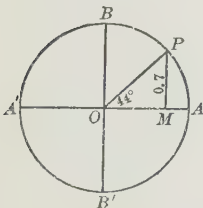


17. $\sin x = 0.7$.

$$\sin x = 0.7.$$

$$x = 44^\circ.$$

$$MP = \sin 44^\circ = 0.7.$$

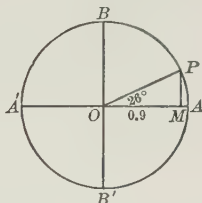


18. $\cos x = 0.9$.

$$\cos x = 0.9.$$

$$x = 26^\circ.$$

$$OM = \cos 26^\circ = 0.9$$

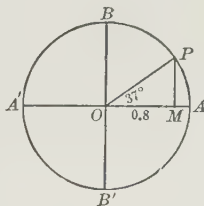


19. $\cos x = 0.8$.

$$\cos x = 0.8.$$

$$x = 37^\circ.$$

$$OM = \cos 37^\circ = 0.8.$$

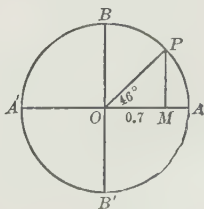


20. $\cos x = 0.7$.

$$\cos x = 0.7.$$

$$x = 46^\circ.$$

$$OM = \cos 46^\circ = 0.7$$

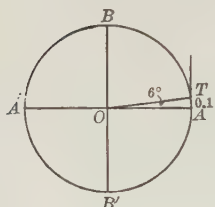


21. $\tan x = 0.1.$

$\tan x = 0.1.$

$x = 6^\circ.$

$AT = \tan 6^\circ = 0.1.$

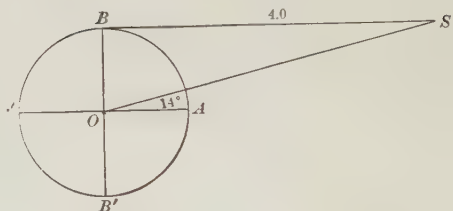


24. $\cot x = 4.0.$

$\cot x = 4.0.$

$x = 14^\circ.$

$BS = \cot 14^\circ = 4.0.$

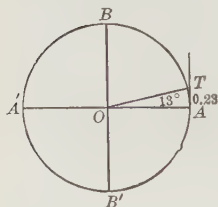


22. $\tan x = 0.23.$

$\tan x = 0.23.$

$x = 13^\circ.$

$AT = \tan 13^\circ = 0.23.$

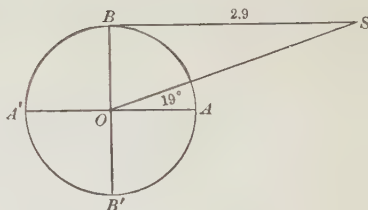


25. $\cot x = 2.9.$

$\cot x = 2.9.$

$x = 19^\circ.$

$BS = \cot 19^\circ = 2.9.$

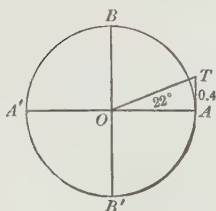


23. $\tan x = 0.4.$

$\tan x = 0.4.$

$x = 22^\circ.$

$AT = \tan 22^\circ = 0.4.$

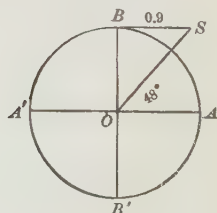


26. $\cot x = 0.9.$

$\cot x = 0.9.$

$x = 48^\circ.$

$BS = \cot 48^\circ = 0.9.$

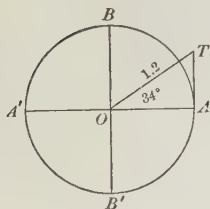


27. $\sec x = 1.2$.

$\sec x = 1.2$.

$x = 34^\circ$.

$OT = \sec 34^\circ = 1.2$.

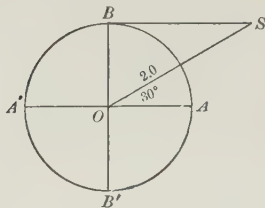


30. $\csc x = 2.0$.

$\csc x = 2.0$.

$x = 30^\circ$.

$OS = \csc 30^\circ = 2.0$.

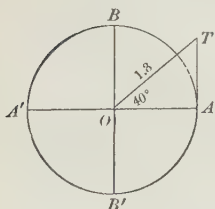


28. $\sec x = 1.3$.

$\sec x = 1.3$.

$x = 40^\circ$.

$OT = \sec 40^\circ = 1.3$.

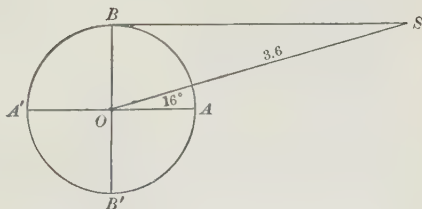


31. $\csc x = 3.6$.

$\csc x = 3.6$.

$x = 16^\circ$.

$OS = \csc 16^\circ = 3.6$.

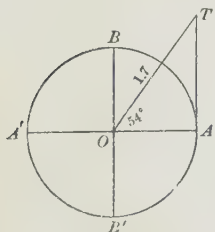


29. $\sec x = 1.7$.

$\sec x = 1.7$.

$x = 54^\circ$.

$OT = \sec 54^\circ = 1.7$.

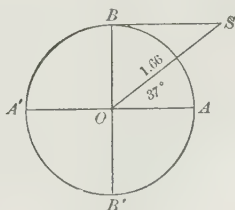


32. $\csc x = 1.66$.

$\csc x = 1.66$.

$x = 37^\circ$.

$OS = \csc 37^\circ = 1.66$.



33. Find the value of $\sin x$ in the equation $\sin x - \frac{1}{\sin x} + 1.5 = 0$. Which root is admissible? Why is the other root impossible?

$$\sin x - \frac{1}{\sin x} + 1.5 = 0$$

$$(\sin x)^2 - 1 + 1\frac{1}{2} \sin x = 0$$

$$2(\sin x)^2 - 2 + 3 \sin x = 0$$

$$2(\sin x)^2 + 3 \sin x - 2 = 0$$

$$(2 \sin x - 1)(\sin x + 2) = 0$$

$$\sin x = \frac{1}{2} \text{ or } -2.$$

$$\frac{1}{2} \text{ is admissible.}$$

-2 is impossible because the sine cannot have an absolute value greater than one.

Exercise 14. Use of the Sexagesimal Table (Page 29)

From the table on page 28 find the values of the following:

1. $\cos 41^\circ = 0.7547$.

2. $\tan 42^\circ = 0.9004$.

3. $\cos 41^\circ 1' = 0.7545$.

4. $\tan 42^\circ 2' = 0.9015$.

5. $\cos 41^\circ 5' = 0.7538$.

6. $\sin 48^\circ 59' = 0.7545$.

7. $\sin 47^\circ 58' = 0.7428$.

8. $\cos 48^\circ 59' = 0.6563$.

9. $\cos 47^\circ 59' = 0.6693$.

10. $\cos 48^\circ 57' = 0.6567$.

11. $\sin 42^\circ 4' = 0.6700$.

12. $\cos 47^\circ 56' = 0.6700$.

13. $\tan 41^\circ 3' = 0.8708$.

14. $\cot 48^\circ 57' = 0.8708$.

15. $\tan 48^\circ 57' = 1.1483$.

In the right triangle ACB , in which $C = 90^\circ$:

16. Given $c = 27$ and $A = 41^\circ 3'$, find a .

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A$$

$$= 27 \times 0.6567$$

$$= 17.7309.$$

$$17.73. \text{ Ans.}$$

18. Given $c = 61$ and $A = 41^\circ 2'$, find b .

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A$$

$$= 61 \times 0.7543$$

$$= 46.0123.$$

$$46.01. \text{ Ans.}$$

20. Given $b = 24$ and $A = 41^\circ 3'$, find a .

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A$$

$$= 24 \times 0.8708$$

$$= 20.8992.$$

$$20.90. \text{ Ans.}$$

17. Given $c = 48$ and $A = 42^\circ 4'$, find a .

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A$$

$$= 48 \times 0.6700$$

$$= 32.1600.$$

$$32.16. \text{ Ans.}$$

19. Given $c = 72$ and $A = 42^\circ 3'$, find b .

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A$$

$$= 72 \times 0.7426$$

$$= 53.4672.$$

$$53.47. \text{ Ans.}$$

21. Given $b = 28$ and $A = 42^\circ 4'$, find a .

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A$$

$$= 28 \times 0.9025$$

$$= 25.2700.$$

$$25.27. \text{ Ans.}$$

22. Given $a = 42$ and $A = 41^\circ 1'$, find b .

$$\cot A = \frac{b}{a}.$$

$$\begin{aligned} b &= a \cot A \\ &= 42 \times 1.1497 \\ &= 48.2874. \end{aligned}$$

48.29. *Ans.*

23. Given $a = 60$ and $A = 42^\circ 4'$, find b .

$$\cot A = \frac{b}{a}.$$

$$\begin{aligned} b &= a \cot A \\ &= 60 \times 1.1080 \\ &= 66.4800. \end{aligned}$$

66.48. *Ans.*

24. Given $c = 86$ and $A = 48^\circ 56'$, find a .

$$\sin A = \frac{a}{c}.$$

$$\begin{aligned} a &= c \sin A \\ &= 86 \times 0.7539 \\ &= 64.8354. \end{aligned}$$

64.84. *Ans.*

25. Given $c = 92$ and $A = 48^\circ 57'$, find a .

$$\sin A = \frac{a}{c}.$$

$$\begin{aligned} a &= c \sin A \\ &= 92 \times 0.7541 \\ &= 69.3772. \end{aligned}$$

69.38. *Ans.*

26. Given $b = 45$ and $A = 47^\circ 55'$, find a .

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned} a &= b \tan A \\ &= 45 \times 1.1074 \\ &= 49.8330. \end{aligned}$$

49.83. *Ans.*

28. Given $a = 86$ and $A = 48^\circ 56'$, find b .

$$\cot A = \frac{b}{a}.$$

$$\begin{aligned} b &= a \cot A \\ &= 86 \times 0.8713 \\ &= 74.9318. \end{aligned}$$

74.93. *Ans.*

29. Given $a = 98$ and $A = 47^\circ 58'$, find b .

$$\cot A = \frac{b}{a}.$$

$$\begin{aligned} b &= a \cot A \\ &= 98 \times 0.9015 \\ &= 88.3470. \end{aligned}$$

88.35. *Ans.*

27. Given $b = 85$ and $A = 47^\circ 59'$, find a .

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned} a &= b \tan A \\ &= 85 \times 1.1100 \\ &= 94.3500. \end{aligned}$$

94.35. *Ans.*

30. Given $b = 67$ and $c = 100$, find A .

$$\cos A = \frac{b}{c}$$

$$\begin{aligned} &= 0.67 \\ &= 0.6700 \\ &= \cos 47^\circ 56'. \end{aligned}$$

47° 56'. *Ans.*

31. A hoisting crane has an arm 30 ft. long. When the arm makes an angle of $41^\circ 3'$ with x , what is the length of y ? what is the length of x ?

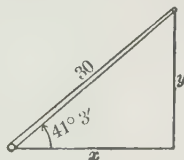
$$\sin A = \frac{a}{c}.$$

$$a = c \sin A = 30 \times 0.6567 = 19.7010.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 30 \times 0.7541 = 22.6230.$$

The length of y is 19.70 ft. and of x is 22.62 ft.



32. In Ex. 31 suppose the arm is raised until it makes an angle of $41^\circ 5'$ with x , what are then the lengths of y and x ?

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A = 30 \times 0.6572 = 19.7160.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 30 \times 0.7538 = 22.6140.$$

The length of y is 19.72 ft. and of x is 22.61 ft.

33. From a point 128 ft. from a building the angle of elevation of the top is observed, by aid of an instrument 5 ft. above the ground, to be $42^{\circ} 4'$. What is the height of the building?

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A$$

$$= 128 \times 0.9025$$

$$= 115.5200.$$

$$115.5 \text{ ft.} + 5 \text{ ft.} = 120.5 \text{ ft.}$$

The height of the building is 120.5 ft.

34. From the top of a building 62 ft. 6 in. high, including the instrument, the angle of depression of the foot of an electric-light pole is observed to be $41^{\circ} 3'$. How far is the pole from the building?

$$\cot A = \frac{b}{a}.$$

$$b = a \cot A$$

$$= 62.5 \times 1.1483$$

$$= 71.76875.$$

The pole is 71.77 ft. from the building.

Exercise 15. Use of the Decimal Table (Page 30)

From the above table find the values of the following :

$$1. \sin 0.5^{\circ} = 0.0087. \quad 8. \tan 4.4^{\circ} = 0.0769. \quad 15. \cot 85.6^{\circ} = 0.0769$$

$$2. \tan 0.4^{\circ} = 0.0070. \quad 9. \cot 4.5^{\circ} = 12.71. \quad 16. \sin 89.5^{\circ} = 1.0000$$

$$3. \sin 4^{\circ} = 0.0698. \quad 10. \cot 4.2^{\circ} = 13.62. \quad 17. \cos 85.9^{\circ} = 0.0715$$

$$4. \cos 4.2^{\circ} = 0.9973. \quad 11. \sin 85.7^{\circ} = 0.9972. \quad 18. \tan 89.6^{\circ} = 143.2.$$

$$5. \tan 4.5^{\circ} = 0.0787. \quad 12. \sin 85.9^{\circ} = 0.9974. \quad 19. \cot 89.7^{\circ} = 0.0052$$

$$6. \sin 4.1^{\circ} = 0.0715. \quad 13. \cos 85.6^{\circ} = 0.0767. \quad 20. \cot 85.8^{\circ} = 0.0734$$

$$7. \cos 4.3^{\circ} = 0.9972. \quad 14. \tan 85.9^{\circ} = 13.95.$$

21. The hypotenuse of a right triangle is 12.7 in., and one acute angle is 85.5° . Find the two perpendicular sides.

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A = 12.7 \times 0.9969 = 12.66063.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A = 12.7 \times 0.0785 = 0.99695.$$

The sides are 12.66 in. and 0.9970 in.

22. From a point on the top of a house the angle of depression of the foot of a tree is observed to be 4.4° . The house, including the instrument is 30 ft. high. How far is the tree from the house?

$$\cot A = \frac{b}{a}.$$

$$b = a \cot A = 30 \times 13.00 = 390.$$

The tree is 390 ft. from the house.

23. A rectangle has a base 9.5 in. long, and the diagonal makes an angle of 4.5° with the base. Find the height of the rectangle and the length of the diagonal.

$$\tan A = \frac{a}{b}.$$

$$\cos A = \frac{b}{c}.$$

$$a = b \tan A = 9.5 \times 0.0787 = 0.74765.$$

$$c = \frac{b}{\cos A} = \frac{9.5}{0.9969} = 9.5295.$$

The height of the rectangle is 0.7477 in. and the length of the diagonal is 9.530 in.

Exercise 16. Use of the Table (Page 33)

Find the values of the following :

1. $\sin 27^\circ 10' 30''.$

$$\sin 27^\circ 11' = 0.4568$$

$$\sin 27^\circ 10' = \frac{0.4566}{0.0002}$$

$$\frac{3}{8} \frac{0}{0} \text{ of } 0.0002 = 0.0001.$$

$$\therefore \sin 27^\circ 10' 30'' = 0.4567.$$

5. $\cos 36^\circ 14' 30''.$

$$\cos 36^\circ 14' = 0.8066$$

$$\cos 36^\circ 15' = \frac{0.8064}{0.0002}$$

$$\frac{3}{8} \frac{0}{0} \text{ of } 0.0002 = 0.0001$$

$$\therefore \cos 36^\circ 14' 30'' = 0.8065$$

2. $\sin 42^\circ 15' 30''.$

$$\sin 42^\circ 16' = 0.6726$$

$$\sin 42^\circ 15' = \frac{0.6724}{0.0002}$$

$$\frac{3}{8} \frac{0}{0} \text{ of } 0.0002 = 0.0001.$$

$$\therefore \sin 42^\circ 15' 30'' = 0.6725.$$

6. $\cos 43^\circ 12' 20''.$

$$\cos 43^\circ 12' = 0.7290$$

$$\cos 43^\circ 13' = \frac{0.7288}{0.0002}$$

$$\frac{2}{6} \frac{0}{0} \text{ of } 0.0002 = 0.0001.$$

$$\therefore \cos 43^\circ 12' 20'' = 0.7289.$$

3. $\sin 56^\circ 29' 40''.$

$$\sin 56^\circ 30' = 0.8339$$

$$\sin 56^\circ 29' = \frac{0.8337}{0.0002}$$

$$\frac{4}{6} \frac{0}{0} \text{ of } 0.0002 = 0.0001.$$

$$\therefore \sin 56^\circ 29' 40'' = 0.8338.$$

7. $\cos 64^\circ 18' 45''.$

$$\cos 64^\circ 18' = 0.4337$$

$$\cos 64^\circ 19' = \frac{0.4334}{0.0003}$$

$$\frac{4}{6} \frac{5}{0} \text{ of } 0.0003 = 0.0002.$$

$$\therefore \cos 64^\circ 18' 45'' = 0.4335.$$

4. $\sin 65^\circ 29' 40''.$

$$\sin 65^\circ 30' = 0.9100$$

$$\sin 65^\circ 29' = \frac{0.9098}{0.0002}$$

$$\frac{4}{6} \frac{0}{0} \text{ of } 0.0002 = 0.0001.$$

$$\therefore \sin 65^\circ 29' 40'' = 0.9099.$$

8. $\tan 28^\circ 32' 20''.$

$$\tan 28^\circ 33' = 0.5441$$

$$\tan 28^\circ 32' = \frac{0.5437}{0.0004}$$

$$\frac{2}{6} \frac{0}{0} \text{ of } 0.0004 = 0.0001.$$

$$\therefore \tan 28^\circ 32' 20'' = 0.5438.$$

9. $\tan 32^\circ 41' 30''$.

$$\tan 32^\circ 42' = 0.6420$$

$$\tan 32^\circ 41' = \underline{0.6416}$$

$$0.0004$$

$$\frac{3}{8} \text{ of } 0.0004 = 0.0002.$$

$$\therefore \tan 32^\circ 41' 30'' = 0.6418.$$

10. $\tan 42^\circ 38' 30''$.

$$\tan 42^\circ 39' = 0.9212$$

$$\tan 42^\circ 38' = \underline{0.9206}$$

$$0.0006$$

$$\frac{3}{8} \text{ of } 0.0006 = 0.0003.$$

$$\therefore \tan 42^\circ 38' 30'' = 0.9209.$$

11. $\tan 52^\circ 10' 45''$.

$$\tan 52^\circ 11' = 1.2884$$

$$\tan 52^\circ 10' = \underline{1.2876}$$

$$0.0008$$

$$\frac{4}{8} \text{ of } 0.0008 = 0.0006.$$

$$\therefore \tan 52^\circ 10' 45'' = 1.2882.$$

12. $\tan 68^\circ 12' 45''$.

$$\tan 68^\circ 13' = 2.5023$$

$$\tan 68^\circ 12' = \underline{2.5002}$$

$$0.0021$$

$$\frac{4}{8} \text{ of } 0.0021 = 0.0016.$$

$$\therefore \tan 68^\circ 12' 45'' = 2.5018.$$

13. $\tan 72^\circ 15' 50''$.

$$\tan 72^\circ 16' = 3.1271$$

$$\tan 72^\circ 15' = \underline{3.1240}$$

$$0.0031$$

$$\frac{5}{8} \text{ of } 0.0031 = 0.0026.$$

$$\therefore \tan 72^\circ 15' 50'' = 3.1266.$$

14. $\tan 85^\circ 17' 45''$.

$$\tan 85^\circ 18' = 12.1632$$

$$\tan 85^\circ 17' = \underline{12.1201}$$

$$0.0431$$

$$\frac{4}{8} \text{ of } 0.0431 = 0.0323.$$

$$\therefore \tan 85^\circ 17' 45'' = 12.1524.$$

15. $\tan 86^\circ 15' 50''$.

$$\tan 86^\circ 16' = 15.3254$$

$$\tan 86^\circ 15' = \underline{15.2571}$$

$$0.0683$$

$$\frac{5}{8} \text{ of } 0.0683 = 0.0569.$$

$$\therefore \tan 86^\circ 15' 50'' = 15.3140.$$

16. $\cot 5^\circ 27' 30''$.

$$\cot 5^\circ 27' = 10.4813$$

$$\cot 5^\circ 28' = \underline{10.4491}$$

$$0.0322$$

$$\frac{3}{8} \text{ of } 0.0322 = 0.0161.$$

$$\therefore \cot 5^\circ 27' 30'' = 10.4652.$$

17. $\cot 6^\circ 32' 45''$.

$$\cot 6^\circ 32' = 8.7317$$

$$\cot 6^\circ 33' = \underline{8.7093}$$

$$0.0224$$

$$\frac{4}{8} \text{ of } 0.0224 = 0.0168.$$

$$\therefore \cot 6^\circ 32' 45'' = 8.7149.$$

18. $\cot 7^\circ 52' 50''$.

$$\cot 7^\circ 52' = 7.2375$$

$$\cot 7^\circ 53' = \underline{7.2220}$$

$$0.0155$$

$$\frac{5}{8} \text{ of } 0.0155 = 0.0129.$$

$$\therefore \cot 7^\circ 52' 50'' = 7.2246.$$

19. $\cot 8^\circ 40' 10''$.

$$\cot 8^\circ 40' = 6.5606$$

$$\cot 8^\circ 41' = \underline{6.5478}$$

$$0.0128$$

$$\frac{1}{8} \text{ of } 0.0128 = 0.0021.$$

$$\therefore \cot 8^\circ 40' 10'' = 6.5585.$$

20. $\cot 9^\circ 20' 10''$.

$$\cot 9^\circ 20' = 6.0844$$

$$\cot 9^\circ 21' = \underline{6.0734}$$

$$0.0110$$

$$\frac{1}{8} \text{ of } 0.0110 = 0.0018.$$

$$\therefore \cot 9^\circ 20' 10'' = 6.0826.$$

21. Given $\sin x = 0.6391$, find x . Then find $\cos x$.

$$\begin{array}{lll}
 0.6392 = \sin 39^\circ 44' & 0.6391 = \sin x & \cos 39^\circ 43' = 0.7692 \\
 \underline{0.6390} = \sin 39^\circ 43' & \underline{0.6390} = \sin 39^\circ 43' & \underline{\cos 39^\circ 44' = 0.7690} \\
 \underline{0.0002} & \underline{0.0001} = & \underline{0.0002} \\
 \frac{1}{2} \text{ of } 60'' = 30'' & \frac{3}{6} \frac{0}{0} \text{ of } 0.0002 = 0.0001 & \\
 \therefore x = 39^\circ 43' 30'' & \therefore \cos 39^\circ 43' 30'' = 0.7691. &
 \end{array}$$

22. Given $\sin x = 0.7691$, find x . Then find $\cos x$.

$$\begin{array}{lll}
 0.7692 = \sin 50^\circ 17' & 0.7691 = \sin x & \cos 50^\circ 16' = 0.6392 \\
 \underline{0.7690} = \sin 50^\circ 16' & \underline{0.7690} = \sin 50^\circ 16' & \underline{\cos 50^\circ 17' = 0.6390} \\
 \underline{0.0002} & \underline{0.0001} & \underline{0.0002} \\
 \frac{1}{2} \text{ of } 60'' = 30'' & \frac{3}{6} \frac{0}{0} \text{ of } 0.0002 = 0.0001 & \\
 \therefore x = 50^\circ 16' 30'' & \therefore \cos 50^\circ 16' 30'' = 0.6391 &
 \end{array}$$

23. Given $\cos x = 0.3174$, find x . Then find $\sin x$.

$$\begin{array}{lll}
 \cos 71^\circ 29' = 0.3176 & 0.3176 = \cos 71^\circ 29' & \sin 71^\circ 30' = 0.9483 \\
 \cos 71^\circ 30' = \underline{0.3173} & \underline{0.3174} = \cos x & \underline{\sin 71^\circ 29' = 0.9482} \\
 \underline{0.0003} & \underline{0.0002} & \underline{0.0001} \\
 \frac{2}{3} \text{ of } 60'' = 40'' & \frac{4}{6} \frac{0}{0} \text{ of } 0.0001 = 0.0001 & \\
 \therefore x = 71^\circ 29' 40'' & \therefore \sin 71^\circ 29' 30'' = 0.9483. &
 \end{array}$$

24. Given $\tan x = 2.8649$, find x . Then find $\cot x$.

$$\begin{array}{lll}
 2.8662 = \tan 70^\circ 46' & 2.8649 = \tan x & \cot 70^\circ 45' = 0.3492 \\
 \underline{2.8636} = \tan 70^\circ 45' & \underline{2.8636} = \tan 70^\circ 45' & \underline{\cot 70^\circ 46' = 0.3489} \\
 \underline{0.0026} & \underline{0.0013} & \underline{0.0003} \\
 \frac{1}{2} \frac{3}{6} \text{ of } 60'' = 30'' & \frac{3}{6} \frac{0}{0} \text{ of } 0.0003 = 0.0002 & \\
 \therefore x = 70^\circ 45' 30'' & \therefore \cot 70^\circ 45' 30'' = 0.3490. &
 \end{array}$$

25. Given $\tan x = 5.3977$, find x . Then find $\cot x$.

$$\begin{array}{lll}
 5.4043 = \tan 79^\circ 31' & 5.3977 = \tan x & \cot 79^\circ 30' = 0.1853 \\
 \underline{5.3955} = \tan 79^\circ 30' & \underline{5.3955} = \tan 79^\circ 30' & \underline{\cot 79^\circ 31' = 0.1850} \\
 \underline{0.0088} & \underline{0.0022} & \underline{0.0003} \\
 \frac{2}{8} \frac{8}{8} \text{ of } 60'' = 15'' & \frac{1}{8} \frac{5}{0} \text{ of } 0.0003 = 0.0001 & \\
 \therefore x = 79^\circ 30' 15'' & \therefore \cot 79^\circ 30' 15'' = 0.1852 &
 \end{array}$$

First converting to sexagesimals, find the following :

Reduce the decimals in Exs. 26-40 by Table X, page 104.

26. $\sin 25.5^\circ$.

27. $\sin 25.55^\circ$.

$$\begin{array}{l}
 25.5^\circ = 25^\circ 30'. \\
 \sin 25^\circ 30' = 0.4305.
 \end{array}$$

$$\begin{array}{l}
 25.55^\circ = 25^\circ 33'. \\
 \sin 25^\circ 33' = 0.4313.
 \end{array}$$

28. $\sin 32.75^\circ$.

$$32.75^\circ = 32^\circ 45'.$$

$$\sin 32^\circ 45' = 0.5410.$$

29. $\sin 41.65^\circ$.

$$41.65^\circ = 41^\circ 39'.$$

$$\sin 41^\circ 39' = 0.6646.$$

30. $\sin 64.75^\circ$.

$$64.75^\circ = 64^\circ 45'.$$

$$\sin 64^\circ 45' = 0.9045.$$

31. $\cos 78.52^\circ$.

$$78.52^\circ = 78^\circ 31' 12''.$$

$$\cos 78^\circ 31' = 0.1991$$

$$\cos 78^\circ 32' = \frac{0.1988}{0.0003}$$

$$\frac{1}{6} \frac{2}{0} \text{ of } 0.0003 = 0.0001.$$

$$\cos 78^\circ 31' 12'' = 0.1990.$$

32. $\tan 78.59^\circ$.

$$78.59^\circ = 78^\circ 35' 24''.$$

$$\tan 78^\circ 36' = 4.9594$$

$$\tan 78^\circ 35' = \frac{4.9520}{0.0074}$$

$$\frac{2}{6} \frac{4}{0} \text{ of } 0.0074 = 0.0030.$$

$$\tan 78^\circ 35' 24'' = 4.9550.$$

33. $\cos 81.43^\circ$.

$$81.43^\circ = 81^\circ 25' 48''.$$

$$\cos 81^\circ 25' = 0.1492$$

$$\cos 81^\circ 26' = \frac{0.1490}{0.0002}$$

$$\frac{4}{6} \frac{8}{0} \text{ of } 0.0002 = 0.0002.$$

$$\cos 81^\circ 25' 48'' = 0.1490.$$

34. $\tan 82.72^\circ$.

$$82.72^\circ = 82^\circ 43' 12''.$$

$$\tan 82^\circ 44' = 7.8424$$

$$\tan 82^\circ 43' = \frac{7.8243}{0.0181}$$

$$\frac{1}{6} \frac{2}{0} \text{ of } 0.0181 = 0.0036.$$

$$\tan 82^\circ 43' 12'' = 7.8279.$$

35. $\tan 84.68^\circ$.

$$84.68^\circ = 84^\circ 40' 48''.$$

$$\tan 84^\circ 41' = 10.7457$$

$$\tan 84^\circ 40' = \frac{10.7119}{0.0338}$$

$$\frac{4}{6} \frac{8}{0} \text{ of } 0.0338 = 0.0270.$$

$$\tan 84^\circ 40' 48'' = 10.7389.$$

36. $\cos 11.25^\circ$.

$$11.25^\circ = 11^\circ 15'.$$

$$\cos 11^\circ 15' = 0.9808.$$

37. $\cot 12.32^\circ$.

$$12.32^\circ = 12^\circ 19' 12''.$$

$$\cot 12^\circ 19' = 4.5800$$

$$\cot 12^\circ 20' = \frac{4.5736}{0.0064}$$

$$\frac{1}{6} \frac{2}{0} \text{ of } 0.0064 = 0.0013.$$

$$\cot 12^\circ 19' 12'' = 4.5787.$$

38. $\cot 13.54^\circ$.

$$13.54^\circ = 13^\circ 32' 24''.$$

$$\cot 13^\circ 32' = 4.1547$$

$$\cot 13^\circ 33' = \frac{4.1493}{0.0054}$$

$$\frac{2}{6} \frac{4}{0} \text{ of } 0.0054 = 0.0022.$$

$$\cot 13^\circ 32' 24'' = 4.1525.$$

39. $\cot 15.48^\circ$.

$$15.48^\circ = 15^\circ 28' 48''.$$

$$\cot 15^\circ 28' = 3.6140$$

$$\cot 15^\circ 29' = \frac{3.6100}{0.0040}$$

$$\frac{4}{6} \frac{8}{0} \text{ of } 0.0040 = 0.0032.$$

$$\cot 15^\circ 28' 48'' = 3.6108.$$

40. $\cot 16.62^\circ$.

$$16.62^\circ = 16^\circ 37' 12''.$$

$$\cot 16^\circ 37' = 3.3509$$

$$\cot 16^\circ 38' = \frac{3.3473}{0.0036}$$

$$\frac{1}{6} \frac{2}{0} \text{ of } 0.0036 = 0.0007.$$

$$\cot 16^\circ 37' 12'' = 3.3502.$$

Find the value of x in each of the following equations :

41. $\sin x = 0.5225$.

$$0.5225 = \sin 31^\circ 30'.$$

$$\therefore x = 31^\circ 30'.$$

42. $\sin x = 0.5771$.

$$0.5771 = \sin 35^\circ 15'.$$

$$\therefore x = 35^\circ 15'.$$

43. $\sin x = 0.6601$.

$$0.6602 = \sin 41^\circ 19'$$

$$\frac{0.6600}{0.0002} = \sin 41^\circ 18'$$

$$0.0002$$

$$0.6601 = \sin x$$

$$\frac{0.6600}{0.0001} = \sin 41^\circ 18'$$

$$0.0001$$

$$\frac{1}{2} \text{ of } 60'' = 30''.$$

$$x = 41^\circ 18' 30''.$$

44. $\sin x = 0.7023$.

$$0.7024 = \sin 44^\circ 37'$$

$$\frac{0.7022}{0.0002} = \sin 44^\circ 36'$$

$$0.0002$$

$$0.7023 = \sin x$$

$$\frac{0.7022}{0.0001} = \sin 44^\circ 36'$$

$$0.0001$$

$$\frac{1}{2} \text{ of } 60'' = 30''.$$

$$x = 44^\circ 36' 30''.$$

45. $\cos x = 0.7853$.

$$0.7853 = \cos 38^\circ 15'.$$

$$\therefore x = 38^\circ 15'.$$

46. $\cos x = 0.7716$.

$$0.7716 = \cos 39^\circ 30'.$$

$$\therefore x = 39^\circ 30'.$$

47. $\cos x = 0.9524$.

$$0.9524 = \cos 17^\circ 45'.$$

$$\therefore x = 17^\circ 45'.$$

48. $\cos x = 0.7115$.

$$0.7116 = \cos 44^\circ 38'$$

$$\frac{0.7114}{0.0002} = \cos 44^\circ 39'$$

$$0.0002$$

$$0.7115 = \cos x$$

$$\frac{0.7114}{0.0001} = \cos 44^\circ 39'$$

$$0.0001$$

$$\frac{1}{2} \text{ of } 60'' = 30''.$$

$$x = 44^\circ 38' 30''.$$

49. $\tan x = 2.6395$.

$$2.6395 = \tan 69^\circ 15'.$$

$$\therefore x = 69^\circ 15'.$$

50. $\tan x = 4.7625$.

$$4.7659 = \tan 78^\circ 9'$$

$$\frac{4.7591}{0.0068} = \tan 78^\circ 8'$$

$$0.0068$$

$$4.7625 = \tan x$$

$$\frac{4.7591}{0.0034} = \tan 78^\circ 8'$$

$$0.0034$$

$$\frac{34}{6} \text{ of } 60'' = 30''.$$

$$x = 78^\circ 8' 30''.$$

51. $\tan x = 4.7608$.

$$4.7659 = \tan 78^\circ 9'$$

$$\frac{4.7591}{0.0068} = \tan 78^\circ 8'$$

$$0.0068$$

$$4.7608 = \tan x$$

$$\frac{4.7591}{0.0017} = \tan 78^\circ 8'$$

$$0.0017$$

$$\frac{17}{6} \text{ of } 60'' = 15''.$$

$$x = 78^\circ 8' 15''.$$

52. $\cot x = 3.7983$.

$$3.7983 = \cot 14^\circ 45'.$$

$$\therefore x = 14^\circ 45'.$$

53. If $\sin x = 0.6431$, what is the value of $\cos x$?

$$0.6432 = \sin 40^\circ 2'$$

$$\underline{0.6430} = \sin 40^\circ 1'$$

$$0.0002$$

$$0.6431 = \sin x$$

$$\underline{0.6430} = \sin 40^\circ 1'$$

$$0.0001$$

$$\frac{1}{2} \text{ of } 60'' = 30''.$$

$$x = 40^\circ 1' 30''.$$

$$\cos 40^\circ 1' = 0.7659$$

$$\cos 40^\circ 2' = \underline{0.7657}$$

$$0.0002$$

$$\frac{3}{8} \text{ of } 0.0002 = 0.0001.$$

$$\therefore \cos 40^\circ 1' 30'' = 0.7658.$$

54. If $\cos x = 0.7652$, what is the value of $\sin x$?

$$0.7653 = \cos 40^\circ 4'$$

$$\underline{0.7651} = \cos 40^\circ 5'$$

$$0.0002$$

$$0.7653 = \cos 40^\circ 4'$$

$$\underline{0.7652} = \cos x$$

$$0.0001$$

$$\frac{1}{2} \text{ of } 60'' = 30''.$$

$$x = 40^\circ 4' 30''.$$

$$\sin 40^\circ 5' = 0.6439$$

$$\sin 40^\circ 4' = \underline{0.6437}$$

$$0.0002$$

$$\frac{3}{8} \text{ of } 0.0002 = 0.0001.$$

$$\therefore \sin 40^\circ 4' 30'' = 0.6438.$$

57. If $\cot x = 1.6550$, what is the value of x ? of $\tan x$?

Verify the second result by the relation $\tan x = 1/\cot x$.

$$1.6555 = \cot 31^\circ 8'$$

$$\underline{1.6545} = \cot 31^\circ 9'$$

$$0.0010$$

$$1.6555 = \cot 31^\circ 8'$$

$$\underline{1.6550} = \cot x$$

$$0.0005$$

$$\frac{5}{10} \text{ of } 60'' = 30''.$$

$$x = 31^\circ 8' 30''. \text{ Ans.}$$

55. If $\tan x = 0.6827$, what is the value of $\sin x$?

$$0.6830 = \tan 34^\circ 20'$$

$$\underline{0.6826} = \tan 34^\circ 19'$$

$$0.0004$$

$$0.6827 = \tan x$$

$$\underline{0.6826} = \tan 34^\circ 19'$$

$$0.0001$$

$$\frac{1}{4} \text{ of } 60'' = 15''.$$

$$x = 34^\circ 19' 15''.$$

$$\sin 34^\circ 20' = 0.5640$$

$$\sin 34^\circ 19' = \underline{0.5638}$$

$$0.0002$$

$$\frac{1}{6} \text{ of } 0.0002 = 0.0001.$$

$$\therefore \sin 34^\circ 19' 15'' = 0.5639.$$

56. If $\tan x = 0.6537$, what is the value of x ? of $\cot x$?

$$0.6540 = \tan 33^\circ 11'$$

$$\underline{0.6536} = \tan 33^\circ 10'$$

$$0.0004$$

$$0.6537 = \tan x$$

$$\underline{0.6536} = \tan 33^\circ 10'$$

$$0.0001$$

$$\frac{1}{4} \text{ of } 60'' = 15''.$$

$$x = 33^\circ 10' 15''. \text{ Ans.}$$

$$\cot 33^\circ 10' = 1.5301$$

$$\cot 33^\circ 11' = \underline{1.5291}$$

$$0.0010$$

$$\frac{1}{6} \text{ of } 0.0010 = 0.0003.$$

$$\therefore \cot 33^\circ 10' 15'' = 1.5298. \text{ Ans.}$$

$$\tan 31^\circ 9' = 0.6044$$

$$\tan 31^\circ 8' = \underline{0.6040}$$

$$0.0004$$

$$\frac{3}{8} \text{ of } 0.0004 = 0.0002.$$

$$\tan 31^\circ 8' 30'' = 0.6042.$$

$$\therefore \tan x = 0.6042. \text{ Ans.}$$

$$\tan x = \frac{1}{\cot x}. \quad 0.6042 = \frac{1}{1.6550} = 0.6042.$$

Exercise 17. The Right Triangle (Page 37)

Solve the right triangle ACB , in which $C = 90^\circ$, given:

1. $a = 3, b = 4.$

$$\tan A = \frac{a}{b} = \frac{3}{4} = 0.7500.$$

$$\therefore A = 36^\circ 52'.$$

$$B = 90^\circ - A = 53^\circ 8'.$$

$$\begin{aligned} c &= \sqrt{a^2 + b^2} = \sqrt{9 + 16} \\ &= \sqrt{25} = 5. \end{aligned}$$

2. $a = 7, c = 13.$

$$\sin A = \frac{a}{c} = \frac{7}{13} = 0.5385.$$

$$\therefore A = 32^\circ 35'.$$

$$B = 90^\circ - A = 57^\circ 25'.$$

$$\begin{aligned} b &= \sqrt{(c + a)(c - a)} = \sqrt{20 \times 6} \\ &= \sqrt{120} = 10.95. \end{aligned}$$

3. $a = 5.3, A = 12^\circ 17'.$

$$B = 90^\circ - A = 77^\circ 43'.$$

$$\frac{a}{c} = \sin A.$$

$$\frac{b}{a} = \cot A.$$

$$\therefore c = \frac{a}{\sin A} = \frac{5.3}{0.2127} = 24.92$$

$$\therefore b = a \cot A = 5.3 \times 4.5928 = 24.34.$$

4. $a = 10.4, B = 43^\circ 18'.$

$$A = 90^\circ - B = 46^\circ 42'.$$

$$\frac{a}{c} = \cos B.$$

$$\frac{b}{a} = \tan B.$$

$$\therefore c = \frac{a}{\cos B} = \frac{10.4}{0.7278} = 14.29.$$

$$\therefore b = a \tan B = 10.4 \times 0.9424 = 9.801.$$

5. $c = 26, A = 37^\circ 42'.$

$$B = 90^\circ - A = 52^\circ 18'.$$

$$\frac{b}{c} = \cos A.$$

$$\frac{a}{c} = \sin A.$$

$$b = c \cos A = 26 \times 0.7912 = 20.57.$$

$$\therefore a = c \sin A = 26 \times 0.6115 = 15.90.$$

6. $c = 140, B = 24^\circ 12'.$

$$A = 90^\circ - B = 65^\circ 48'.$$

$$\frac{b}{c} = \sin B.$$

$$\frac{a}{c} = \cos B.$$

$$\therefore b = c \sin B = 140 \times 0.4099 = 57.39$$

$$\therefore a = c \cos B = 140 \times 0.9121 = 127.7.$$

7. $b = 19, c = 23.$

$$\sin B = \frac{b}{c} = \frac{19}{23} = 0.8261.$$

$$B = 55^\circ 42'.$$

$$A = 90^\circ - B = 34^\circ 18'.$$

$$\begin{aligned} a &= \sqrt{(c + b)(c - b)} = \sqrt{42 \times 4} \\ &= \sqrt{168} = 12.960. \end{aligned}$$

8. $b = 98, c = 135.2$.

$$\sin B = \frac{b}{c} = \frac{98}{135.2} = 0.7249, \quad a = \sqrt{(c+a)(c-a)} = \sqrt{233.2 \times 37.2}$$

$$= \sqrt{8675.04} = 93.14.$$

$$B = 46^\circ 27'.$$

$$A = 90^\circ - B = 43^\circ 33'.$$

9. $b = 42.4, A = 32^\circ 14'.$

$$B = 90^\circ - A = 57^\circ 46', \quad \frac{b}{c} = \cos A.$$

$$\frac{a}{b} = \tan A, \quad \therefore c = \frac{b}{\cos A} = \frac{42.4}{0.8459} = 50.12.$$

$$\therefore a = b \tan A = 42.4 \times 0.6305 = 26.73.$$

10. $b = 200, B = 46^\circ 11'.$

$$A = 90^\circ - B = 43^\circ 49', \quad \frac{b}{c} = \sin B.$$

$$\frac{a}{b} = \cot B, \quad \therefore c = \frac{b}{\sin B} = \frac{200}{0.7216} = 277.2.$$

$$\therefore a = b \cot B = 200 \times 0.9595 = 191.9.$$

11. $a = 95, b = 37.$

$$\tan A = \frac{a}{b} = \frac{95}{37} = 2.5676.$$

$$c = \sqrt{a^2 + b^2} = \sqrt{9025 + 1369}$$

$$= \sqrt{10394} = 102.0.$$

$$\therefore A = 68^\circ 43'.$$

$$B = 90^\circ - A = 21^\circ 17'.$$

12. $a = 6, c = 103.$

$$\sin A = \frac{a}{c} = \frac{6}{103} = 0.0583.$$

$$b = \sqrt{(c+a)(c-a)} = \sqrt{109 \times 97}$$

$$= \sqrt{10573} = 102.8.$$

$$\therefore A = 3^\circ 20'.$$

$$B = 90^\circ - A = 86^\circ 40'.$$

13. $a = 3.12, B = 5^\circ 8'.$

$$A = 90^\circ - B = 84^\circ 52', \quad \frac{a}{c} = \cos B.$$

$$\frac{b}{a} = \tan B, \quad \therefore c = \frac{a}{\cos B} = \frac{3.12}{0.9960} = 3.133.$$

$$\therefore b = a \tan B = 3.12 \times 0.0898 = 0.2802.$$

14. $a = 17, c = 18.$

$$\sin A = \frac{a}{c} = \frac{17}{18} = 0.9444.$$

$$b = \sqrt{(c+a)(c-a)} = \sqrt{35 \times 1}$$

$$= \sqrt{35} = 5.916.$$

$$\therefore A = 70^\circ 48'.$$

$$B = 90^\circ - A = 19^\circ 12'.$$

$$15. c = 57, A = 38^\circ 29'.$$

$$B = 90^\circ - A = 51^\circ 31'.$$

$$\frac{a}{c} = \sin A.$$

$$\frac{b}{c} = \cos A.$$

$$\therefore b = c \cos A = 57 \times 0.7828 = 44.62$$

$$\therefore a = c \sin A = 57 \times 0.6223 = 35.47.$$

$$16. a + c = 18, b = 12.$$

$$c^2 - a^2 = b^2.$$

$$(c + a)(c - a) = b^2.$$

$$18(c - a) = 144.$$

$$c - a = 8.$$

$$\therefore c = 13.$$

$$a = 5.$$

$$\sin A = \frac{a}{c} = \frac{5}{13} = 0.3846.$$

$$\therefore A = 22^\circ 37'.$$

$$B = 90^\circ - A = 67^\circ 23'.$$

$$17. a + c = 90, b = 30.$$

$$c^2 - a^2 = b^2.$$

$$(c + a)(c - a) = b^2.$$

$$90(c - a) = 900.$$

$$c - a = 10.$$

$$\therefore c = 50.$$

$$a = 40.$$

$$\sin A = \frac{a}{c} = \frac{40}{50} = 0.8000.$$

$$\therefore A = 53^\circ 8'.$$

$$B = 90^\circ - A = 36^\circ 52'.$$

$$18. a + c = 45, b = 30.$$

$$c^2 - a^2 = b^2.$$

$$(c + a)(c - a) = b^2.$$

$$45(c - a) = 900.$$

$$c - a = 20.$$

$$\therefore c = 32.5.$$

$$a = 12.5.$$

$$\sin A = \frac{a}{c} = \frac{12.5}{32.5} = 0.3846.$$

$$\therefore A = 22^\circ 37'.$$

$$B = 90^\circ - A = 67^\circ 23'.$$

Solve the right triangle ACB , in which $C = 90^\circ$, given:

$$19. a = 2.5, A = 35^\circ 10' 30''.$$

$$B = 90^\circ - A = 54^\circ 49' 30''.$$

$$\frac{a}{c} = \sin A.$$

$$\frac{b}{a} = \cot A.$$

$$\therefore b = a \cot A = 2.5 \times 1.4189 = 3.547$$

$$\therefore c = \frac{a}{\sin A} = \frac{2.5}{0.5761} = 4.340.$$

$$20. a = 5.7, A = 42^\circ 12' 30''.$$

$$B = 90^\circ - A = 47^\circ 47' 30''.$$

$$\frac{b}{a} = \cot A.$$

$$\frac{a}{c} = \sin A.$$

$$\therefore b = a \cot A = 5.7 \times 1.1025 = 6.284. \quad \therefore c = \frac{a}{\sin A} = \frac{5.7}{0.6718} = 8.485.$$

21. $a = 6.4$, $B = 29^\circ 18' 30''$.

$$A = 90^\circ - B = 60^\circ 41' 30''.$$

$$\frac{b}{a} = \tan B.$$

$$\therefore b = a \tan B = 6.4 \times 0.5614 = 3.593.$$

$$\frac{a}{c} = \cos B.$$

$$\therefore c = \frac{a}{\cos B} = \frac{6.4}{0.8720} = 7.339.$$

22. $a = 7.9$, $B = 36^\circ 20' 30''$.

$$A = 90^\circ - B = 53^\circ 39' 30''.$$

$$\frac{b}{a} = \tan B.$$

$$\therefore b = a \tan B = 7.9 \times 0.7357 = 5.812.$$

$$\frac{a}{c} = \cos B.$$

$$\therefore c = \frac{a}{\cos B} = \frac{7.9}{0.8055} = 9.808.$$

23. $c = 6.8$, $A = 29^\circ 42' 30''$.

$$B = 90^\circ - A = 60^\circ 17' 30''.$$

$$\frac{a}{c} = \sin A.$$

$$\therefore a = c \sin A = 6.8 \times 0.4956 = 3.370.$$

$$\frac{b}{c} = \cos A.$$

$$\therefore b = c \cos A = 6.8 \times 0.8685 = 5.906.$$

24. $c = 360$, $A = 34^\circ 20' 30''$.

$$B = 90^\circ - A = 55^\circ 39' 30''.$$

$$\frac{a}{c} = \sin A.$$

$$\therefore a = c \sin A = 360 \times 0.5641 = 203.08.$$

$$\frac{b}{c} = \cos A.$$

$$\therefore b = c \cos A = 360 \times 0.8257 = 297.25.$$

25. $b = 250$, $A = 41^\circ 10' 40''$.

$$B = 90^\circ - A = 48^\circ 49' 20''.$$

$$\frac{a}{b} = \tan A.$$

$$\therefore a = b \tan A = 250 \times 0.8747 = 218.68.$$

$$\frac{b}{c} = \cos A.$$

$$\therefore c = \frac{b}{\cos A} = \frac{250}{0.7527} = 332.14.$$

26. $a = 48$, $A = 25.5^\circ$.

$$B = 90^\circ - A = 64.5^\circ.$$

$$\frac{b}{a} = \cot A.$$

$$\therefore b = a \cot A = 48 \times 2.0965 = 100.6.$$

$$\frac{a}{c} = \sin A.$$

$$\therefore c = \frac{a}{\sin A} = \frac{48}{0.4305} = 111.5.$$

27. $c = 25$, $A = 24.5^\circ$.

$$B = 90^\circ - A = 65.5^\circ.$$

$$\frac{a}{c} = \sin A.$$

$$\therefore a = c \sin A = 25 \times 0.4147 = 10.37$$

$$\frac{b}{c} = \cos A.$$

$$\therefore b = c \cos A = 25 \times 0.9100 = 22.75.$$

28. $c = 40$, $A = 32.55^\circ$.

$$B = 90^\circ - A = 57.45^\circ.$$

$$\frac{b}{c} = \cos A.$$

$$\frac{a}{c} = \sin A.$$

$$\therefore b = c \cos A = 40 \times 0.8429 = 33.72.$$

$$\therefore a = c \sin A = 40 \times 0.5380 = 21.52.$$

29. $c = 80$, $A = 55.51^\circ$.

$$B = 90^\circ - A = 34.49^\circ.$$

$$\frac{b}{c} = \cos A.$$

$$\frac{a}{c} = \sin A.$$

$$\therefore b = c \cos A = 80 \times 0.5663 = 45.30.$$

$$\therefore a = c \sin A = 80 \times 0.8242 = 65.94.$$

30. $c = 75$, $A = 63.46^\circ$.

$$B = 90^\circ - A = 26.54^\circ.$$

$$\frac{b}{c} = \cos A.$$

$$\frac{a}{c} = \sin A.$$

$$\therefore b = c \cos A = 75 \times 0.4468 = 33.51.$$

$$\therefore a = c \sin A = 75 \times 0.8946 = 67.10.$$

31. $a = 45$, $B = 50.59^\circ$.

$$A = 90^\circ - B = 39.41^\circ.$$

$$\frac{a}{c} = \cos B.$$

$$\frac{b}{a} = \tan B.$$

$$\therefore c = \frac{a}{\cos B} = \frac{45}{0.6349} = 70.88.$$

$$\therefore b = a \tan B = 45 \times 1.2170 = 54.77.$$

32. $b = 90$, $A = 68.25^\circ$.

$$B = 90^\circ - A = 21.75^\circ.$$

$$\frac{b}{c} = \cos A.$$

$$\frac{a}{b} = \tan A.$$

$$\therefore c = \frac{b}{\cos A} = \frac{90}{0.3706} = 242.8.$$

$$\therefore a = b \tan A = 90 \times 2.5065 = 225.6.$$

33. Each equal side of an isosceles triangle is 16 in., and one of the equal angles is $24^\circ 10'$. What is the length of the base?

$$\frac{b}{c} = \cos A.$$

$$\therefore b = c \cos A = \frac{1}{2} \text{ base}.$$

$$2b = 2c \cos A = 2 \times 16 \times 0.9124 = 29.20.$$

The length of the base is 29.20 in.

34. Each equal side of an isosceles triangle is 25 in., and the vertical angle is $36^\circ 40'$. What is the altitude of the triangle?

$$\frac{1}{2} \text{ of } 36^\circ 40' = 18^\circ 20'.$$

$$\frac{a}{c} = \cos B.$$

$$\therefore a = c \cos B = 25 \times 0.9492 = 23.73.$$

The altitude is 23.73 in.

35. Each equal side of an isosceles triangle is 25 in., and one of the equal angles is $32^{\circ} 20' 30''$. What is the length of the base?

$$\frac{b}{c} = \cos A.$$

$$\therefore b = c \cos A = \frac{1}{2} \text{ base.}$$

$$2b = 2c \cos A = 2 \times 25 \times 0.8449 = 42.25.$$

The length of the base is 42.25 in.

36. Each equal side of an isosceles triangle is 60 in., and the vertical angle is $50^{\circ} 30' 30''$. What is the altitude of the triangle?

$$\frac{1}{2} \text{ of } 50^{\circ} 30' 30'' = 25^{\circ} 15' 15''.$$

$$\frac{a}{c} = \cos B.$$

$$\therefore a = c \cos B = 60 \times 0.9044 = 54.26.$$

The altitude is 54.26 in.

37. Find the altitude of an equilateral triangle of which the side is 50 in. Show three methods of finding the altitude.

$$(1) a = \sqrt{c^2 - \frac{1}{4}b^2} = \text{the altitude.} \quad (3) \frac{1}{2} B = 30^{\circ}.$$

$$(2) \text{ Each angle} = 60^{\circ}.$$

$$\frac{a}{c} = \sin A.$$

$$\frac{a}{c} = \cos B$$

$$\therefore a = c \cos 30^{\circ}.$$

$$\therefore a = c \sin 60^{\circ}.$$

Using (2) or (3),

$$a = 50 \times 0.8660 = 43.3.$$

The altitude is 43.3 in.

38. What is the side of an equilateral triangle of which the altitude is 52 in.?

$$A = 60^{\circ}.$$

$$\frac{a}{c} = \sin A.$$

$$\therefore c = \frac{a}{\sin A} = \frac{52}{0.8660} = 60.05.$$

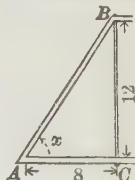
The side is 60.05 in.

39. In planning a truss for a bridge it is necessary to have the upright $BC = 12$ ft., and the horizontal $AC = 8$ ft., as shown in the figure. What angle does AB make with AC ? with BC ?

$$\tan A = \frac{a}{b} = \frac{12}{8} = 1.5000.$$

$$A = 56^{\circ} 18' 36''.$$

$$B = 90^{\circ} - A = 33^{\circ} 41' 24''.$$



AB makes an angle of $56^{\circ} 18' 36''$ with AC and an angle of $33^{\circ} 41' 24''$ with BC .

40. In Ex. 39 what are the angles if $AB = 12$ ft. and $AC = 9$ ft.?

$$\cos A = \frac{b}{c} = \frac{9}{12} = 0.7500.$$

$$A = 41^\circ 24' 30''.$$

$$B = 90^\circ - A = 48^\circ 35' 30''.$$

$$A = 41^\circ 24' 30'' \text{ and } B = 48^\circ 35' 30''.$$

41. In the figure of Ex. 39, what is the length of BC if $AB = 15$ ft. and $x = 62^\circ 10'$?

$$\frac{a}{c} = \sin A.$$

$$\therefore a = c \sin A$$

$$= 15 \times 0.8843 = 13.26.$$

The length of BC is 13.26 ft.

42. Two angles of a triangle are $42^\circ 17'$ and $47^\circ 43'$ respectively, and the included side is 25 in. Find the other two sides.

$$180^\circ - (42^\circ 17' + 47^\circ 43') = 90^\circ.$$

The triangle is a right triangle. Let $A = 42^\circ 17'$, $B = 47^\circ 43'$.

$$\frac{a}{c} = \sin A.$$

$$\therefore a = c \sin A = 25 \times 0.6728 = 16.82.$$

$$\frac{b}{c} = \cos A.$$

$$\therefore b = c \cos A = 25 \times 0.7398 = 18.50.$$

The remaining two sides are 16.82 in. and 18.50 in.

43. A tangent AB , drawn from a point A to a circle, makes an angle of $51^\circ 10'$ with a line from A through the center. If $AB = 10$ ft., what is the length of the radius?

Let C be the center of the circle.

Then BC is the radius and B is a right angle.

$$\frac{BC}{AB} = \tan A.$$

$$\therefore BC = AB \tan A = 10 \times 1.2423 = 12.42.$$

The length of the radius is 12.42 ft.

44. How far from the center of a circle of radius 12 in. will a tangent meet a diameter with which it makes an angle of $10^\circ 20'$?

Let C be the center of the circle, B the point of tangency, and A the point in which the diameter and tangent meet.

$$\frac{BC}{AC} = \sin A.$$

$$\therefore AC = \frac{BC}{\sin A} = \frac{12}{0.1794} = 66.89.$$

The distance AC is 66.89 in.

45. Two circles of radii 10 in. and 14 in. are externally tangent. What angle does their line of centers make with their common exterior tangent?

Let A be the center of the smaller circle and C be the point in which the line of centers meets the common exterior tangent.

$$\begin{aligned}\frac{AC + 24}{AC} &= \frac{14}{10}. & \sin C &= \frac{10}{AC} = \frac{10}{60} \\ 10AC + 240 &= 14AC. & &= 0.1667. \\ 4AC &= 240. & C &= 9^\circ 35' 40''. \\ AC &= 60.\end{aligned}$$

The required angle is $9^\circ 35' 40''$.

Exercise 18. Logarithms (Page 41)

1. Since $2^5 = 32$, what is $\log_2 32$? 3. Since $10^4 = 10,000$, what is $\log 10,000$?
2. Since $4^2 = 16$, what is $\log_4 16$? $\log 10,000 = 4$.
- $\log_2 32 = 5$. $\log_4 16 = 2$.

Write the following logarithms:

- | | | |
|-----------------------|----------------------------|-------------------------|
| 4. $\log_2 16 = 4$. | 8. $\log_3 243 = 5$. | 12. $\log_6 36 = 2$. |
| 5. $\log_2 64 = 6$. | 9. $\log_3 729 = 6$. | 13. $\log_7 343 = 3$. |
| 6. $\log_2 128 = 7$. | 10. $\log_4 256 = 4$. | 14. $\log_8 512 = 3$. |
| 7. $\log_2 256 = 8$. | 11. $\log_5 125 = 3$. | 15. $\log_9 6561 = 4$. |
| 16. $\log 100 = 2$. | 18. $\log 100,000 = 5$. | |
| 17. $\log 1000 = 3$. | 19. $\log 1,000,000 = 6$. | |

20. Since $10^{-1} = \frac{1}{10}$, or 0.1, what is $\log 0.1$?

$$\log 0.1 = -1.$$

21. What is $\log \frac{1}{100}$, or $\log 0.01$? $\log 0.001$? $\log 0.0001$?

$$\log 0.01 = -2; \log 0.001 = -3; \log 0.0001 = -4.$$

22. Between what consecutive integers is $\log 52$? $\log 726$? $\log 2400$? $\log 24,000$? $\log 175,000$? $\log 175,000,000$?

$\log 52$ is between 1 and 2;	$\log 24,000$ is between 4 and 5;
$\log 726$ is between 2 and 3;	$\log 175,000$ is between 5 and 6;
$\log 2400$ is between 3 and 4;	$\log 175,000,000$ is between 8 and 9.

23. Between what consecutive negative integers is $\log 0.08$? $\log 0.008$? $\log 0.0008$? $\log 0.1238$? $\log 0.0123$? $\log 0.002768$?

$\log 0.08$ is between -2 and -1 ;	$\log 0.1238$ is between -1 and 0 ;
$\log 0.008$ is between -3 and -2 ;	$\log 0.0123$ is between -2 and -1 ;
$\log 0.0008$ is between -4 and -3 ;	$\log 0.002768$ is between -3 and -2 .

24. To the base 2, write the logarithms of 2, 4, 8, 64, 512, 1024, $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, $\frac{1}{256}$.

$$\begin{aligned}\log_2 2 &= 1; & \log_2 64 &= 6; & \log_2 \frac{1}{4} &= -2; & \log_2 \frac{1}{64} &= -6; \\ \log_2 4 &= 2; & \log_2 512 &= 9; & \log_2 \frac{1}{16} &= -4; & \log_2 \frac{1}{128} &= -7; \\ \log_2 8 &= 3; & \log_2 1024 &= 10; & \log_2 \frac{1}{32} &= -5; & \log_2 \frac{1}{256} &= -8.\end{aligned}$$

25. To the base 3, write the logarithms of 3, 81, 729, 2187, 6561, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, $\frac{1}{243}$, $\frac{1}{729}$, $\frac{1}{2187}$.

$$\begin{aligned}\log_3 3 &= 1; & \log_3 2187 &= 7; & \log_3 \frac{1}{9} &= -2; & \log_3 \frac{1}{243} &= -5; \\ \log_3 81 &= 4; & \log_3 6561 &= 8; & \log_3 \frac{1}{27} &= -3; & \log_3 \frac{1}{729} &= -6; \\ \log_3 729 &= 6; & \log_3 \frac{1}{3} &= -1; & \log_3 \frac{1}{81} &= -4; & \log_3 \frac{1}{2187} &= -7.\end{aligned}$$

26. To the base 10, write the logarithms of 1, 0.0001, 0.00001, 10,000,000, 100,000,000.

$$\begin{aligned}\log 1 &= 0; & \log 10,000,000 &= 7; \\ \log 0.0001 &= -4; & \log 100,000,000 &= 8. \\ \log 0.00001 &= -5;\end{aligned}$$

Write the consecutive integers between which the logarithms of the following numbers lie:

27. 75.

$\log 75$ is between 1 and 2.

35. 7346.

$\log 7346$ is between 3 and 4.

28. 75.9.

$\log 75.9$ is between 1 and 2.

36. 7346.9.

$\log 7346.9$ is between 3 and 4.

29. 75.05.

$\log 75.05$ is between 1 and 2.

37. 7346.09.

$\log 7346.09$ is between 3 and 4.

30. 82.95.

$\log 82.95$ is between 1 and 2.

38. 9182.735.

$\log 9182.735$ is between 3 and 4

31. 642.

$\log 642$ is between 2 and 3.

39. 243,481.

$\log 243,481$ is between 5 and 6.

32. 642.75.

$\log 642.75$ is between 2 and 3.

40. 5,276,192.

$\log 5,276,192$ is between 6 and 7

33. 642.005.

$\log 642.005$ is between 2 and 3.

41. 7,286,348.5.

$\log 7,286,348.5$ is between 6 and 7.

34. 793.175.

$\log 793.175$ is between 2 and 3.

42. 19,423,076.

$\log 19,423,076$ is between 7 and 8.

Show that the following statements are true :

$$43. \log_2 4 + \log_2 8 + \log_2 16 + \log_2 64 + \log_2 2 + \log_2 32 = 21.$$

$$\log_2 4 = 2; \log_2 8 = 3; \log_2 16 = 4; \log_2 64 = 6; \log_2 2 = 1; \log_2 32 = 5 \\ 2 + 3 + 4 + 6 + 1 + 5 = 21. \\ 21 = 21.$$

$$44. \log_3 3 + \log_3 9 + \log_3 81 + \log_3 729 + \log_3 27 + \log_3 243 = 21.$$

$$\log_3 3 = 1; \log_3 9 = 2; \log_3 81 = 4; \log_3 729 = 6; \log_3 27 = 3; \log_3 243 = 5. \\ 1 + 2 + 4 + 6 + 3 + 5 = 21. \\ 21 = 21.$$

$$45. \log_{11} 11 + \log_{11} 121 + \log_{11} 1331 + \log_{11} 14,641 = 10.$$

$$\log_{11} 11 = 1; \log_{11} 121 = 2; \log_{11} 1331 = 3; \log_{11} 14,641 = 4. \\ 1 + 2 + 3 + 4 = 10. \\ 10 = 10.$$

$$46. \log 1 + \log 10 + \log 1000 + \log 0.1 + \log 0.001 = 0.$$

$$\log 1 = 0; \log 10 = 1; \log 1000 = 3; \log 0.1 = -1; \log 0.001 = -3 \\ 0 + 1 + 3 - 1 - 3 = 0. \\ 0 = 0.$$

$$47. \log 1 + \log 100 + \log 10,000 + \log 0.01 + \log 0.0001 = 0.$$

$$\log 1 = 0; \log 100 = 2; \log 10,000 = 4; \log 0.01 = -2; \log 0.0001 = -4. \\ 0 + 2 + 4 - 2 - 4 = 0. \\ 0 = 0.$$

$$48. \log 10,000 - \log 1000 + \log 100,000 - \log 100 = 4.$$

$$\log 10,000 = 4; \log 1000 = 3; \log 100,000 = 5; \log 100 = 2. \\ 4 - 3 + 5 - 2 = 4. \\ 4 = 4.$$

Exercise 19. Logarithms (Page 45)

Write the characteristics of the logarithms of the following :

1. 75.	5. 7540.	9. 2.578.	13. 0.88.	17. 0.0077.
1.	3.	0.	-1.	-3.
2. 75.4.	6. 2578.	10. 25,780.	14. 0.885.	18. 0.00007
1.	3.	4.	-1.	-5.
3. 754.	7. 257.8.	11. 0.8.	15. 0.005.	19. 0.10097.
2.	2.	-1.	-3.	-1.
4. 7.54.	8. 25.78.	12. 0.08.	16. 0.0007.	20. 0.07007.
0.	1.	-2.	-4.	-2.

Given 3.58681 as the logarithm of 3862, find the following:

- | | |
|--------------------------------|--|
| 21. $\log 38.62 = 1.58681$. | 26. $\log 38,620,000 = 7.58681$. |
| 22. $\log 3.862 = 0.58681$. | 27. $\log 0.3862 = \bar{1}.58681$. |
| 23. $\log 386.2 = 2.58681$. | 28. $\log 0.03862 = \bar{2}.58681$. |
| 24. $\log 38,620 = 4.58681$. | 29. $\log 0.0003862 = \bar{4}.58681$. |
| 25. $\log 386,200 = 5.58681$. | |

Given $\bar{1}.67724$ as the logarithm of 0.4756, find the following:

- | | |
|-------------------------------|--|
| 30. $\log 4756 = 3.67724$. | 33. $\log 47,560,000 = 7.67724$. |
| 31. $\log 4.756 = 0.67724$. | 34. $\log 0.04756 = 2.67724$. |
| 32. $\log 47,560 = 4.67724$. | 35. $\log 0.0004756 = \bar{5}.67724$. |

Given 3.40603 as the logarithm of 2547, find the following:

- | | |
|-------------------------------------|---------------------------------------|
| 36. $\log 2.547 = 0.40603$. | 39. $\log 0.002547 = \bar{3}.40603$. |
| 37. $\log 25.47 = 1.40603$. | 40. $\log 25,470 = 4.40603$. |
| 38. $\log 0.2547 = \bar{1}.40603$. | 41. $\log 25,470,000 = 7.40603$. |

Given 1.39794 as the logarithm of 25, find the following:

- | | |
|--|------------------------------------|
| 42. $\log 2\frac{1}{2} = \log 2.5 = 0.39794$. | 45. $\log 0.025 = \bar{2}.39794$. |
| 43. $\log \frac{1}{4} = \log 0.25 = \bar{1}.39794$. | 46. $\log 25,000 = 4.39794$. |
| 44. $\log 0.25 = \bar{1}.39794$. | 47. $\log 25,000,000 = 7.39794$. |

Exercise 20. Using the Table (Page 47)

Using the table, find the logarithms of the following:

- | | | |
|-------------------------------------|-------------------------------------|-----------------------------|
| 1. $\log 2 = 0.30103$. | 4. $\log 0.002 = \bar{3}.30103$. | 7. $\log 2156 = 3.33365$. |
| 2. $\log 20 = 1.30103$. | 5. $\log 2100 = 3.32222$. | 8. $\log 2.156 = 0.33365$. |
| 3. $\log 200 = 2.30103$. | 6. $\log 2150 = 3.33244$. | 9. $\log 3485 = 3.54220$. |
| 10. $\log 4462 = 3.64953$. | 23. $\log 0.0009 = \bar{4}.95424$. | |
| 11. $\log 5581 = 3.74671$. | 24. $\log 0.0178 = \bar{2}.25042$. | |
| 12. $\log 7007 = 3.84553$. | 25. $\log 12,340 = 4.09132$. | |
| 13. $\log 5285 = 3.72304$. | 26. $12,345$. | |
| 14. $\log 68.48 = 1.83556$. | | |
| 15. $\log 7.926 = 0.89905$. | | |
| 16. $\log 834.8 = 2.92158$. | | |
| 17. $\log 0.7 = \bar{1}.84510$. | | |
| 18. $\log 0.75 = \bar{1}.87506$. | | |
| 19. $\log 0.756 = \bar{1}.87852$. | | |
| 20. $\log 0.7567 = \bar{1}.87892$. | | |
| 21. $\log 0.0255 = 2.40654$. | | |
| 22. $\log 0.0036 = \bar{3}.55630$. | | |
-
- | | |
|--|---|
| | $\log 12,350 = 4.09167$ |
| | $\log 12,340 = \frac{4.09132}{0.00035}$ |
| | $\frac{.5}{0.000175}$ |
| | $= 0.00018$. |
| | $\log 12,345 = 4.09150$ |

27. 12,347.

$$\log 12,350 = 4.09167$$

$$\log 12,340 = \underline{4.09132}$$

$$0.00035$$

$$\underline{.7}$$

$$0.000245$$

$$= 0.00025.$$

$$\log 12,347 = 4.09157.$$

28. 123.47.

$$\log 123.50 = 2.09167$$

$$\log 123.40 = \underline{2.09132}$$

$$0.00035$$

$$\underline{.7}$$

$$0.000245$$

$$= 0.00025.$$

$$\log 123.47 = 2.09157.$$

29. 234.62.

$$\log 234.70 = 2.37051$$

$$\log 234.60 = \underline{2.37033}$$

$$0.00018$$

$$\underline{.2}$$

$$0.000036$$

$$= 0.00004.$$

$$\log 234.62 = 2.37037.$$

30. 41.327.

$$\log 41.330 = 1.61627$$

$$\log 41.320 = \underline{1.61616}$$

$$0.00011$$

$$\underline{.7}$$

$$0.000077$$

$$= 0.00008.$$

$$\log 41.327 = 1.61624.$$

31. 56.283.

$$\log 56.290 = 1.75043$$

$$\log 56.280 = \underline{1.75035}$$

$$0.00008$$

$$\underline{.3}$$

$$0.000024$$

$$= 0.00002.$$

$$\log 56.283 = 1.75037.$$

32. 0.41282.

$$\log 0.41290 = 0.61584 - 1$$

$$\log 0.41280 = \underline{0.61574 - 1}$$

$$0.00010$$

$$\underline{.2}$$

$$0.000020$$

$$\log 0.41282 = \bar{1}.61576.$$

33. In a certain computation it is necessary to find the sum of the logarithms of 45.6, 72.8, and 98.4.

What is this sum?

$$\log 45.6 = 1.65896$$

$$\log 72.8 = 1.86213$$

$$\log 98.4 = \underline{1.99300}$$

$$5.51409$$

34. In a certain computation it is necessary to subtract the logarithm of 3.84 from the sum of the logarithms of 52.8 and 26.5. What is the resulting logarithm?

$$\log 52.8 = 1.72263$$

$$\log 26.5 = \underline{1.42325}$$

$$3.14588$$

$$\log 3.84 = \underline{0.58433}$$

$$2.56155$$

Perform the following operations:

85. $\log 275 + \log 321 + \log 4.26 + \log 3.87 + \log 46.4.$

$$\log 275 = 2.43933$$

$$\log 321 = 2.50651$$

$$\log 4.26 = 0.62941$$

$$\log 3.87 = 0.58771$$

$$\log 46.4 = \underline{1.66652}$$

$$7.82948$$

36. $\log 2643 + \log 3462 + \log 4926 + \log 5376 + \log 2194.$

$$\log 2643 = 3.42210$$

$$\log 3462 = 3.53933$$

$$\log 4926 = 3.69249$$

$$\log 5376 = 3.73046$$

$$\log 2194 = \underline{3.34124}$$

$$17.72562$$

37. $\log 51.82 + \log 7.263 + \log 5.826 + \log 218.7 + \log 3275.$

$$\log 51.82 = 1.71450$$

$$\log 7.263 = 0.86112$$

$$\log 5.826 = 0.76537$$

$$\log 218.7 = 2.33985$$

$$\log 3275 = \underline{3.51521}$$

$$9.19605$$

38. $\log 8263 + \log 2179 + \log 3972 - \log 2163 - \log 178.$

$$\log 8263 = 3.91714$$

$$\log 2163 = 3.33506$$

$$\log 2179 = 3.33826$$

$$\log 178 = \underline{2.25042}$$

$$\log 3972 = \underline{3.59901}$$

$$5.58548$$

$$10.85441$$

$$\underline{5.58548}$$

$$5.26893 \text{ Ans.}$$

39. $\log 37.42 + \log 61.73 + \log 5.823 - \log 1.46 - \log 27.83.$

$$\log 37.42 = 1.57310$$

$$\log 1.46 = 0.16435$$

$$\log 61.73 = 1.79050$$

$$\log 27.83 = \underline{1.44451}$$

$$\log 5.823 = \underline{0.76515}$$

$$1.60886$$

$$4.12875$$

$$\underline{1.60886}$$

$$2.51989 \text{ Ans.}$$

40. $\log 3.427 + \log 38.46 + \log 723.8 - \log 2.73 - \log 21.68.$

$$\log 3.427 = 0.53491$$

$$\log 2.73 = 0.43616$$

$$\log 38.46 = 1.58501$$

$$\log 21.68 = \underline{1.33606}$$

$$\log 723.8 = \underline{2.85962}$$

$$1.77222$$

$$4.97954$$

$$\underline{1.77222}$$

$$3.20732 \text{ Ans.}$$

41. In a certain operation it is necessary to find three times $\log 41.75$. What is the resulting logarithm?

$$\log 41.75 = 1.62066$$

$$\underline{3}$$

$$4.86198$$

42. In a certain operation it is necessary to find one fifth of $\log 254.8$. What is the resulting logarithm?

$$\log 254.8 = 2.40620.$$

$$\frac{1}{5} \text{ of } 2.40620 = 0.48124.$$

Perform the following operations:

43. $2 \times \log 3.$

$$\begin{array}{r} \log 3 = 0.47712 \\ 2 \\ \hline 0.95424 \end{array}$$

45. $3 \times \log 25.6.$

$$\begin{array}{r} \log 25.6 = 1.40824 \\ 3 \\ \hline 4.22472 \end{array}$$

47. $4 \times \log 21.42.$

$$\begin{array}{r} \log 21.42 = 1.33082 \\ 4 \\ \hline 5.32328 \end{array}$$

44. $3 \times \log 2.$

$$\begin{array}{r} \log 2 = 0.30103 \\ 3 \\ \hline 0.90309 \end{array}$$

46. $5 \times \log 3.76.$

$$\begin{array}{r} \log 3.76 = 0.57519 \\ 5 \\ \hline 2.87595 \end{array}$$

48. $5 \times \log 346.8.$

$$\begin{array}{r} \log 346.8 = 2.54008 \\ 5 \\ \hline 12.70040 \end{array}$$

49. $12 \times \log 42.86.$

$$\begin{array}{r} \log 42.86 = 1.63205 \\ 12 \\ \hline 19.58460 \end{array}$$

56. $\frac{3}{4} \log 763.8.$

$$\begin{array}{r} \log 763.8 = 2.88298 \\ 3 \\ 4 \overline{) 8.64894} \\ 2.16224 \end{array}$$

50. $\frac{1}{2} \log 2.$

$$\begin{array}{l} \log 2 = 0.30103. \\ \frac{1}{2} \text{ of } 0.30103 = 0.15052. \end{array}$$

57. $0.3 \log 431.$

$$\begin{array}{r} \log 431 = 2.63448 \\ 0.3 \\ \hline 0.790344 \end{array}$$

51. $\frac{1}{2} \log 2000.$

$$\begin{array}{l} \log 2000 = 3.30103. \\ \frac{1}{2} \text{ of } 3.30103 = 1.65052. \end{array}$$

58. $0.7 \log 43.19.$

$$\begin{array}{r} \log 43.19 = 1.63538 \\ 0.7 \\ \hline 1.144766 \\ 1.14477. \text{ Ans.} \end{array}$$

52. $\frac{1}{3} \log 3460.$

$$\begin{array}{l} \log 3460 = 3.53908. \\ \frac{1}{3} \text{ of } 3.53908 = 1.17969. \end{array}$$

59. $0.9 \log 4.007.$

$$\begin{array}{r} \log 4.007 = 0.60282 \\ 0.9 \\ \hline 0.542538 \\ 0.54254. \text{ Ans.} \end{array}$$

53. $\frac{1}{3} \log 24.76.$

$$\begin{array}{l} \log 24.76 = 1.39375. \\ \frac{1}{3} \text{ of } 1.39375 = 0.46458. \end{array}$$

54. $\frac{1}{4} \log 368.7.$

$$\begin{array}{l} \log 368.7 = 2.56667. \\ \frac{1}{4} \text{ of } 2.56667 = 0.64167. \end{array}$$

60. $1.4 \log 5.108.$

$$\begin{array}{r} \log 5.108 = 0.70825 \\ 1.4 \\ \hline 283300 \\ 70825 \\ \hline 0.991550 \\ 0.99155. \text{ Ans.} \end{array}$$

55. $\frac{2}{3} \log 41.73.$

$$\begin{array}{r} \log 41.73 = 1.62045 \\ 2 \\ 3 \overline{) 3.24090} \\ 1.08030 \end{array}$$

61. $2.3 \log 7.411$.

$\log 7.411 = 0.86988$

$$\begin{array}{r} 2.3 \\ 260964 \\ 173976 \\ \hline 2.000724 \end{array}$$

2.00072. *Ans.*

62. $\frac{5}{8} \log 16.05$.

$\log 16.05 = 1.20548$

$$\begin{array}{r} 5 \\ 8 \overline{) 6.02740} \\ 0.75343 \end{array}$$

63. $\frac{7}{8} \log 23.43$.

$\log 23.43 = 1.36977$

$$\begin{array}{r} 7 \\ 8 \overline{) 9.58839} \\ 1.19855 \end{array}$$

Exercise 21. Antilogarithms (Page 49)*Find the antilogarithms of the following:*

1. 0.47712.

3.

5. 4.56844.

37,020.

9. 3 74076.

5505.

13. 0.82575.

6.695.

2. 3.47712.

3000.

6. 1.66276.

46.

10. $\bar{2}.76305$.

0.05795.

14. 0.88081.

7.6.

3. $\bar{3}.47712$.

0.003.

7. 2.66978.

467.5.

11. $\bar{4}.78497$.

0.0006095.

15. 9.89237.

7,805,000,000

4. 2.48359

304.5.

8. $\bar{5}.74819$.

0.000056.

12. $\bar{1}.81954$.

0.66.

16. 7.90282.

79,950,000

17. 0.23305.

$\log 1.711 = 0.23325$

$\log 1.710 = 0.23300$

$\frac{0.00025}{0.00025}$

antilog 0.23305 = 1.7102.

20. $\bar{1}.58041$.

$\log 0.3806 = \bar{1}.58047$

$\log 0.3805 = \bar{1}.58035$

$\frac{0.00012}{0.00012}$

antilog $\bar{1}.58041 = 0.38055$.

18. 1.43144.

$\log 27.01 = 1.43152$

$\log 27.00 = 1.43136$

$\frac{0.00016}{0.00016}$

antilog 1.43144 = 27.005.

21. $\bar{3}.63490$.

$\log 0.004315 = \bar{3}.63498$

$\log 0.004314 = \bar{3}.63488$

$\frac{0.00010}{0.00010}$

antilog $\bar{3}.63490 = 0.0043142$

19. 2.56838.

$\log 370.2 = 2.56844$

$\log 370.1 = 2.56832$

$\frac{0.00012}{0.00012}$

antilog 2.56838 = 370.15.

22. 4.63492.

$\log 43,150 = 4.63498$

$\log 43,140 = 4.63488$

$\frac{0.00010}{0.00010}$

antilog 4.63492 = 43,144.

42. In a certain operation it is necessary to find one fifth of $\log 254.8$. What is the resulting logarithm?

$$\begin{aligned}\log 254.8 &= 2.40620. \\ \frac{1}{5} \text{ of } 2.40620 &= 0.48124.\end{aligned}$$

Perform the following operations:

43. $2 \times \log 3.$	45. $3 \times \log 25.6.$	47. $4 \times \log 21.42.$
$\log 3 = 0.47712$	$\log 25.6 = 1.40824$	$\log 21.42 = 1.33082$
$\begin{array}{r} 2 \\ 0.95424 \end{array}$	$\begin{array}{r} 3 \\ 4.22472 \end{array}$	$\begin{array}{r} 4 \\ 5.32328 \end{array}$

44. $3 \times \log 2.$	46. $5 \times \log 3.76.$	48. $5 \times \log 346.8.$
$\log 2 = 0.30103$	$\log 3.76 = 0.57519$	$\log 346.8 = 2.54008$
$\begin{array}{r} 3 \\ 0.90309 \end{array}$	$\begin{array}{r} 5 \\ 2.87595 \end{array}$	$\begin{array}{r} 5 \\ 12.70040 \end{array}$

49. $12 \times \log 42.86.$	56. $\frac{3}{4} \log 763.8.$
$\log 42.86 = 1.63205$	$\log 763.8 = 2.88298$
$\begin{array}{r} 12 \\ 19.58460 \end{array}$	$\begin{array}{r} 3 \\ 4 \overline{) 8.64894} \\ 2.16224 \end{array}$

50. $\frac{1}{2} \log 2.$	57. $0.3 \log 431.$
$\log 2 = 0.30103.$	$\log 431 = 2.63448$
$\frac{1}{2} \text{ of } 0.30103 = 0.15052.$	$\begin{array}{r} 0.3 \\ 0.790344 \end{array}$
51. $\frac{1}{2} \log 2000.$	$0.79034. \text{ Ans.}$
$\log 2000 = 3.30103.$	
$\frac{1}{2} \text{ of } 3.30103 = 1.65052.$	

52. $\frac{1}{3} \log 3460.$	58. $0.7 \log 43.19.$
$\log 3460 = 3.53908.$	$\log 43.19 = 1.63538$
$\frac{1}{3} \text{ of } 3.53908 = 1.17969.$	$\begin{array}{r} 0.7 \\ 1.144766 \end{array}$
	$1.14477. \text{ Ans.}$

53. $\frac{1}{3} \log 24.76.$	59. $0.9 \log 4.007.$
$\log 24.76 = 1.39375.$	$\log 4.007 = 0.60282$
$\frac{1}{3} \text{ of } 1.39375 = 0.46458.$	$\begin{array}{r} 0.9 \\ 0.542538 \end{array}$
	$0.54254. \text{ Ans}$

54. $\frac{1}{4} \log 368.7.$	60. $1.4 \log 5.108.$
$\log 368.7 = 2.56667.$	$\log 5.108 = 0.70825$
$\frac{1}{4} \text{ of } 2.56667 = 0.64167.$	$\begin{array}{r} 1.4 \\ 283300 \\ 70825 \\ 0.991550 \end{array}$
55. $\frac{2}{3} \log 41.73.$	$0.99155. \text{ Ans.}$
$\log 41.73 = 1.62045$	
$\begin{array}{r} 2 \\ 3 \overline{) 3.24090} \\ 1.08030 \end{array}$	

61. $2.3 \log 7.411.$

$\log 7.411 = 0.86988$

$$\begin{array}{r} 2.3 \\ 26.9964 \\ \hline 173976 \\ 2.000724 \end{array}$$

$2.00072. \text{ Ans.}$

62. $\frac{5}{8} \log 16.05.$

$\log 16.05 = 1.20548$

$$\begin{array}{r} 5 \\ 8 \overline{) 6.02740} \\ 0.75343 \end{array}$$

63. $\frac{7}{8} \log 23.43.$

$\log 23.43 = 1.36977$

$$\begin{array}{r} 7 \\ 8 \overline{) 9.58839} \\ 1.19855 \end{array}$$

Exercise 21. Antilogarithms (Page 49)*Find the antilogarithms of the following:*

1. 0.47712.

3.

5. 4.56844.

37,020.

9. 3 74076.

5505.

13. 0.82575.

6.695.

2. 3.47712.

3000.

6. 1.66276.

46.

10. $\bar{2}.76305.$

0.05795.

14. 0.88081.

7.6.

3. $\bar{3}.47712.$

0.003.

7. 2.66978.

467.5.

11. $\bar{4}.78497.$

0.0006095.

15. 9.89237.

7,805,000,000

4. 2.48359

304.5.

8. $\bar{5}.74819.$

0.000056.

12. $\bar{1}.81954.$

0.66.

16. 7.90282.

79,950,000

17. 0.23305.

$\log 1.711 = 0.23325$

$\log 1.710 = 0.23300$

0.00025

$\text{antilog } 0.23305 = 1.7102.$

20. $\bar{1}.58041.$

$\log 0.3806 = \bar{1}.58047$

$\log 0.3805 = \bar{1}.58035$

0.00012

$\text{antilog } \bar{1}.58041 = 0.38055.$

18. 1.43144.

$\log 27.01 = 1.43152$

$\log 27.00 = 1.43136$

0.00016

$\text{antilog } 1.43144 = 27.005.$

21. $\bar{3}.63490.$

$\log 0.004315 = \bar{3}.63498$

$\log 0.004314 = \bar{3}.63488$

0.00010

$\text{antilog } \bar{3}.63490 = 0.0043142$

19. 2.56838.

$\log 370.2 = 2.56844$

$\log 370.1 = 2.56832$

0.00012

$\text{antilog } 2.56838 = 370.15.$

22. 4.63492.

$\log 43,150 = 4.63498$

$\log 43,140 = 4.63488$

0.00010

$\text{antilog } 4.63492 = 43,144.$

23. 0.63994.

$$\log 4.365 = 0.63998$$

$$\log 4.364 = \underline{0.63988}$$

$$0.00010$$

$$\text{antilog } 0.63994 = 4.3646.$$

24. $\bar{2}.69085$.

$$\log 0.04908 = \bar{2}.69090$$

$$\log 0.04907 = \underline{\bar{2}.69082}$$

$$0.00008$$

$$\text{antilog } \bar{2}.69085 = 0.049074.$$

25. 8.77425.

$$\log 594,700,000 = 8.77430$$

$$\log 594,600,000 = \underline{8.74422}$$

$$0.00008$$

$$\text{antilog } 8.77425 = 594,640,000.$$

26. $\bar{4}.82966$.

$$\log 0.0006756 = \bar{4}.82969$$

$$\log 0.0006755 = \underline{\bar{4}.82963}$$

$$0.00006$$

$$\text{antilog } \bar{4}.82966 = 0.00067555.$$

27. 3.83547.

$$\log 6847 = 3.83550$$

$$\log 6846 = \underline{3.83544}$$

$$0.00006$$

$$\text{antilog } 3.83547 = 6846.5.$$

28. 2.83604.

$$\log 685.6 = 2.83607$$

$$\log 685.5 = \underline{2.83601}$$

$$0.00006$$

$$\text{antilog } 2.83604 = 685.55.$$

29. 4.88960.

$$\log 77,560 = 4.88964$$

$$\log 77,550 = \underline{4.88958}$$

$$0.00006$$

$$\text{antilog } 4.88960 = 77,553.$$

30. 2.89523.

$$\log 785.7 = 2.89526$$

$$\log 785.6 = \underline{2.89520}$$

$$0.00006$$

$$\text{antilog } 2.89523 = 785.65.$$

31. 3.89858.

$$\log 7918 = 3.89862$$

$$\log 7917 = \underline{3.89856}$$

$$0.00006$$

$$\text{antilog } 3.89858 = 7917.3.$$

32. 0.93223.

$$\log 8.556 = 0.93227$$

$$\log 8.555 = \underline{0.93222}$$

$$0.00005$$

$$\text{antilog } 0.93223 = 8.5552.$$

33. If the logarithm of the product of two numbers is 2.94210, what is the product of the numbers?

$$\log 875.2 = 2.94211$$

$$\log 875.1 = \underline{2.94206}$$

$$0.00005$$

$$\text{antilog } 2.94210 = 875.18.$$

34. If the logarithm of the quotient of two numbers is 0.30103, what is the quotient of the numbers?

$$\text{antilog } 0.30103 = 2.$$

35. If we wish to multiply 2857 by 2875, what logarithms do we need? What are these logarithms?

$$\log 2857 = 3.45591.$$

$$\log 2875 = 3.45864.$$

36. If we know that the logarithm of a result which we are seeking is 3.47056, what is that result?

$$\text{antilog } 3.47056 = 2955.$$

37. If we know that $\log \sqrt{0.000043641}$ is $\bar{3}.81995$, what is the value of $\sqrt{0.000043641}$?

$$\log 0.006607 = \bar{3}.82000$$

$$\log 0.006606 = \bar{3}.81994$$

$$0.00006$$

$$\text{antilog } \bar{3}.81995 = 0.0066062.$$

38. If we know that $\log \sqrt[6]{0.076553}$ is $\bar{1}.81400$, what is the value of $\sqrt[6]{0.076553}$?

$$\log 0.6517 = \bar{1}.81405$$

$$\log 0.6516 = \bar{1}.81398$$

$$0.00007$$

$$\text{antilog } \bar{1}.81400 = 0.65163.$$

39. The logarithm of $\sqrt{8322}$ is 1.96012. Find $\sqrt{8322}$ to three decimal places.

$$\log 91.23 = 1.96014$$

$$\log 91.22 = 1.96009$$

$$0.00005$$

$$\text{antilog } 1.96012 = 91.226.$$

40. The logarithm of the cube of 376 is 7.72557. Find the cube of 376 to five significant figures.

$$\log 53,160,000 = 7.72558$$

$$\log 53,150,000 = 7.72550$$

$$0.00008$$

$$\text{antilog } 7.72557 = 53,159,000.$$

41. If we know that $\log 0.003278^2$ is $\bar{5}.03122$, what is the value of 0.003278^2 ?

$$\log 0.00001075 = \bar{5}.03141$$

$$\log 0.00001074 = \bar{5}.03100$$

$$0.00041$$

$$\text{antilog } \bar{5}.03122 = 0.000010745.$$

42. Find twice $\log 731$, and find the antilogarithm of the result.

$$\log 731 = 2.86392$$

$$\log 534,400 = 5.72787$$

$$\begin{array}{r} 2 \\ \hline 5.72784 \end{array}$$

$$\log 534,300 = 5.72779$$

$$0.00008$$

$$\text{antilog } 5.72784 = 534,360.$$

43. Find the antilogarithm of the sum of $\log 27.8 + \log 34.6 + \log 367.8$.

$$\log 27.8 = 1.44404$$

$$\log 353,800 = 5.54876$$

$$\log 34.6 = 1.53908$$

$$\log 353,700 = 5.54864$$

$$\log 367.8 = 2.56561$$

$$0.00012$$

$$\begin{array}{r} 5.54873 \end{array}$$

$$\text{antilog } 5.54873 = 353,780.$$

Find the antilogarithms of the following :

44. $\log 7 + \log 2 - \log 1.934$.

$$\log 7 = 0.84510$$

$$\log 2 = 0.30103$$

$$\underline{1.14613}$$

$$\log 1.934 = 0.28646$$

$$\underline{0.85967}$$

$$\log 7.239 = 0.85968$$

$$\log 7.238 = 0.85962$$

$$\underline{0.00006}$$

$$\text{antilog } 0.85967 = 7.2388.$$

45. $\log 63 + \log 5.8 - \log 3.415$.

$$\log 63 = 1.79934$$

$$\log 5.8 = 0.76343$$

$$\underline{2.56277}$$

$$\log 3.415 = 0.53339$$

$$\underline{2.02938}$$

$$\text{antilog } 2.02938 = 107.$$

46. $\log 728 + \log 96.8 - \log 2.768$.

$$\log 728 = 2.86213$$

$$\log 96.8 = 1.98588$$

$$\underline{4.84801}$$

$$\log 2.768 = 0.44217$$

$$\underline{4.40584}$$

$$\log 25,460 = 4.40586$$

$$\log 25,450 = 4.40569$$

$$\underline{0.00017}$$

$$\text{antilog } 4.40584 = 25,459.$$

47. $5 \log 27.83$.

$$\log 27.83 = 1.44451$$

$$\underline{5}$$

$$\underline{7.22255}$$

$$\log 16,700,000 = 7.22272$$

$$\log 16,690,000 = 7.22246$$

$$\underline{0.00026}$$

$$\text{antilog } 7.22255 = 16,693,000.$$

48. $2.8 \log 5.683$.

$$\log 5.683 = 0.75458$$

$$\underline{2.8}$$

$$\underline{603664}$$

$$\underline{150916}$$

$$\underline{2.112824}$$

$$= 2.11282.$$

$$\log 129.7 = 2.11294$$

$$\log 129.6 = 2.11261$$

$$\underline{0.00033}$$

$$\text{antilog } 2.11282 = 129.66.$$

49. $\frac{3}{4}(\log 2 + \log 4.2)$.

$$\log 2 = 0.30103$$

$$\log 4.2 = 0.62325$$

$$\underline{0.92428}$$

$$\underline{3}$$

$$\underline{4)2.77284}$$

$$\underline{0.69321}$$

$$\log 4.935 = 0.69329$$

$$\log 4.934 = 0.69320$$

$$\underline{0.00009}$$

$$\text{antilog } 0.69321 = 4.9341.$$

Exercise 22. Multiplication by Logarithms (Page 50)

Using logarithms, find the following products :

1. 2×5 .

$$\log 2 = 0.30103$$

$$\log 5 = 0.69897$$

$$\log x = 1.00000$$

$$x = 10.$$

2. 4×6 .

$$\log 4 = 0.60206$$

$$\log 6 = 0.77815$$

$$\log x = 1.38021$$

$$x = 24.$$

3. 3×5 .

$$\log 3 = 0.47712$$

$$\log 5 = 0.69897$$

$$\log x = 1.17609$$

$$x = 15.$$

4. 5×7 .

$$\log 5 = 0.69897$$

$$\log 7 = 0.84510$$

$$\log x = 1.54407$$

$$x = 35.$$

11. 2×50 .

$$\log 2 = 0.30103$$

$$\log 50 = 1.69897$$

$$\log x = 2.00000$$

$$x = 100.$$

18. 30×600 .

$$\log 30 = 1.47712$$

$$\log 600 = 2.77815$$

$$\log x = 4.25527$$

$$x = 18,000.$$

5. 2×4 .

$$\log 2 = 0.30103$$

$$\log 4 = 0.60206$$

$$\log x = 0.90309$$

$$x = 8.$$

12. 40×60 .

$$\log 40 = 1.60206$$

$$\log 60 = 1.77815$$

$$\log x = 3.38021$$

$$x = 2400.$$

19. $7 \times 80,000$.

$$\log 7 = 0.84510$$

$$\log 80,000 = 4.90309$$

$$\log x = 5.74819$$

$$x = 560,000.$$

6. 3×7 .

$$\log 3 = 0.47712$$

$$\log 7 = 0.84510$$

$$\log x = 1.32222$$

$$x = 21.$$

13. 3×500 .

$$\log 3 = 0.47712$$

$$\log 500 = 2.69897$$

$$\log x = 3.17609$$

$$x = 1500.$$

20. 200×900 .

$$\log 200 = 2.30103$$

$$\log 900 = 2.95424$$

$$\log x = 5.25527$$

$$x = 180,000.$$

7. 2×6 .

$$\log 2 = 0.30103$$

$$\log 6 = 0.77815$$

$$\log x = 1.07918$$

$$x = 12.$$

14. 50×70 .

$$\log 50 = 1.69897$$

$$\log 70 = 1.84510$$

$$\log x = 3.54407$$

$$x = 3500.$$

21. 35.8×28.9 .

$$\log 35.8 = 1.55388$$

$$\log 28.9 = 1.46090$$

$$\log x = 3.01478$$

$$x = 1034.6.$$

8. 3×6 .

$$\log 3 = 0.47712$$

$$\log 6 = 0.77815$$

$$\log x = 1.25527$$

$$x = 18.$$

15. 2×4000 .

$$\log 2 = 0.30103$$

$$\log 4000 = 3.60206$$

$$\log x = 3.90309$$

$$x = 8000.$$

22. 52.7×41.6 .

$$\log 52.7 = 1.72181$$

$$\log 41.6 = 1.61909$$

$$\log x = 3.34090$$

$$x = 2192.3.$$

9. 7×8 .

$$\log 7 = 0.84510$$

$$\log 8 = 0.90309$$

$$\log x = 1.74819$$

$$x = 56.$$

16. 30×700 .

$$\log 30 = 1.47712$$

$$\log 700 = 2.84510$$

$$\log x = 4.32222$$

$$x = 21,000.$$

23. 2.75×4.84 .

$$\log 2.75 = 0.43933$$

$$\log 4.84 = 0.68485$$

$$\log x = 1.12418$$

$$x = 13.31.$$

10. 2×9 .

$$\log 2 = 0.30103$$

$$\log 9 = 0.95424$$

$$\log x = 1.25527$$

$$x = 18.$$

17. 200×60 .

$$\log 200 = 2.30103$$

$$\log 60 = 1.77815$$

$$\log x = 4.07918$$

$$x = 12,000.$$

24. 5.25×3.86 .

$$\log 5.25 = 0.72016$$

$$\log 3.86 = 0.58659$$

$$\log x = 1.30675$$

$$x = 20.265.$$

25. $14.26 \times 42.35.$

$\log 14.26 = 1.15412$

$\log 42.35 = \underline{1.62685}$

$\log x = \underline{2.78097}$

$x = 603.9.$

27. $529.6 \times 348.7.$

$\log 529.6 = 2.72395$

$\log 348.7 = \underline{2.54245}$

$\log x = \underline{5.26640}$

$x = 184,670.$

29. $34.81 \times 46.25.$

$\log 34.81 = 1.54170$

$\log 46.25 = \underline{1.66511}$

$\log x = \underline{3.20681}$

$x = 1609.9.$

26. $43.28 \times 29.64.$

$\log 43.28 = 1.63629$

$\log 29.64 = \underline{1.47188}$

$\log x = \underline{3.10817}$

$x = 1282.8.$

28. $240.8 \times 46.09.$

$\log 240.8 = 2.38166$

$\log 46.09 = \underline{1.66361}$

$\log x = \underline{4.04527}$

$x = 11,099.$

30. $5028 \times 3.472.$

$\log 5028 = 3.70140$

$\log 3.472 = \underline{0.54058}$

$\log x = \underline{4.24198}$

$x = 17,458.$

31. Taking the circumference of a circle to be 3.14 times the diameter, find the circumference of a steel shaft of diameter 5.8 in.

$\log 3.14 = 0.49693$

$\log 5.8 = \underline{0.76343}$

$\log x = \underline{1.26036}$

$x = 18.212.$

32. Taking the ratio of the circumference to the diameter as given in Ex. 31, find the circumference of a water tank of diameter 36 ft.

$\log 3.14 = 0.49693$

$\log 36 = \underline{1.55630}$

$\log x = \underline{2.05323}$

$x = 113.04.$

The circumference is 18.212 in.

The circumference is 113.04 ft.

Using logarithms, find the following products :

33. $2 \times 3 \times 5 \times 7.$

$\log 2 = 0.30103$

$\log 3 = 0.47712$

$\log 5 = 0.69897$

$\log 7 = \underline{0.84510}$

$\log x = \underline{2.32222}$

$x = 210.$

35. $5 \times 7 \times 11 \times 13.$

$\log 5 = 0.69897$

$\log 7 = 0.84510$

$\log 11 = 1.04139$

$\log 13 = \underline{1.11394}$

$\log x = \underline{3.69940}$

$x = 5005.$

37. $527.6 \times 283.4 \times 4.196.$

$\log 527.6 = 2.72230$

$\log 283.4 = 2.45240$

$\log 4.196 = \underline{0.62284}$

$\log x = \underline{5.79754}$

$x = 627,400.$

34. $3 \times 5 \times 7 \times 9.$

$\log 3 = 0.47712$

$\log 5 = 0.69897$

$\log 7 = 0.84510$

$\log 9 = \underline{0.95424}$

$\log x = \underline{2.97543}$

$x = 945.$

36. $43.8 \times 26.9 \times 32.8.$

$\log 43.8 = 1.64147$

$\log 26.9 = 1.42975$

$\log 32.8 = \underline{1.51587}$

$\log x = \underline{4.58709}$

$x = 38,645.$

38. $7.283 \times 6.987 \times 5.437.$

$\log 7.283 = 0.86231$

$\log 6.987 = 0.84429$

$\log 5.437 = \underline{0.73536}$

$\log x = \underline{2.44196}$

$x = 276.67.$

Exercise 23. Negative Characteristics (Page 51)*Add the following logarithms:*

1. $2.41283 + 5.27681.$

$$2.41283$$

$$\underline{5.27681}$$

$$7.68964$$

2. $\bar{2}.41283 + 5.27681.$

$$\bar{2}.41283 = 0.41283 - 2$$

$$5.27681 = 5.27681$$

$$\underline{5.68964 - 2}$$

$$= 3.68964.$$

3. $\bar{2}.41283 + \bar{5}.27681.$

$$\bar{2}.41283 = 0.41283 - 2$$

$$\bar{5}.27681 = 0.27681 - 5$$

$$\underline{0.68964 - 7}$$

$$= \bar{7}.68964.$$

4. $0.38264 + \bar{4}.71233.$

$$0.38264 = 0.38264$$

$$\bar{4}.71233 = 0.71233 - 4$$

$$\underline{1.09497 - 4}$$

$$= \bar{3}.09497.$$

5. $0.57121 + \bar{1}.42879.$

$$0.57121 = 0.57121$$

$$\bar{1}.42879 = 0.42879 - 1$$

$$\underline{1.00000 - 1}$$

$$= 0.00000.$$

6. $\bar{2}.63841 + 1.36158.$

$$\bar{2}.63841 = 0.63841 - 2$$

$$1.36158 = 1.36158$$

$$\underline{1.99999 - 2}$$

$$= \bar{1}.99999.$$

7. $\bar{2}.41238 + \bar{3}.62701.$

$$\bar{2}.41238 = 0.41238 - 2$$

$$\bar{3}.62701 = 0.62701 - 3$$

$$\underline{1.03939 - 5}$$

$$= \bar{4}.03939.$$

8. $\bar{5}.58623 + 6.41387.$

$$\bar{5}.58623 = 0.58623 - 5$$

$$6.41387 = 6.41387$$

$$\underline{7.00010 - 5}$$

$$= 2.00010.$$

9. $\bar{6}.41382 + 7.58617.$

$$\bar{6}.41382 = 0.41382 - 6$$

$$7.58617 = 7.58617$$

$$\underline{7.99999 - 6}$$

$$= 1.99999.$$

10. $\bar{4}.22334 + 3.77666.$

$$\bar{4}.22334 = 0.22334 - 4$$

$$3.77666 = 3.77666$$

$$\underline{4.00000 - 4}$$

$$= 0.00000.$$

Using logarithms, find the following products:

11. $256 \times 4875.$

$$\log 256 = 2.40824$$

$$\log 4875 = \underline{3.68797}$$

$$\log x = 6.09621$$

$$x = 1,248,000.$$

12. $2.56 \times 48.75.$

$$\log 2.56 = 0.40824$$

$$\log 48.75 = \underline{1.68797}$$

$$\log x = 2.09621$$

$$x = 124.8.$$

13. 0.256×0.4875 .

$$\log 0.256 = 0.40824 - 1$$

$$\log 0.4875 = \underline{0.68797 - 1}$$

$$\log x = \underline{1.09621 - 2}$$

$$= \bar{1}.09621.$$

$$x = 0.1248.$$

14. 0.0256×0.004875 .

$$\log 0.0256 = 0.40824 - 2$$

$$\log 0.004875 = \underline{0.68797 - 3}$$

$$\log x = \underline{1.09621 - 5}$$

$$= \bar{4}.09621.$$

$$x = 0.0001248.$$

15. 0.1275×0.03428 .

$$\log 0.1275 = 0.10551 - 1$$

$$\log 0.03428 = \underline{0.53504 - 2}$$

$$\log x = \underline{0.64055 - 3}$$

$$= \bar{3}.64055.$$

$$x = 0.0043707.$$

16. 0.2763×0.4134 .

$$\log 0.2763 = 0.44138 - 1$$

$$\log 0.4134 = \underline{0.61637 - 1}$$

$$\log x = \underline{1.05775 - 2}$$

$$= \bar{1}.05775.$$

$$x = 0.11422.$$

17. 0.00025×0.00125 .

$$\log 0.00025 = 0.39794 - 4$$

$$\log 0.00125 = \underline{0.09691 - 3}$$

$$\underline{0.49485 - 7}$$

$$= \bar{7}.49485.$$

$$x = 0.0000003125.$$

18. 0.725×0.3465 .

$$\log 0.725 = 0.86034 - 1$$

$$\log 0.3465 = \underline{0.53970 - 1}$$

$$\log x = \underline{1.40004 - 2}$$

$$= \bar{1}.40004.$$

$$x = 0.25121.$$

19. 0.256×0.0875 .

$$\log 0.256 = 0.40824 - 1$$

$$\log 0.0875 = \underline{0.94201 - 2}$$

$$\log x = \underline{1.35025 - 3}$$

$$= \bar{2}.35025.$$

$$x = 0.02240.$$

20. 0.037×0.00425 .

$$\log 0.037 = 0.56820 - 2$$

$$\log 0.00425 = \underline{0.62839 - 3}$$

$$\log x = \underline{1.19659 - 5}$$

$$= \bar{4}.19659.$$

$$x = 0.00015725.$$

21. 47.26×0.02755 .

$$\log 47.26 = 1.67449$$

$$\log 0.02755 = \underline{0.44012 - 2}$$

$$\log x = \underline{2.11461 - 2}$$

$$= 0.11461.$$

$$x = 1.3020.$$

22. 296.8×0.1283 .

$$\log 296.8 = 2.47246$$

$$\log 0.1283 = \underline{0.10823 - 1}$$

$$\log x = \underline{2.58069 - 1}$$

$$= 1.58069.$$

$$x = 38.079.$$

23. $45,650 \times 0.0725$.

$$\log 45,650 = 4.65944$$

$$\log 0.0725 = \underline{0.86034 - 2}$$

$$\log x = \underline{5.51978 - 2}$$

$$= 3.51978.$$

$$x = 3309.6.$$

24. $127,400 \times 0.00355$.

$$\log 127,400 = 5.10517$$

$$\log 0.00355 = \underline{0.55023 - 3}$$

$$\log x = \underline{5.65540 - 3}$$

$$= 2.65540.$$

$$x = 452.27.$$

25. Given $\sin 25.75^\circ = 0.4344$, find $52.8 \sin 25.75^\circ$.

$$\begin{aligned}\log 52.8 &= 1.72263 \\ \log 0.4344 &= \underline{0.63789 - 1} \\ \log x &= \underline{2.36052 - 1} \\ &= 1.36052. \\ x &= 22.936.\end{aligned}$$

26. Given $\cos 37.25^\circ = 0.7960$, find $42.85 \cos 37.25^\circ$.

$$\begin{aligned}\log 42.85 &= 1.63195 \\ \log 0.7960 &= \underline{0.90091 - 1} \\ \log x &= \underline{2.53286 - 1} \\ &= 1.53286. \\ x &= 34.108.\end{aligned}$$

27. Given $\tan 30^\circ 50' 30'' = 0.5971$, find $27.65 \tan 30^\circ 50' 30''$.

$$\begin{aligned}\log 27.65 &= 1.44170 \\ \log 0.5971 &= \underline{0.77605 - 1} \\ \log x &= \underline{2.21775 - 1} \\ &= 1.21775. \\ x &= 16.51.\end{aligned}$$

Exercise 24. Division by Logarithms (Page 53)

Add the following logarithms:

1. $\bar{2}.14755 + 3.82764$.

$$\begin{aligned}\bar{2}.14755 &= 0.14755 - 2 \\ 3.82764 &= \underline{3.82764} \\ &= \underline{3.97519 - 2} \\ &= 1.97519.\end{aligned}$$

5. $\bar{4}.18755 + \bar{2}.81245$.

$$\begin{aligned}\bar{4}.18755 &= 0.18755 - 4 \\ \bar{2}.81245 &= \underline{0.81245 - 2} \\ &= \underline{1.00000 - 6} \\ &= \bar{5}.00000.\end{aligned}$$

2. $\bar{4}.07256 + 1.58822$.

$$\begin{aligned}\bar{4}.07256 &= 0.07256 - 4 \\ 1.58822 &= \underline{1.58822} \\ &= \underline{1.66078 - 4} \\ &= \bar{3}.66078.\end{aligned}$$

6. $\bar{6}.28742 + \bar{3}.41258$.

$$\begin{aligned}\bar{6}.28742 &= 0.28742 - 6 \\ \bar{3}.41258 &= \underline{0.41258 - 3} \\ &= \underline{0.70000 - 9} \\ &= \bar{9}.70000.\end{aligned}$$

3. $0.21783 + \bar{1}.46835$.

$$\begin{aligned}0.21783 &= 0.21783 \\ \bar{1}.46835 &= \underline{0.46835 - 1} \\ &= \underline{0.68618 - 1} \\ &= \bar{1}.68618.\end{aligned}$$

7. $\bar{4}.21722 + \bar{4}.78278$.

$$\begin{aligned}\bar{4}.21722 &= 0.21722 - 4 \\ \bar{4}.78278 &= \underline{0.78278 - 4} \\ &= \underline{1.00000 - 8} \\ &= \bar{7}.00000.\end{aligned}$$

4. $0.41722 + \bar{3}.28682$.

$$\begin{aligned}0.41722 &= 0.41722 \\ \bar{3}.28682 &= \underline{0.28682 - 3} \\ &= \underline{0.70404 - 3} \\ &= \bar{3}.70404.\end{aligned}$$

8. $\bar{5}.28720 + \bar{3}.71280$.

$$\begin{aligned}\bar{5}.28720 &= 0.28720 - 5 \\ \bar{3}.71280 &= \underline{0.71280 - 3} \\ &= \underline{1.00000 - 8} \\ &= \bar{7}.00000.\end{aligned}$$

9. Find the sum of $\bar{2}.41280$, 4.17623 , $\bar{5}.26453$, 6.21020 , 7.36423 , 2.63577 , $\bar{6}.41323$, and 3.28740 .

$$\begin{array}{r}
 \bar{2}.41280 = 0.41280 - 2 \\
 \bar{4}.17623 = 0.17623 - 4 \\
 \bar{5}.26453 = 0.26453 - 5 \\
 0.21020 = 0.21020 \\
 7.36423 = 7.36423 \\
 2.63577 = 2.63577 \\
 \bar{6}.41323 = 0.41323 - 6 \\
 3.28740 = 3.28740 \\
 \hline
 14.76439 - 17 \\
 \hline
 = \bar{3}.76439.
 \end{array}$$

From the first of these logarithms subtract the second:

10. 0.21250 , $\bar{2}.21250$.

$$\begin{array}{r}
 0.21250 = 0.21250 \\
 \bar{2}.21250 = 0.21250 - 2 \\
 \hline
 0.00000 + 2 \\
 \hline
 = 2.00000.
 \end{array}$$

14. $\bar{4}.17325$, $\bar{2}.17325$.

$$\begin{array}{r}
 \bar{4}.17325 = 0.17325 - 4 \\
 \bar{2}.17325 = 0.17325 - 2 \\
 \hline
 0.00000 - 2 \\
 \hline
 = \bar{2}.00000.
 \end{array}$$

11. 0.17286 , $\bar{3}.27286$.

$$\begin{array}{r}
 0.17286 = 0.17286 - 10 \\
 \bar{3}.27286 = 0.27286 - 3 \\
 \hline
 9.90000 - 7 \\
 \hline
 = 2.90000.
 \end{array}$$

15. $\bar{5}.82340$, $\bar{3}.71120$.

$$\begin{array}{r}
 \bar{5}.82340 = 0.82340 - 5 \\
 \bar{3}.71120 = 0.71120 - 3 \\
 \hline
 0.11220 - 2 \\
 \hline
 = \bar{2}.11220.
 \end{array}$$

12. 2.34222 , $\bar{5}.44222$.

$$\begin{array}{r}
 2.34222 = 2.34222 \\
 \bar{5}.44222 = 0.44222 - 5 \\
 \hline
 1.90000 + 5 \\
 \hline
 = 6.90000.
 \end{array}$$

16. $\bar{3}.14286$, $\bar{1}.14000$.

$$\begin{array}{r}
 \bar{3}.14286 = 0.14286 - 3 \\
 \bar{1}.14000 = 0.14000 - 1 \\
 \hline
 0.00286 - 2 \\
 \hline
 = \bar{2}.00286.
 \end{array}$$

13. 3.14725 , $\bar{1}.25625$.

$$\begin{array}{r}
 3.14725 = 3.14725 \\
 \bar{1}.25625 = 0.25625 - 1 \\
 \hline
 2.89100 + 1 \\
 \hline
 = 3.89100.
 \end{array}$$

17. $\bar{3}.27283$, $\bar{5}.56111$.

$$\begin{array}{r}
 \bar{3}.27283 = 10.27283 - 13 \\
 \bar{5}.56111 = 0.56111 - 5 \\
 \hline
 9.71172 - 8 \\
 \hline
 = 1.71172.
 \end{array}$$

Using logarithms, divide as follows:

18. $10 \div 2$.

$$\begin{array}{r}
 \log 10 = 1.00000 \\
 \log 2 = 0.30103 \\
 \log x = 0.69897 \\
 \hline
 x = 5.
 \end{array}$$

19. $15 \div 3$.

$$\begin{array}{r}
 \log 15 = 1.17609 \\
 \log 3 = 0.47712 \\
 \log x = 0.69897 \\
 \hline
 x = 5.
 \end{array}$$

20. $15 \div 5$.

$$\begin{array}{r}
 \log 15 = 1.17609 \\
 \log 5 = 0.69897 \\
 \log x = 0.47712 \\
 \hline
 x = 3.
 \end{array}$$

21. $12 \div 3.$

$\log 12 = 1.07918$

$\log 3 = \underline{0.47712}$

$\log x = 0.60206$

$x = 4.$

25. $125 \div 25.$

$\log 125 = 2.09691$

$\log 25 = \underline{1.39794}$

$\log x = 0.69897$

$x = 5.$

29. $19,224 \div 540.$

$\log 19,224 = 4.28384$

$\log 540 = \underline{2.73239}$

$\log x = 1.55145$

$x = 35.6.$

22. $12 \div 4.$

$\log 12 = 1.07918$

$\log 4 = \underline{0.60206}$

$\log x = 0.47712$

$x = 3.$

26. $25,284 \div 301.$

$\log 25,284 = 4.40285$

$\log 301 = \underline{2.47857}$

$\log x = \underline{1.92428}$

$x = 84.$

30. $37,960 \div 520.$

$\log 37,960 = 4.57933$

$\log 520 = \underline{2.71600}$

$\log x = 1.86333$

$x = 73.002.$

23. $60 \div 12.$

$\log 60 = 1.77815$

$\log 12 = \underline{1.07918}$

$\log x = 0.69897$

$x = 5.$

27. $51,742 \div 631.$

$\log 51,742 = 4.71385$

$\log 631 = \underline{2.80003}$

$\log x = 1.91382$

$x = 82.002.$

31. $84,640 \div 920.$

$\log 84,640 = 4.92758$

$\log 920 = \underline{2.96379}$

$\log x = 1.96379$

$x = 92.$

24. $75 \div 25.$

$\log 75 = 1.87506$

$\log 25 = \underline{1.39794}$

$\log x = \underline{0.47712}$

$x = 3.$

28. $47,348 \div 623.$

$\log 47,348 = 4.67530$

$\log 623 = \underline{2.79449}$

$\log x = \underline{1.88081}$

$x = 76.$

32. $65,100 \div 620.$

$\log 65,100 = 4.81358$

$\log 620 = \underline{2.79239}$

$\log x = \underline{2.02119}$

$x = 105.$

33. $45,990 \div 730.$

$\log 45,990 = 4.66266$

$\log 730 = \underline{2.86332}$

$\log x = 1.79934$

$x = 63.$

36. $851.4 \div 0.66.$

$\log 851.4 = 2.93013$

$\log 0.66 = \underline{0.81954 - 1}$

$\log x = \underline{2.11059 + 1}$

$= 3.11059.$

$x = 1290.$

34. $59.29 \div 0.77.$

$\log 59.29 = 1.77298$

$\log 0.77 = \underline{0.88649 - 1}$

$\log x = \underline{0.88649 + 1}$

$= 1.88649.$

$x = 77.$

37. $0.98902 \div 99.$

$\log 0.98902 = 9.99521 - 10$

$\log 99 = \underline{1.99564}$

$\log x = \underline{7.99957 - 10}$

$= \bar{3}.99957.$

$x = 0.00999.$

35. $2.451 \div 190.$

$\log 2.451 = 10.38934 - 10$

$\log 190 = \underline{2.27875}$

$\log x = \underline{8.11059 - 10}$

$= \bar{2}.11059.$

$x = 0.0129.$

38. $0.41831 \div 5.9.$

$\log 0.41831 = 9.62150 - 10$

$\log 5.9 = \underline{0.77085}$

$\log x = \underline{8.85065 - 10}$

$= \bar{2}.85065.$

$x = 0.0709.$

39. $0.08772 \div 4.3.$

$$\log 0.08772 = 8.94310 - 10$$

$$\log 4.3 = \frac{0.63347}{}$$

$$\log x = \frac{8.30963 - 10}{}$$

$$= \bar{2}.30963.$$

$$x = 0.0204.$$

40. $0.02275 \div 0.35.$

$$\log 0.02275 = 8.35698 - 10$$

$$\log 0.35 = \frac{0.54437 - 1}{}$$

$$\log x = \frac{7.81291 - 9}{}$$

$$= \bar{2}.81291.$$

$$x = 0.065.$$

41. $0.02736 \div 0.057.$

$$\log 0.02736 = 8.43712 - 10$$

$$\log 0.057 = \frac{0.75587 - 2}{}$$

$$\log x = \frac{7.68125 - 8}{}$$

$$= \bar{1}.68125.$$

$$x = 0.48001.$$

Using logarithms, divide to four significant figures:

42. $15 \div 7.$

$$\log 15 = 1.17609$$

$$\log 7 = \frac{0.84510}{}$$

$$\log x = \frac{0.33099}{}$$

$$x = 2.143.$$

43. $7 \div 15.$

$$\log 7 = 10.84510 - 10$$

$$\log 15 = \frac{1.17609}{}$$

$$\log x = \frac{9.66901 - 10}{}$$

$$= \bar{1}.66901.$$

$$x = 0.4667.$$

44. $0.7 \div 150.$

$$\log 0.7 = 9.84510 - 10$$

$$\log 150 = \frac{2.17609}{}$$

$$\log x = \frac{7.66901 - 10}{}$$

$$= \bar{3}.66901.$$

$$x = 0.004667.$$

45. $26.4 \div 13.8.$

$$\log 26.4 = 1.42160$$

$$\log 13.8 = \frac{1.13988}{}$$

$$\log x = \frac{0.28172}{}$$

$$x = 1.913.$$

48. $17.625 \div 3.4.$

$$\log 17.625 = 1.24613$$

$$\log 3.4 = \frac{0.53148}{}$$

$$\log x = \frac{0.71465}{}$$

$$x = 5.184.$$

46. $4.21 \div 3.75.$

$$\log 4.21 = 0.62428$$

$$\log 3.75 = \frac{0.57403}{}$$

$$\log x = \frac{0.05025}{}$$

$$x = 1.123.$$

49. $43.826 \div 0.72.$

$$\log 43.826 = 1.64173$$

$$\log 0.72 = \frac{0.85733 - 1}{}$$

$$\log x = \frac{0.78440 + 1}{}$$

$$= 1.78440.$$

$$x = 60.87$$

47. $63.25 \div 4.92.$

$$\log 63.25 = 1.80106$$

$$\log 4.92 = \frac{0.69197}{}$$

$$\log x = \frac{1.10909}{}$$

$$x = 12.86.$$

50. $5.483 \div 8.4.$

$$\log 5.483 = 10.73902 - 10$$

$$\log 8.4 = \frac{0.92428}{}$$

$$\log x = \frac{9.81474 - 10}{}$$

$$= \bar{1}.81474.$$

$$x = 0.6527.$$

Taking $\log 3.1416$ as 0.49715 and interpolating for six figures on the same principle as for five, find the diameters of circles with circumferences as follows:

51. 62.832.

$$62.832 \div 3.1416 = 20.$$

$$\log 62.832 = 1.79818$$

$$\log 3.1416 = \underline{0.49715}$$

$$\log x = \underline{1.30103}$$

$$x = 20.$$

52. 157.08.

$$157.08 \div 3.1416 = 50.$$

$$\log 157.08 = 2.19612$$

$$\log 3.1416 = \underline{0.49715}$$

$$\log x = \underline{1.69897}$$

$$x = 50.$$

53. 2199.12.

$$2199.12 \div 3.1416 = 700.$$

$$\log 2199.12 = 3.34225$$

$$\log 3.1416 = \underline{0.49715}$$

$$\log x = \underline{2.84510}$$

$$x = 700.$$

54. 2513.28.

$$2513.28 \div 3.1416 = 800.$$

$$\log 2513.28 = 3.40024$$

$$\log 3.1416 = \underline{0.49715}$$

$$\log x = \underline{2.90309}$$

$$x = 800.$$

55. 28,274.2.

$$28,274.2 \div 3.1416 = 9000.$$

$$\log 28,274.2 = 4.45139$$

$$\log 3.1416 = \underline{0.49715}$$

$$\log x = \underline{3.95424}$$

$$x = 9000.$$

56. 34,557.6.

$$34,557.6 \div 3.1416 = 11,000.$$

$$\log 34,557.6 = 4.53854$$

$$\log 3.1416 = \underline{0.49715}$$

$$\log x = \underline{4.04139}$$

$$x = 11,000.$$

57. 376,992.

$$376,992 \div 3.1416 = 120,000$$

$$\log 376,992 = 5.57623$$

$$\log 3.1416 = \underline{0.49715}$$

$$\log x = \underline{5.07918}$$

$$x = 120,000.$$

58. 0.031416.

$$0.031416 \div 3.1416 = 0.01.$$

$$\log 0.031416 = 0.49715 - 2$$

$$\log 3.1416 = \underline{0.49715}$$

$$\log x = \underline{0.00000 - 2}$$

$$= \underline{2.00000}.$$

$$x = 0.01.$$

59. By using logarithms find the product of 41.74×20.87 , and the quotient of $41.74 \div 20.87$.

$$41.74 \times 20.87 = 871.1.$$

$$\log 20.87 = 1.31952$$

$$\log 41.74 = \underline{1.62055}$$

$$\log x = \underline{2.94007}$$

$$x = 871.1. \text{ Ans.}$$

$$41.74 \div 20.87 = 2.$$

$$\log 41.74 = 1.62055$$

$$\log 20.87 = \underline{1.31952}$$

$$\log x = \underline{0.30103}$$

$$x = 2. \text{ Ans.}$$

Exercise 25. Cologarithms (Page 54)

Write the cologarithms of the following numbers :

- | | |
|--|---|
| 1. 25.
$\log 25 = 1.39794.$
$\text{colog } 25 = 8.60206 - 10.$ | 9. 0.5.
$\log 0.5 = 9.69897 - 10$
$\text{colog } 0.5 = 0.30103.$ |
| 2. 130.
$\log 130 = 2.11394.$
$\text{colog } 130 = 7.88606 - 10.$ | 10. 0.72.
$\log 0.72 = 9.85733 - 10.$
$\text{colog } 0.72 = 0.14267.$ |
| 3. 27.4.
$\log 27.4 = 1.43775.$
$\text{colog } 27.4 = 8.56225 - 10.$ | 11. 0.083.
$\log 0.083 = 8.91908 - 10.$
$\text{colog } 0.083 = 1.08092.$ |
| 4. 5.83.
$\log 5.83 = 0.76567.$
$\text{colog } 5.83 = 9.23433 - 10.$ | 12. 0.00726.
$\log 0.00726 = 7.86094 - 10.$
$\text{colog } 0.00726 = 2.13906.$ |
| 5. 3751.
$\log 3751 = 3.57415.$
$\text{colog } 3751 = 6.42585 - 10.$ | 13. 3.007.
$\log 3.007 = 0.47813.$
$\text{colog } 3.007 = 9.52187 - 10.$ |
| 6. 427.3.
$\log 427.3 = 2.63073.$
$\text{colog } 427.3 = 7.36927 - 10.$ | 14. 62.09.
$\log 62.09 = 1.79302$
$\text{colog } 62.09 = 8.20698 - 10.$ |
| 7. 51.61.
$\log 51.61 = 1.71273.$
$\text{colog } 51.61 = 8.28727 - 10.$ | 15. 0.0006.
$\log 0.0006 = 6.77815 - 10.$
$\text{colog } 0.0006 = 3.22185.$ |
| 8. 7.213.
$\log 7.213 = 0.85812.$
$\text{colog } 7.213 = 9.14188 - 10.$ | 16. 0.00007.
$\log 0.00007 = 5.84510 - 10.$
$\text{colog } 0.00007 = 4.15490.$ |

17. What number has 0 for its cologarithm ?

The number which has 0 for its cologarithm is 1.

18. What number has 1 for its cologarithm ?

The number which has 1 for its cologarithm is 0.1.

19. What number has ∞ for its cologarithm ?

The number which has ∞ for its cologarithm is 0.

20. Find the number whose cologarithm equals its logarithm ?

The number whose cologarithm equals its logarithm is the number whose reciprocal equals the number and hence is 1.

Exercise 26. Use of Cologarithms (Page 55)*Using cologarithms, find the value of the following to five figures :*

$$1. \frac{3 \times 2}{4 \times 1.5}.$$

$$\begin{aligned} \log 3 &= 0.47712 \\ \log 2 &= 0.30103 \\ \text{colog } 4 &= 9.39794 - 10 \\ \text{colog } 1.5 &= 9.82391 - 10 \\ \hline &0.00000 \end{aligned}$$

$$= \log 1.$$

$$2. \frac{8 \times 9}{3 \times 4}.$$

$$\begin{aligned} \log 8 &= 0.90309 \\ \log 9 &= 0.95424 \\ \text{colog } 3 &= 9.52288 - 10 \\ \text{colog } 4 &= 9.39794 - 10 \\ \hline &0.77815 \end{aligned}$$

$$= \log 6.$$

$$3. \frac{6 \times 12}{3 \times 8}.$$

$$\begin{aligned} \log 6 &= 0.77815 \\ \log 12 &= 1.07918 \\ \text{colog } 3 &= 9.52288 - 10 \\ \text{colog } 8 &= 9.09691 - 10 \\ \hline &0.47712 \end{aligned}$$

$$= \log 3.$$

$$4. \frac{4 \times 24}{12 \times 16}.$$

$$\begin{aligned} \log 4 &= 0.60206 \\ \log 24 &= 1.38021 \\ \text{colog } 12 &= 8.92082 - 10 \\ \text{colog } 16 &= 8.79588 - 10 \\ \hline &9.69897 - 10 \end{aligned}$$

$$= \log 0.5,$$

$$5. \frac{12 \times 15}{9 \times 20}.$$

$$\begin{aligned} \log 12 &= 1.07918 \\ \log 15 &= 1.17609 \\ \text{colog } 9 &= 9.04576 - 10 \\ \text{colog } 20 &= 8.69897 - 10 \\ \hline &0.00000 \end{aligned}$$

$$= \log 1.$$

$$6. \frac{12 \times 28}{8 \times 21}.$$

$$\begin{aligned} \log 12 &= 1.07918 \\ \log 28 &= 1.44716 \\ \text{colog } 8 &= 9.09691 - 10 \\ \text{colog } 21 &= 8.67778 - 10 \\ \hline &0.30103 \end{aligned}$$

$$= \log 2.$$

$$7. \frac{3 \times 22}{18 \times 33}.$$

$$\begin{aligned} \log 3 &= 0.47712 \\ \log 22 &= 1.34242 \\ \text{colog } 18 &= 8.74473 - 10 \\ \text{colog } 33 &= 8.48149 - 10 \\ \hline &9.04576 - 10 \end{aligned}$$

$$= \log 0.11111.$$

$$8. \frac{11 \times 13}{17 \times 19}.$$

$$\begin{aligned} \log 11 &= 1.04139 \\ \log 13 &= 1.11394 \\ \text{colog } 17 &= 8.76955 - 10 \\ \text{colog } 19 &= 8.72125 - 10 \\ \hline &9.64613 - 10 \end{aligned}$$

$$= \log 0.44272.$$

$$9. \frac{15 \times 17}{11 \times 13}.$$

$$\begin{aligned} \log 15 &= 1.17609 \\ \log 17 &= 1.23045 \\ \text{colog } 11 &= 8.95861 - 10 \\ \text{colog } 13 &= 8.88606 - 10 \\ \hline &0.25121 \end{aligned}$$

$$= \log 1.7833.$$

$$10. \frac{172.8 \times 1.44}{0.288 \times 0.864}.$$

$$\begin{aligned} \log 172.8 &= 2.23754 \\ \log 1.44 &= 0.15836 \\ \text{colog } 0.288 &= 0.54061 \\ \text{colog } 0.864 &= 0.06349 \\ \hline &3.00000 \end{aligned}$$

$$= \log 1000.$$

Exercise 27. Raising to Powers (Page 56)

By logarithms, find the value of each of the following to five significant figures :

1. 2^2 .

$$\log 2 = 0.30103$$

$$\log 2^2 = \frac{2}{0.60206}$$

$$= \log 4.$$

2. 2^3 .

$$\log 2 = 0.30103$$

$$\log 2^3 = \frac{3}{0.90309}$$

$$= \log 8.$$

3. 2^5 .

$$\log 2 = 0.30103$$

$$\log 2^5 = \frac{5}{1.50515}$$

$$= \log 32.$$

10. 7^9 .

$$\log 7 = 0.84510$$

$$\log 7^9 = \frac{9}{7.60590}$$

$$= \log 40,355,000.$$

11. 9^7 .

$$\log 9 = 0.95424$$

$$\log 9^7 = \frac{7}{6.67968}$$

$$= \log 4,782,800.$$

12. 8^8 .

$$\log 8 = 0.90309$$

$$\log 8^8 = \frac{8}{7.22472}$$

$$= \log 16,777,000.$$

13. 11^7 .

$$\log 11 = 1.04139$$

$$\log 11^7 = \frac{7}{7.28973}$$

$$= \log 19,486,000.$$

4. 2^{10} .

$$\log 2 = 0.30103$$

$$\log 2^{10} = \frac{10}{3.01030}$$

$$= \log 1024.$$

5. 3^4 .

$$\log 3 = 0.47712$$

$$\log 3^4 = \frac{4}{1.90848}$$

$$= \log 80.998.$$

6. 3^5 .

$$\log 3 = 0.47712$$

$$\log 3^5 = \frac{6}{2.86272}$$

$$= \log 728.98.$$

7. 4^3 .

$$\log 4 = 0.60206$$

$$\log 4^3 = \frac{3}{1.80618}$$

$$= \log 64.$$

8. 5^3 .

$$\log 5 = 0.69897$$

$$\log 5^3 = \frac{3}{2.09691}$$

$$= \log 125.$$

9. 1^{10} .

$$\log 1 = 0.00000$$

$$\log 1^{10} = \frac{10}{0.00000}$$

$$= \log 1.$$

14. 15^6 .

$$\log 15 = 1.17609$$

$$\log 15^6 = \frac{6}{7.05654}$$

$$= \log 11,391,000.$$

15. 1.5^6 .

$$\log 1.5 = 0.17609$$

$$\log 1.5^6 = \frac{6}{1.05654}$$

$$= \log 11.391.$$

16. 17^4 .

$$\log 17 = 1.23045$$

$$\log 17^4 = \frac{4}{4.92180}$$

$$= \log 83,522.$$

17. 25^3 .

$$\log 25 = 1.39794$$

$$\log 25^3 = \frac{3}{4.19382}$$

$$= \log 15,625.$$

18. 25^7 .

$$\log 25 = 1.39794$$

$$\log 25^7 = \frac{7}{9.78558}$$

$$= \log 6,103,600,000.$$

19. 125^2 .

$$\log 125 = 2.09691$$

$$\log 125^2 = \frac{2}{4.19382}$$

$$= \log 15,625.$$

20. 625^3 .

$$\log 625 = 2.79588$$

$$\log 625^3 = \frac{3}{8.38764}$$

$$= \log 244,140,000.$$

21. 1750^5 .

$$\log 1750 = 3.24304$$

$$\log 1750^5 = \frac{5}{16.21520}$$

$$= \log 16,413,000,000,000,000.$$

22. 2775^2 .

$$\log 2775 = 3.44326$$

$$\log 2775^2 = \frac{2}{6.88652}$$

$$= \log 7,700,500.$$

23. 3146^3 .

$$\log 3146 = 3.49776$$

$$\log 3146^3 = \frac{3}{10.49328}$$

$$= \log 31,137,000,000.$$

24. 4135^4 .

$$\log 4135 = 3.61648$$

$$\log 4135^4 = \frac{4}{14.46592}$$

$$= \log 292,360,000,000,000.$$

25. 1.1^8 .

$$\log 1.1 = 0.04139$$

$$\log 1.1^8 = \frac{8}{0.33112}$$

$$= \log 2.1435.$$

26. 2.1^7 .

$$\log 2.1 = 0.32222$$

$$\log 2.1^7 = \frac{7}{2.25554}$$

$$= \log 180.11.$$

27. 0.1^{12} .

$$\log 0.1 = 0.00000 - 1$$

$$\log 0.1^{12} = \frac{12}{0.00000 - 12}$$

$$= \overline{12.00000}$$

$$= \log 0.000000000001.$$

28. 0.2^{11} .

$$\log 0.2 = 0.30103 - 1$$

$$\log 0.2^{11} = \frac{11}{3.31133 - 11}$$

$$= \overline{8.31133}$$

$$= \log 0.00000002048$$

29. 0.7^8 .

$$\log 0.7 = 0.84510 - 1$$

$$\log 0.7^8 = \frac{8}{6.76080 - 8}$$

$$= \overline{2.76080}$$

$$= \log 0.05765.$$

30. 0.07^6 .

$$\log 0.07 = 0.84510 - 2$$

$$\log 0.07^6 = \frac{6}{5.07060 - 12}$$

$$= \overline{7.07060}$$

$$= \log 0.00000011765$$

31. 0.37^4 .

$$\log 0.37 = 0.56820 - 1$$

$$\log 0.37^4 = \frac{4}{2.27280 - 4}$$

$$= \overline{2.27280}$$

$$= \log 0.018741$$

32. 5.37^3 .

$$\log 5.37 = 0.72997$$

$$\log 5.37^3 = \frac{3}{2.18991}$$

$$= \log 154.8\frac{1}{2}$$

33. 12.55^2 .

$$\log 12.55 = 1.09864$$

$$\log 12.55^2 = \frac{2}{2.19728}$$

$$= \log 157.5.$$

34. 34.75^3 .

$$\log 34.75 = 1.54095$$

$$\log 34.75^3 = \frac{3}{4.62285}$$

$$= \log 41,961.$$

35. 1.275^3 .

$$\log 1.275 = 0.10551$$

$$\log 1.275^3 = \frac{3}{0.31653}$$

$$= \log 2.0727.$$

36. 0.1254^3 .

$$\log 0.1254 = 0.09830 - 1$$

$$\log 0.1254^3 = \frac{3}{0.29490 - 3}$$

$$= \bar{3}.29490$$

$$= \log 0.0019720.$$

37. 0.4725^5 .

$$\log 0.4725 = 0.67440 - 1$$

$$\log 0.4725^5 = \frac{5}{3.37200 - 5}$$

$$= \bar{2}.37200$$

$$= \log 0.023551.$$

38. 0.01234^2 .

$$\log 0.01234 = 0.09132 - 2$$

$$\log 0.01234^2 = \frac{2}{0.18264 - 4}$$

$$= \bar{4}.18264$$

$$= \log 0.00015228.$$

39. 0.00275^2 .

$$\log 0.00275 = 0.43933 - 3$$

$$\log 0.00275^2 = \frac{2}{0.87866 - 6}$$

$$= \bar{6}.87866$$

$$= \log 0.0000075624.$$

40. 0.000355^2 .

$$\log 0.000355 = 0.55023 - 4$$

$$\log 0.000355^2 = \frac{2}{1.10046 - 8}$$

$$= \bar{7}.10046$$

$$= 0.00000012603.$$

41. If $\log \pi = 0.49715$, what is the value of π^2 ? of π^3 ?

$$\log \pi = 0.49715$$

$$\log \pi^2 = \frac{2}{0.99430}$$

$$= \log 9.8696.$$

$$\log \pi = 0.49715$$

$$\log \pi^3 = \frac{3}{1.49145}$$

$$= \log 31.006.$$

42. Using $\log \pi$ as in Ex. 41, what is the value of πr when $r = 7$? of πr^2 when $r = 7$? of $\frac{4}{3}\pi r^3$ when $r = 9$?

$$\log \pi = 0.49715$$

$$\log 7 = \frac{0.84510}{1.34225}$$

$$= \log 21.991.$$

$$\log \pi = 0.49715$$

$$\log 7^2 = \frac{1.69020}{2.18735}$$

$$= \log 153.94.$$

$$\log 4 = 0.60206$$

$$\log 3 = 9.52288 - 10$$

$$\log \pi = 0.49715$$

$$\log 9^3 = \frac{2.86272}{3.48481}$$

$$= \log 3053.6.$$

Exercise 28. Extracting Roots (Page 57)

By logarithms, find the value of each of the following:

1. $\sqrt{2}$.

$\log 2 = 0.30103$

$$\begin{array}{r} 2)0.30103 \\ \hline \end{array}$$

$\log \sqrt{2} = 0.15052$

$= \log 1.4142.$

2. $\sqrt[3]{5}$.

$\log 5 = 0.69897$

$$\begin{array}{r} 3)0.69897 \\ \hline \end{array}$$

$\log \sqrt[3]{5} = 0.23299$

$= \log 1.71.$

3. $\sqrt[7]{7}$.

$\log 7 = 0.84510$

$$\begin{array}{r} 7)0.84510 \\ \hline \end{array}$$

$\log \sqrt[7]{7} = 0.12073$

$= \log 1.3205$

4. $\sqrt[15]{25}$.

$\log 25 = 1.39794$

$$\begin{array}{r} 15)1.39794 \\ \hline \end{array}$$

$\log \sqrt[15]{25} = 0.09320$

$= 1.2394.$

11. $\sqrt[3]{22}$.

$\log 22 = 1.34242.$

$$\begin{array}{r} 3)1.34242 \\ \hline \end{array}$$

$\log \sqrt[3]{22} = 0.44747$

$= \log 2.802.$

5. $2^{\frac{1}{5}}$.

$\log 2 = 0.30103.$

$\log 2^{\frac{1}{5}} = 0.06021 = \log 1.1487.$

12. $\sqrt[25]{100}$.

$\log 100 = 2.00000.$

$$\begin{array}{r} 25)2.00000 \\ \hline \end{array}$$

$\log \sqrt[25]{100} = 0.08000$

$= \log 1.2023.$

6. $3^{\frac{3}{4}}$.

$\log 3 = 0.47712.$

$\log 3^{\frac{3}{4}} = 0.35784 = 2.2795.$

13. $0.3^{\frac{1}{2}}$.

$\log 0.3 = 0.47712 - 1$

$= 1.47712 - 2.$

$\log 0.3^{\frac{1}{2}} = 0.73856 - 1$

$= \bar{1}.73856$

$= \log 0.54773.$

7. $8^{\frac{5}{8}}$.

$\log 8 = 0.90309.$

$\log 8^{\frac{5}{8}} = 0.75258 = \log 5.6569.$

8. $7^{\frac{4}{7}}$.

$\log 7 = 0.84510.$

$\log 7^{\frac{4}{7}} = 0.48291 = \log 3.0403.$

14. $0.05^{\frac{1}{3}}$.

$\log 0.05 = 0.69897 - 2$

$= 1.69897 - 3.$

$\log 0.05^{\frac{1}{3}} = 0.56632 - 1$

$= \bar{1}.56632$

$= \log 0.3684.$

9. $\sqrt{11}$.

$\log 11 = 1.04139.$

$$\begin{array}{r} 2)1.04139 \\ \hline \end{array}$$

$\log \sqrt{11} = 0.52070$

$= \log 3.3166.$

15. $0.0175^{\frac{2}{3}}$.

$\log 0.0175 = 0.24304 - 2$

$= 1.24304 - 3.$

$\log 0.0175^{\frac{2}{3}} = 0.82869 - 2$

$= \bar{2}.82869$

$= \log 0.067405.$

10. $\sqrt[3]{3}$.

$\log 3 = 0.47712.$

$$\begin{array}{r} 3)0.47712 \\ \hline \end{array}$$

$\log \sqrt[3]{3} = 0.15904$

$= \log 1.4422.$

16. $0.0325^{\frac{4}{5}}$.

$$\begin{aligned}\log 0.0325 &= 0.51188 - 2 \\ &= 3.51188 - 5.\end{aligned}$$

$$\begin{aligned}\log 0.0325^{\frac{4}{5}} &= 2.80950 - 4 \\ &= \bar{2}.80950 \\ &= \log 0.064491.\end{aligned}$$

17. $127.8^{\frac{5}{8}}$.

$$\log 127.8 = 2.10653.$$

$$\begin{aligned}\log 127.8^{\frac{5}{8}} &= 1.31658 \\ &= \log 20.729.\end{aligned}$$

18. $2.475^{\frac{3}{4}}$.

$$\log 2.475 = 0.39358.$$

$$\begin{aligned}\log 2.475^{\frac{3}{4}} &= 0.29519 \\ &= \log 1.9733.\end{aligned}$$

19. $5.135^{\frac{5}{8}}$.

$$\log 5.135 = 0.71054.$$

$$\begin{aligned}\log 5.135^{\frac{5}{8}} &= 0.59212 \\ &= \log 3.9095.\end{aligned}$$

20. $0.00125^{\frac{7}{8}}$.

$$\begin{aligned}\log 0.00125 &= 0.09691 - 3 \\ &= 5.09691 - 8.\end{aligned}$$

$$\begin{aligned}\log 0.00125^{\frac{7}{8}} &= 4.45980 - 7. \\ &= \bar{3}.45980 \\ &= \log 0.0028827.\end{aligned}$$

21. If $\log \pi = 0.49715$, what is the value of $\sqrt{\pi}$? of $\sqrt[3]{\pi}$?

$$\log \pi = 0.49715.$$

$$\log \pi^{\frac{1}{2}} = 0.24858 = \log 1.7725.$$

$$\log \pi^{\frac{1}{3}} = 0.16572 = \log 1.4645.$$

22. Using the value of $\log \pi$ given in Ex. 21, what is the value of $\pi^{\frac{1}{4}}$? of $\pi^{\frac{2}{5}}$? of $\pi^{\frac{3}{8}}$? of $\pi^{-\frac{3}{4}}$? of $\pi^{-\frac{4}{5}}$? of $\pi^{-0.2}$?

$$\log \pi = 0.49715.$$

$$\log \pi^{\frac{1}{4}} = 0.12429 = \log 1.3313.$$

$$\log \pi^{\frac{2}{5}} = 0.33143 = \log 2.1450.$$

$$\log \pi^{\frac{3}{8}} = 0.74573 = \log 5.5684.$$

$$\begin{aligned}\log \pi^{-\frac{3}{4}} &= \log \frac{1}{\pi^{\frac{3}{4}}} = \log 1 + \operatorname{colog} \pi^{\frac{3}{4}} \\ &= 0.00000 + 9.62714 - 10 \\ &= \log 0.42378.\end{aligned}$$

$$\begin{aligned}\log \pi^{-\frac{4}{5}} &= \log \frac{1}{\pi^{\frac{4}{5}}} = \log 1 + \operatorname{colog} \pi^{\frac{4}{5}} \\ &= 0.00000 + 9.60228 - 10 \\ &= \log 0.40020.\end{aligned}$$

$$\begin{aligned}\log \pi^{-0.2} &= \log \frac{1}{\pi^{0.2}} = \log 1 + \operatorname{colog} \pi^{0.2} \\ &= 0.00000 + 9.90057 - 10 \\ &= 0.79537.\end{aligned}$$

Exercise 29. Exponential Equations (Page 59)

By logarithms, solve the following exponential equations:

1. $2^x = 8.$

2. $3^x = 81.$

$$x \log 2 = \log 8.$$

$$x \log 3 = \log 81.$$

$$x = \frac{\log 8}{\log 2} = \frac{0.90309}{0.30103} = 3.$$

$$x = \frac{\log 81}{\log 3} = \frac{1.90849}{0.47712} = 4.$$

$$3. 5^x = 625.$$

$$x \log 5 = \log 625.$$

$$x = \frac{\log 625}{\log 5} = \frac{2.79588}{0.69897} = 4.$$

$$4. 4^x = 256.$$

$$x \log 4 = \log 256.$$

$$x = \frac{\log 256}{\log 4} = \frac{2.40824}{0.60206} = 4.$$

$$5. 11^x = 1331.$$

$$x \log 11 = \log 1331.$$

$$x = \frac{\log 1331}{\log 11} = \frac{3.12418}{1.04139} = 3.$$

$$6. 2^x = 19.$$

$$x \log 2 = \log 19.$$

$$x = \frac{\log 19}{\log 2} = \frac{1.27875}{0.30103} = 4.2479.$$

$$7. 3^x = 75.$$

$$x \log 3 = \log 75.$$

$$x = \frac{\log 75}{\log 3} = \frac{1.87506}{0.47712} = 3.9300.$$

$$8. 5^x = 1000.$$

$$x \log 5 = \log 1000.$$

$$x = \frac{\log 1000}{\log 5} = \frac{3.00000}{0.69897} = 4.2920.$$

$$13. 0.3^{-x} = 0.9.$$

$$\frac{1}{0.3^x} = 0.9.$$

$$\log 1 - x \log 0.3 = \log 0.9.$$

$$\begin{aligned} x &= \frac{\log 1 - \log 0.9}{\log 0.3} \\ &= \frac{0.00000 - (0.95424 - 1)}{0.47712 - 1} \\ &= \frac{0.04576}{0.47712 - 1} \\ &= -0.087515. \end{aligned}$$

$$9. 4 = 2560.$$

$$x \log 4 = \log 2560.$$

$$x = \frac{\log 2560}{\log 4} = \frac{3.40824}{0.60206} = 5.6610.$$

$$10. 11^x = 1500.$$

$$x \log 11 = \log 1500.$$

$$x = \frac{\log 1500}{\log 11} = \frac{3.17609}{1.04139} = 3.0499$$

$$11. 2^{-x} = \frac{1}{8}.$$

$$\frac{1}{2^x} = \frac{1}{8}.$$

$$2^x = 8.$$

$$2^x = 2^3.$$

$$x \log 2 = 3 \log 2.$$

$$x = 3.$$

$$12. 2^{-x} = 0.1.$$

$$\frac{1}{2^x} = 0.1.$$

$$2^x = 10.$$

$$x \log 2 = \log 10.$$

$$\begin{aligned} x &= \frac{\log 10}{\log 2} \\ &= \frac{1.00000}{0.30103} \\ &= 3.3219. \end{aligned}$$

$$14. 2^{x+1} = 3^{x-1}.$$

$$(x+1) \log 2 = (x-1) \log 3.$$

$$x \log 2 + \log 2 = x \log 3 - \log 3.$$

$$x(\log 2 - \log 3) = -\log 2 - \log 3.$$

....

$$\begin{aligned} x &= \frac{\log 2 + \log 3}{\log 3 - \log 2} \\ &= \frac{0.30103 + 0.47712}{0.47712 - 0.30103} \\ &= \frac{0.77815}{0.17609} = 4.4190. \end{aligned}$$

$$15. 9^{x+5} = 53,143.$$

$$x \log 9 + 5 \log 9 = \log 53,143.$$

$$\begin{aligned} x &= \frac{\log 53,143 - 5 \log 9}{\log 9} \\ &= \frac{4.72544 - 4.77120}{0.95424} \\ &= -0.047954. \end{aligned}$$

Solve the following simultaneous equations:

$$16. a^{x+y} = a^4 \quad (1)$$

$$a^{x-y} = a^2 \quad (2)$$

$$x + y = 4. \quad (3)$$

$$x - y = 2. \quad (4)$$

Adding (3) and (4),

$$2x = 6.$$

$$x = 3.$$

Subtracting (4) from (3),

$$2y = 2.$$

$$y = 1.$$

$$18. 3^x \cdot 4^y = 12 \quad (1)$$

$$5^x \cdot 7^y = 35 \quad (2)$$

$$x \log 3 + y \log 4 = \log 12. \quad (3)$$

$$x \log 5 + y \log 7 = \log 35. \quad (4)$$

$$\text{From (3),} \quad x = \frac{\log 12 - y \log 4}{\log 3}. \quad (5)$$

$$\text{From (4),} \quad x = \frac{\log 35 - y \log 7}{\log 5}. \quad (6)$$

Equating (5) and (6),

$$\frac{\log 12 - y \log 4}{\log 3} = \frac{\log 35 - y \log 7}{\log 5}.$$

$$\log 5 \cdot \log 12 - \log 5 \cdot y \log 4 = \log 3 \cdot \log 35 - \log 3 \cdot y \log 7.$$

$$\log 5 (\log 4 + \log 3) - \log 5 \cdot y \log 4 = \log 3 (\log 7 + \log 5) - \log 3 \cdot y \log 7$$

$$y (\log 3 \log 7 - \log 5 \log 4) = \log 3 \log 7 - \log 5 \log 4.$$

$$y = 1.$$

Substituting in (1),

$$3^x \cdot 4 = 12.$$

$$3^x = 3.$$

$$x \log 3 = \log 3.$$

$$x = 1.$$

$$19. 2^x \cdot 3^y = 36 \quad (1)$$

$$4^x \cdot 5^y = 400 \quad (2)$$

$$x \log 2 + y \log 3 = \log 36. \quad (3)$$

$$x \log 4 + y \log 5 = \log 400. \quad (4)$$

$$\log 4 = \log 2^2 = 2 \log 2.$$

$$2x \log 2 + y \log 5 = \log 400. \quad (5)$$

Multiplying (3) by 2 and subtracting (3) and (5),

$$2y \log 3 - y \log 5 = 2 \log 36 - \log 400,$$

$$y(2 \log 3 - \log 5) = 2 \log 36 - \log 400.$$

$$\begin{aligned} y &= \frac{2 \log 36 - \log 400}{2 \log 3 - \log 5} \\ &= \frac{2 \times 1.55630 - 2.60206}{2 \times 0.47712 - 0.69897} \\ &= \frac{0.51054}{0.25527} = 2. \end{aligned}$$

$$2^x \cdot 9 = 36.$$

$$2^x = 4.$$

$$x = \frac{\log 4}{\log 2} = \frac{0.60206}{0.30103} = 2.$$

$$20. 2^x \cdot 5^y = 200 \quad (1)$$

$$3^x \cdot 3^y = 243 \quad (2)$$

$$x \log 2 + y \log 5 = \log 200. \quad (3)$$

$$x \log 3 + y \log 3 = \log 243 = 5 \log 3. \quad (4)$$

From (4),

$$x + y = 5.$$

$$x = 5 - y.$$

Substituting the value of x in (3),

$$5 \log 2 - y \log 2 + y \log 5 = \log 200.$$

$$y(\log 5 - \log 2) = \log 200 - 5 \log 2.$$

$$\begin{aligned} y &= \frac{\log 200 - 5 \log 2}{\log 5 - \log 2} \\ &= \frac{\log 2^3 + \log 5^2 - 5 \log 2}{\log 5 - \log 2} \\ &= \frac{3 \log 2 + 2 \log 5 - 5 \log 2}{\log 5 - \log 2} \\ &= \frac{2 \log 5 - 2 \log 2}{\log 5 - \log 2} = 2. \end{aligned}$$

$$x = 3.$$

$$21. 2^x \cdot 8^y = 256 \quad (1)$$

$$8^x \cdot 32^y = 65,536 \quad (2)$$

$$x \log 2 + y \log 8 = \log 256. \quad (3)$$

$$x \log 8 + y \log 32 = \log 65,536. \quad (4)$$

From (3) and (4),

$$x \log 2 + 3y \log 2 = 8 \log 2. \quad x + 3y = 8. \quad (5)$$

$$3x \log 2 + 5y \log 2 = 16 \log 2. \quad 3x + 5y = 16. \quad (6)$$

Multiplying (5) by 3 and subtracting (6),

$$4y = 8.$$

$$y = 2.$$

$$x = 2.$$

Solve the following equations by logarithms:

$$22. a = p(1+r)^x.$$

$$24. 2^{x^2+2x} = 8.$$

$$\log a = \log p + x \log(1+r).$$

$$2^{x(x+2)} = 8.$$

$$x \log(1+r) = \log a - \log p.$$

$$x(x+2) \log 2 = 3 \log 2.$$

$$x = \frac{\log a - \log p}{\log(1+r)}.$$

$$x(x+2) = 3.$$

$$x^2 + 2x - 3 = 0.$$

$$(x+3)(x-1) = 0.$$

$$23. l = ar^{x-1}.$$

$$x = 1, -3.$$

$$\log l = \log a + (x-1) \log r.$$

$$\log l = \log a + x \log r - \log r.$$

$$x \log r = \log r + \log l - \log a.$$

$$25. a = p(1+rt)^x.$$

$$\log a = \log p + x \log(1+rt).$$

$$x = \frac{\log r + \log l - \log a}{\log r}$$

$$x \log(1+rt) = \log a - \log p.$$

$$= 1 + \frac{\log l - \log a}{\log r}.$$

$$x = \frac{\log a - \log p}{\log(1+rt)}.$$

$$26. s(r-1) = ar^x - a.$$

$$ar^x = s(r-1) + a.$$

$$\log a + x \log r = \log [s(r-1) + a].$$

$$x = \frac{\log [s(r-1) + a] - \log a}{\log r}.$$

$$27. 3^{x^2-x+1} = 27.$$

$$(x^2 - x + 1) \log 3 = 3 \log 3.$$

$$x^2 - x + 1 = 3.$$

$$x^2 - x - 2 = 0.$$

$$(x-2)(x+1) = 0.$$

$$x = 2, -1.$$

Perform the following operations by logarithms:

$$28. \frac{2.47 \times 84.96}{34.8 \times 96.55}.$$

$$\log 2.47 = 0.39270$$

$$\log 84.96 = 1.92921$$

$$\text{colog } 34.8 = 8.45842 - 10$$

$$\text{colog } 96.55 = \frac{8.01525 - 10}{18.79558 - 20}$$

$$= \bar{2}.79558$$

$$= \log 0.062457.$$

$$29. \sqrt[4]{\frac{42.4 \times 0.075}{3.64 \times 0.009}}.$$

$$\log 42.4 = 1.62737$$

$$\log 0.075 = 8.87506 - 10$$

$$\text{colog } 3.64 = 9.43890 - 10$$

$$\text{colog } 0.009 = \frac{2.04576}{21.98709 - 20}$$

$$4) 1.98709$$

$$\underline{0.49677}$$

$$= \log 3.1389.$$

$$31. \sqrt[5]{\left(\frac{0.07 \times 0.00964}{3.426 \times 0.875}\right)^2}.$$

$$\log 0.07 = 8.84510 - 10$$

$$\log 0.00964 = 7.98408 - 10$$

$$\text{colog } 3.426 = 9.46521 - 10$$

$$\text{colog } 0.875 = \frac{0.05799}{6.35238 - 10}$$

$$\frac{2}{5) 12.70476 - 20}$$

$$\underline{2.54095 - 4}$$

$$= \log 0.03475.$$

$$30. \left(\frac{5.75 \times 3.428}{59.62 \times 48.08}\right)^{\frac{2}{3}}.$$

$$\log 5.75 = 0.75967$$

$$\log 3.428 = 0.53504$$

$$\text{colog } 59.62 = 8.22461 - 10$$

$$\text{colog } 48.08 = \frac{8.31804 - 10}{17.83736 - 20}$$

$$= 0.83736 - 3$$

$$\frac{2}{3) 1.67472 - 6}$$

$$\underline{0.55824 - 2}$$

$$= \bar{2}.55824$$

$$= \log 0.036161.$$

32. To what power must 7 be raised to equal 117,649?

$$7^x = 117,649.$$

$$x \log 7 = \log 117,649.$$

$$x = \frac{\log 117,649}{\log 7}$$

$$= \frac{5.07059}{0.84510} = 6.$$

33. To what power must a be raised to equal b ?

$$a^x = b.$$

$$x \log a = \log b.$$

$$x = \frac{\log b}{\log a}.$$

34. To what power must 5 be raised to equal n ?

$$5^x = n.$$

$$x \log 5 = \log n.$$

$$x = \frac{\log n}{\log 5}.$$

35. Find the value of x when $\sqrt[3]{9} = 3$; when $\sqrt[3]{2} = 1.1$; when $\sqrt[3]{2} = 1.414$; when $\sqrt[3]{3} = 1.73$.

$$\sqrt[3]{9} = 3.$$

$$\frac{1}{x} \log 9 = \log 3.$$

$$\frac{2}{x} \log 3 = \log 3.$$

$$\frac{2}{x} = 1.$$

$$x = 2.$$

$$\sqrt[3]{2} = 1.1.$$

$$\frac{1}{x} \log 2 = \log 1.1.$$

$$\log 2 = x \log 1.1.$$

$$x = \frac{\log 2}{\log 1.1}$$

$$= \frac{0.30103}{0.04139}$$

$$= 7.2730.$$

$$\sqrt[3]{2} = 1.414.$$

$$\frac{1}{x} \log 2 = \log 1.414.$$

$$x = \frac{\log 2}{\log 1.414}$$

$$= \frac{0.30103}{0.15045}$$

$$= 2.0009.$$

$$\sqrt[3]{3} = 1.73.$$

$$\frac{1}{x} \log 3 = \log 1.73.$$

$$x = \frac{\log 3}{\log 1.73}$$

$$= \frac{0.47712}{0.23805}$$

$$= 2.0043.$$

36. Find the value of x when $\sqrt[3]{3} = 3$; when $\sqrt[3]{a} = b$; when $\sqrt[3]{a} = a$; when $\sqrt[3]{1331} = 11$; when $\sqrt[3]{20736} = 12$

$$\sqrt[3]{3} = 3.$$

$$\frac{1}{x} \log 3 = \log 3.$$

$$x = 1.$$

$$\sqrt[3]{a} = b.$$

$$\frac{1}{x} \log a = \log b.$$

$$x = \frac{\log a}{\log b}.$$

$$\sqrt[3]{a} = a.$$

$$\frac{1}{x} \log a = \log a$$

$$x = 1.$$

$$\sqrt[3]{1331} = 11.$$

$$\frac{1}{x} \log 1331 = \log 11.$$

$$\frac{3}{x} \log 11 = \log 11.$$

$$x = 3.$$

$$\sqrt[3]{20736} = 12.$$

$$\frac{1}{x} \log 20,736 = \log 12.$$

$$\frac{4}{x} \log 12 = \log 12.$$

$$x = 4.$$

37. Solve the equations

$$\sqrt[x]{y} = a \quad (1)$$

$$x+1\sqrt{y} = b \quad (2)$$

$$\frac{1}{x} \log y = \log a. \quad (3)$$

$$\frac{1}{x+1} \log y = \log b. \quad (4)$$

From (3),

$$\log y = x \log a.$$

From (4),

$$\log y = (x+1) \log b.$$

$$x \log a = (x+1) \log b.$$

$$x \log a = x \log b + \log b.$$

$$x(\log a - \log b) = \log b.$$

$$x = \frac{\log b}{\log a - \log b}.$$

38. What value of x satisfies the equation $\frac{1}{a^{x^2+2x+4}} = \sqrt[3]{a}$.

$$\frac{1}{a^{x^2+2x+4}} = \sqrt[3]{a}.$$

$$\frac{1}{x^2+2x+4} \log a = \frac{1}{3} \log a.$$

$$x^2+2x+4 = 3.$$

$$x^2+2x+1 = 0.$$

$$(x+1)^2 = 0.$$

$$x = -1.$$

Exercise 30. Use of the Tables (Page 62)

Find the value of each of the following:

1. $\log \sin 27^\circ = 9.65705 - 10.$

13. $\log \sin 4.5^\circ.$

2. $\log \sin 69^\circ = 9.97015 - 10.$

$4.5^\circ \quad 4^\circ 30'.$

3. $\log \cos 36^\circ = 9.90796 - 10.$

$\log \sin 4^\circ 30' = 8.89464 - 10.$

4. $\log \cos 48^\circ = 9.82551 - 10.$

14. $\log \cos 7.25^\circ.$

5. $\log \tan 75^\circ = 10.57195 - 10.$

$7.25^\circ = 7^\circ 15'.$

6. $\log \tan 12^\circ = 9.32747 - 10.$

$\log \cos 7^\circ 15' = 9.99651 - 10.$

7. $\log \cot 15^\circ = 10.57195 - 10.$

8. $\log \cot 78^\circ = 9.32747 - 10.$

15. $\log \tan 9.75^\circ.$

9. $\log \sin 9^\circ 15' = 9.20613 - 10.$

$9.75^\circ = 9^\circ 45'.$

10. $\log \cos 8^\circ 27' = 9.99526 - 10.$

11. $\log \tan 7^\circ 56' = 9.14412 - 10.$

$\log \tan 9^\circ 45' = 9.23510 - 10.$

12. $\log \cot 82^\circ 4' = 9.14412 - 10.$

16. $\log \cos 42^\circ 45''$.

$$\log \cos 42^\circ = 9.87107 - 10.$$

Tabular difference is 0.00011.

$$\frac{4}{6} \frac{5}{0} \text{ of } 0.00011 = 0.000082.$$

$$\log \cos 42^\circ 45'' = 9.87099 - 10.$$

17. $\log \tan 26^\circ 15''$.

$$\log \tan 26^\circ = 9.68818 - 10.$$

Tabular difference is 0.00032.

$$\frac{1}{6} \frac{5}{0} \text{ of } 0.00032 = 0.00008.$$

$$\log \tan 26^\circ 15'' = 9.68826 - 10.$$

18. $\log \cot 38^\circ 30''$.

$$\log \cot 38^\circ = 10.10719 - 10.$$

Tabular difference is 0.00026.

$$\frac{3}{6} \frac{0}{0} \text{ of } 0.00026 = 0.00013.$$

$$\log \cot 38^\circ 30'' = 10.10706 - 10.$$

19. $\log \sin 21^\circ 10' 4''$.

$$\log \sin 21^\circ 10' = 9.55761 - 10.$$

Tabular difference is 0.00032.

$$\frac{4}{6} \frac{0}{0} \text{ of } 0.00032 = 0.00002.$$

$$\log \sin 21^\circ 10' 4'' = 9.55763 - 10.$$

20. $\log \sin 68^\circ 49' 56''$.

$$\log \sin 68^\circ 49' = 9.96962 - 10.$$

Tabular difference = 0.00004.

$$\frac{5}{6} \frac{6}{0} \text{ of } 0.00004 = 0.000037.$$

$$\log \sin 68^\circ 49' 56'' = 9.96966 - 10.$$

21. $\log \cos 15^\circ 17' 3''$.

$$\log \cos 15^\circ 17' = 9.98436 - 10.$$

Tabular difference is 0.00003.

$$\frac{3}{6} \frac{0}{0} \text{ of } 0.00003 = 0.000002.$$

$$\log \cos 15^\circ 17' 3'' = 9.984358 - 10.$$

$$= 9.98436 - 10.$$

22. $\log \cos 74^\circ 42' 57''$.

$$\log \cos 74^\circ 42' = 9.42140 - 10.$$

Tabular difference is 0.00047.

$$\frac{5}{6} \frac{7}{0} \text{ of } 0.00047 = 0.00045.$$

$$\log \cos 74^\circ 42' 57'' = 9.42095 - 10.$$

23. $\log \tan 17^\circ 2' 10''$.

$$\log \tan 17^\circ 2' = 9.48624 - 10$$

Tabular difference is 0.00045.

$$\frac{1}{6} \frac{0}{0} \text{ of } 0.00045 = 0.00008.$$

$$\log \tan 17^\circ 2' 10'' = 9.48632 - 10.$$

24. $\log \tan 26^\circ 3' 4''$.

$$\log \tan 26^\circ 3' = 9.68914 - 10.$$

Tabular difference is 0.00032.

$$\frac{4}{6} \frac{0}{0} \text{ of } 0.00032 = 0.00002.$$

$$\log \tan 26^\circ 3' 4'' = 9.68916 - 10.$$

25. $\log \cot 48^\circ 4' 5''$.

$$\log \cot 48^\circ 4' = 9.95342 - 10.$$

Tabular difference is 0.00025.

$$\frac{5}{6} \frac{0}{0} \text{ of } 0.00025 = 0.00002.$$

$$\log \cot 48^\circ 4' 5'' = 9.95340 - 10.$$

26. $\log \cot 4^\circ 10' 7''$.

$$\log \cot 4^\circ 10' = 11.13757 - 10$$

Tabular difference is 0.00174.

$$\frac{7}{6} \frac{0}{0} \text{ of } 0.00174 \text{ is } 0.00020.$$

$$\log \cot 4^\circ 10' 7'' = 11.13737 - 10$$

27. $\log \sin 34^\circ 30''$.

$$\log \sin 34^\circ = 9.74756 - 10.$$

Tabular difference is 0.00019.

$$\frac{3}{6} \frac{0}{0} \text{ of } 0.00019 = 0.00010.$$

$$\log \sin 34^\circ 30'' = 9.74766 - 10.$$

28. $\log \sin 27.45^\circ$.

$$27.45^\circ = 27^\circ 27'.$$

$$\log \sin 27^\circ 27' = 9.66368 - 10.$$

29. $\log \tan 56.35^\circ$.

$$56.35^\circ = 56^\circ 21'.$$

$$\log \tan 56^\circ 21' = 10.17675 - 10$$

30. $\log \cos 48.26^\circ$.

$$48.26^\circ = 48^\circ 15' 36''.$$

$$\log \cos 48^\circ 15' = 9.82340 - 10.$$

Tabular difference is 0.00014.

$$\frac{3}{6} \frac{6}{0} \text{ of } 0.00014 = 0.00008.$$

$$\log \cos 48^\circ 15' 36'' = 9.82332 - 10.$$

31. $\log \sin 0^\circ 1' 7''.$

$$\log \sin 0^\circ 1' 7'' = 6.51165 - 10.$$

32. $\log \sin 1^\circ 2' 5''.$

$$\log \sin 1^\circ 2' = 8.25609.$$

Tabular difference is 0.00117.

$$\frac{5}{10} \text{ of } 0.00117 = 0.00058.$$

$$\log \sin 1^\circ 2' 5'' = 8.25667 - 10.$$

33. $\log \tan 0^\circ 2' 8''.$

$$\log \tan 0^\circ 2' 8'' = 6.79257 - 10.$$

34. $\log \tan 2^\circ 7' 7''.$

$$\log \tan 2^\circ 7' = 8.56773 - 10.$$

Tabular difference is 0.00341.

$$\frac{7}{10} \text{ of } 0.00341 = 0.00040.$$

$$\log \tan 2^\circ 7' 7'' = 8.56813 - 10.$$

35. $\log \cos 89^\circ 50' 10''.$

$$\log \cos 89^\circ 50' 10'' = 7.45643 - 10.$$

36. $\log \cos 89^\circ 10' 45''.$

$$\log \cos 89^\circ 10' 40'' = 8.15685 - 10.$$

Tabular difference is 0.00147.

$$\frac{5}{10} \text{ of } 0.00147 = 0.00074.$$

$$\log \cos 89^\circ 10' 45'' = 8.15611 - 10.$$

37. $\log \cot 89^\circ 15' 12''.$

$$\log \cot 89^\circ 15' 10'' = 8.11535 - 10.$$

Tabular difference is 0.00162.

$$\frac{2}{10} \text{ of } 0.00162 = 0.00032.$$

$$\log \cot 89^\circ 15' 12'' = 8.11503 - 10.$$

38. $\log \cot 89^\circ 25' 15''.$

$$\log \cot 89^\circ 25' 10'' = 8.00574 - 10.$$

Tabular difference is 0.00209.

$$\frac{5}{10} \text{ of } 0.00209 = 0.00105.$$

$$\log \cot 89^\circ 25' 15'' = 8.00469 - 10.$$

39. $\log \sin 1^\circ 1' 1''.$

$$\log \sin 1^\circ 1' = 8.24903 - 10.$$

Tabular difference is 0.00119.

$$\frac{1}{10} \text{ of } 0.00119 = 0.00012.$$

$$\log \sin 1^\circ 1' 1'' = 8.24915 - 10.$$

40. $\log \cos 88^\circ 58' 59''.$

$$\log \cos 88^\circ 58' 50'' = 8.25022$$

$$= 8.24903.$$

Tabular difference is 0.00119.

$$\frac{9}{10} \text{ of } 0.00119 = 0.00107.$$

$$\log \cos 88^\circ 58' 59'' = 8.24915 - 10.$$

41. $\log \tan 2^\circ 27' 25''.$

$$\log \tan 2^\circ 27' = 8.63131 - 10.$$

Tabular difference is 0.00295.

$$\frac{5}{10} \text{ of } 0.00295 = 0.00123.$$

$$\log \tan 2^\circ 27' 25'' = 8.63254 - 10.$$

42. $\log \cot 87^\circ 32' 45''.$

$$\log \cot 87^\circ 32' = 8.63426 - 10.$$

Tabular difference is 0.00295.

$$\frac{5}{10} \text{ of } 0.00295 = 0.00123.$$

$$\log \cot 87^\circ 32' 45'' = 8.63205 - 10.$$

43. $\log \sin 12^\circ 12' 12''.$

$$\log \sin 12^\circ 12' = 9.32495 - 10.$$

Tabular difference is 0.00058.

$$\frac{2}{10} \text{ of } 0.00058 = 0.00012.$$

$$\log \sin 12^\circ 12' 12'' = 9.32507 - 10.$$

44. $\log \cos 77^\circ 47' 48''.$

$$\log \cos 77^\circ 47' = 9.32553 - 10.$$

Tabular difference is 0.00058.

$$\frac{8}{10} \text{ of } 0.00058 = 0.00046.$$

$$\log \cos 77^\circ 47' 48'' = 9.32507 - 10.$$

45. $\log \tan 68^\circ 6' 43''.$

$$\log \tan 68^\circ 6' = 10.39578 - 10$$

Tabular difference is 0.00036.

$$\frac{3}{10} \text{ of } 0.00036 = 0.00026.$$

$$\log \tan 68^\circ 6' 43'' = 10.39604 - 10.$$

Find the value of x , given the following logarithms, each of which is too large:

46. $\log \sin x = 9.11570$.

$$\log \sin 7^\circ 30' = 9.11570 - 10.$$

$$x = 7^\circ 30'.$$

47. $\log \sin x = 9.72843$.

$$\log \sin 32^\circ 21' = 9.72843 - 10.$$

$$x = 32^\circ 21'.$$

48. $\log \sin x = 9.93053$.

$$\log \sin 58^\circ 27' = 9.93053 - 10.$$

$$x = 58^\circ 27'.$$

49. $\log \sin x = 9.99866$.

$$\log \sin 85^\circ 30' = 9.99866 - 10.$$

$$x = 85^\circ 30'$$

50. $\log \cos x = 9.99866$.

$$\log \cos 4^\circ 30' = 9.99866 - 10.$$

$$x = 4^\circ 30'.$$

51. $\log \cos x = 9.93053$.

$$\log \cos 31^\circ 33' = 9.93053 - 10.$$

$$x = 31^\circ 33'.$$

52. $\log \cos x = 9.71705$.

$$\log \cos 58^\circ 35' = 9.71705 - 10.$$

$$x = 58^\circ 35'.$$

53. $\log \cos x = 9.80320$.

$$\log \cos 50^\circ 32' = 9.80320 - 10.$$

$$x = 50^\circ 32'.$$

54. $\log \tan x = 9.90889$.

$$\log \tan 39^\circ 2' = 9.90889 - 10.$$

$$x = 39^\circ 2'.$$

55. $\log \tan x = 10.30587$.

$$\log \tan 63^\circ 41' = 10.30575 - 10.$$

Tabular difference is 32.

$\frac{1}{2}$ of $60'' = 22\frac{1}{2}''$, angular difference.

$$x = 63^\circ 41' 23''.$$

56. $\log \tan x = 10.64011$.

$$\log \tan 77^\circ 6' = 10.64011 - 10.$$

$$x = 77^\circ 6'.$$

57. $\log \cot x = 9.28865$.

$$\log \cot 79^\circ = 9.28865 - 10.$$

$$x = 79^\circ.$$

58. $\log \cot x = 9.56107$.

$$\log \cot 70^\circ = 9.56107 - 10.$$

$$x = 70^\circ.$$

59. $\log \sin x = 9.53871$.

$$\log \sin 20^\circ 13' = 9.53854 - 10.$$

Tabular difference is 34.

$\frac{1}{4}$ of $60'' = 30''$, angular difference.

$$x = 20^\circ 13' 30''.$$

60. $\log \sin x = 9.72868$.

$$\log \sin 32^\circ 22' = 9.72863 - 10.$$

Tabular difference is 20.

$\frac{5}{2}$ of $60'' = 15''$, angular difference.

$$x = 32^\circ 22' 15''.$$

61. $\log \sin x = 9.88150$.

$$\log \sin 49^\circ 34' = 9.88148 - 10.$$

Tabular difference is 10.

$\frac{2}{10}$ of $60'' = 12''$, angular difference.

$$x = 49^\circ 34' 12''.$$

62. $\log \sin x = 9.89530$.

$$\log \sin 51^\circ 47' = 9.89524 - 10.$$

Tabular difference is 10.

$\frac{6}{10}$ of $60'' = 36''$, angular difference.

$$x = 51^\circ 47' 36''.$$

63. $\log \cos x = 9.90151$.

$$\log \cos 37^\circ 8' = 9.90159 - 10.$$

Tabular difference is 10.

$\frac{8}{10}$ of $60'' = 48''$, angular difference.

$$x = 37^\circ 8' 48''.$$

64. $\log \cos x = 9.80070$.

$\log \cos 50^\circ 48' = 9.80074 - 10$.

Tabular difference is 16.

$\frac{4}{16}$ of $60'' = 15''$, angular difference.
 $x = 50^\circ 48' 15''$.

65. $\log \cos x = 9.99483$.

$\log \cos 8^\circ 49' = 9.99484 - 10$.

Tabular difference is 2.

$\frac{1}{2}$ of $60'' = 30''$, angular difference.
 $x = 8^\circ 49' 30''$.

66. $\log \tan x = 9.18854$.

$\log 8^\circ 46' = 9.18812 - 10$.

Tabular difference is 84.

$\frac{4\frac{3}{4}}{84}$ of $60'' = 30''$, angular difference.
 $x = 8^\circ 46' 30''$.

67. $\log \tan x = 10.18750$.

$\log \tan 57^\circ = 10.18748 - 10$.

Tabular difference is 28.

$\frac{2\frac{1}{8}}{28}$ of $60'' = 4\frac{3}{4}''$, angular difference.
 $x = 57^\circ 4\frac{3}{4}''$.

68. $\log \tan x = 10.06725$.

$\log \tan 49^\circ 25' = 10.06722 - 10$.

Tabular difference is 26.

$\frac{3}{26}$ of $60'' = 6\frac{1}{3}''$, angular difference.
 $x = 49^\circ 25' 7''$.

69. $\log \cot x = 10.10134$.

$\log \cot 38^\circ 22' = 10.10147 - 10$.

Tabular difference is 26.

$\frac{1\frac{3}{6}}{26}$ of $60'' = 30''$, angular difference.
 $x = 38^\circ 22' 30''$.

70. $\log \cot x = 11.44442$.

$\log \cot 2^\circ 3' = 11.44618 - 10$.

Tabular difference is 352.

$\frac{1\frac{7}{5}\frac{6}{2}}{352}$ of $60'' = 30''$, angular difference.
 $x = 2^\circ 3' 30''$.

71. $\log \cot x = 7.49849$.

$\log \cot 89^\circ 49' 10'' = 7.49849 - 10$.

$x = 89^\circ 49' 10''$.

Exercise 31. The Right Triangle (Page 67)

Using logarithms, solve the following right triangles, finding the sides and areas to four figures, and the angles to minutes:

1. $a = 6, c = 12$.

$\sin A = \frac{a}{c} = \frac{1}{2}$.

$\therefore A = 30^\circ$.

$B = (90^\circ - A) = 60^\circ$.

$\cos A = \frac{b}{c}$.

$b = c \cos A$.

$\log \cos A = 9.93753 - 10$

$\log 12 = 1.07918$

$\log b = 1.01671$

$b = 10.392$

$= 10.39$.

$\log S = \text{colog } 2 + \log a + \log b$

$\text{colog } 2 = 9.69897 - 10$

$\log a = 0.77815$

$\log b = 1.01671$

$\log S = 1.49383$

$S = 31.176$

$= 31.18$.

2. $b = 4$, $A = 60^\circ$.

$$B = 30^\circ.$$

$$\cos A = \frac{b}{c}.$$

$$c = \frac{b}{\cos A}.$$

$$\log b = 0.60206$$

$$\text{colog } \cos A = \frac{0.30103}{}$$

$$\log c = 0.90309$$

$$c = 8.$$

$$\frac{a}{b} = \tan A.$$

$$a = 4 \tan 60^\circ.$$

$$\log a = \log b + \log \tan 60^\circ.$$

$$\log b = 0.60206$$

$$\log \tan A = \frac{10.23855 - 10}{}$$

$$\log a = 0.84062$$

$$a = 6.9282$$

$$= 6.928.$$

$$\log S = \text{colog } 2 + \log a + \log b.$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 0.84062$$

$$\log b = 0.60206$$

$$\log S = 1.14165$$

$$S = 13.856$$

$$= 13.86.$$

3. $a = 3$, $A = 30^\circ$.

$$B = 60^\circ.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log a = 0.47712$$

$$\text{colog } \sin A = \frac{0.30103}{}$$

$$\log c = 0.77315$$

$$c = 6.$$

$$\frac{b}{a} = \cot A.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot 30^\circ.$$

$$\log a = 0.47712$$

$$\log \cot A = \frac{10.23856 - 10}{}$$

$$\log b = 0.71568$$

$$b = 5.1961$$

$$= 5.196.$$

$$\log S = \text{colog } 2 + \log a + \log b$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 0.47712$$

$$\log b = 0.71568$$

$$\log S = 0.89177$$

$$S = 7.7941 = 7.794.$$

4. $a = 4$, $b = 4$.

Since a and b each = 4, the $\triangle$ is isosceles and $\angle A = \angle B$.

$$A = \frac{1}{2} \text{ of } 90^\circ = 45^\circ,$$

and $B = \frac{1}{2} \text{ of } 90^\circ = 45^\circ.$

$$\frac{a}{c} = \sin A.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog } \sin 45^\circ$$

$$\log a = 0.60206$$

$$\text{colog } \sin A = \frac{0.15051}{}$$

$$\log c = 0.75257$$

$$c = 5.6568$$

$$= 5.657.$$

$$S = \frac{1}{2} ab$$

$$= \frac{1}{2} \times 4 \times 4$$

$$= 8.$$

5. $a = 2, c = 2.89.$

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \text{colog } c.$$

$$\log a = 0.30103$$

$$\text{colog } c = \frac{9.53910 - 10}{}$$

$$\log \sin A = 9.84013$$

$$A = 43^\circ 47'.$$

$$B = 46^\circ 13'.$$

$$\frac{b}{c} = \cos A.$$

∴

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 0.46090$$

$$\log \cos A = \frac{9.87839 - 10}{}$$

$$\log b = 0.31929$$

$$b = 2.0859$$

$$= 2.086.$$

$$\log S = \text{colog } 2 + \log a + \log b.$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 0.30103$$

$$\log b = 0.31935$$

$$\log S = 0.31935$$

$$S = 2.0861$$

$$= 2.086.$$

6. $c = 627, A = 23^\circ 30'.$

$$B = 66^\circ 30'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\log c = 2.79727$$

$$\log \sin A = \frac{9.60070 - 10}{}$$

$$\log a = 2.39797$$

$$a = 250.02$$

$$= 250.0.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 2.79727$$

$$\log \cos A = \frac{9.96240 - 10}{}$$

$$\log b = 2.75967$$

$$b = 575.0.$$

$$\log S = \text{colog } 2 + \log a + \log b$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 2.39797$$

$$\log b = 2.75967$$

$$\log S = 4.85661$$

$$S = 71,880.$$

7. $c = 2280, A = 28^\circ 5'.$

$$B = 61^\circ 55'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\log c = 3.35793$$

$$\log \sin A = \frac{9.67280 - 10}{}$$

$$\log a = 3.03073$$

$$a = 1073.3$$

$$= 1073.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 3.35793$$

$$\log \cos A = \frac{9.94560 - 10}{}$$

$$\log b = 3.30353$$

$$b = 2011.5$$

$$= 2012.$$

$$\log S = \text{colog } 2 + \log a + \log b$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 3.03073$$

$$\log b = 3.30353$$

$$\log S = 6.03323$$

$$S = 1,079,500.$$

$$8. c = 72.15, A = 39^\circ 34'.$$

$$B = 50^\circ 26'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\log c = 1.85824$$

$$\log \sin A = 9.80412 - 10$$

$$\log a = 1.66236$$

$$a = 45.958$$

$$= 45.96.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 1.85824$$

$$\log \cos A = 9.88699 - 10$$

$$\log b = 1.74523$$

$$b = 55.620$$

$$= 55.62$$

$$\log S = \text{colog } 2 + \log a + \log b.$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 1.66236$$

$$\log b = 1.74523$$

$$\log S = 3.10656$$

$$S = 1278.$$

$$9. c = 1, A = 36^\circ.$$

$$B = 54^\circ.$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\log c = 0.00000$$

$$\log \sin A = 9.76922 - 10$$

$$\log a = 9.76922 - 10$$

$$a = 0.58779$$

$$= 0.5878.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 0.00000$$

$$\log \cos A = 9.90796 - 10$$

$$\log b = 9.90796 - 10$$

$$b = 0.80902$$

$$= 0.8090.$$

$$\log S = \text{colog } 2 + \log a + \log b$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 9.76922 - 10$$

$$\log b = 9.90796 - 10$$

$$\log S = 29.37615 - 30$$

$$= 1.37615.$$

$$S = 0.23776$$

$$= 0.2378.$$

$$10. c = 200, B = 21^\circ 47'.$$

$$A = 68^\circ 13'.$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\log c = 2.30103$$

$$\log \sin A = 9.96783 - 10$$

$$\log a = 2.26886$$

$$a = 185.72$$

$$= 185.7.$$

$$\cos a = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 2.30103$$

$$\log \cos A = 9.56949 - 10$$

$$\log b = 1.87052$$

$$b = 74.220$$

$$= 74.22.$$

$$\log S = \text{colog } 2 + \log a + \log b$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 2.26886$$

$$\log b = 1.87052$$

$$\log S = 3.83835$$

$$S = 6892.$$

$$11. c = 93.4, B = 76^\circ 25'.$$

$$A = 13^\circ 35'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\log c = 1.97035$$

$$\log \sin A = \frac{9.37081 - 10}{}$$

$$\log a = 1.34116$$

$$a = 21.936$$

$$= 21.94.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\log a = 1.34116$$

$$\log \cot A = \frac{10.61687 - 10}{}$$

$$\log b = 1.95803$$

$$b = 90.788$$

$$= 90.79.$$

$$\log S = \text{colog } 2 + \log a + \log b.$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 1.34116$$

$$\log b = \frac{1.95803}{}$$

$$\log S = 2.99816$$

$$S = 995.77$$

$$= 995.8.$$

$$12. a = 637, A = 4^\circ 35'.$$

$$B = 85^\circ 25'.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\log a = 2.80414$$

$$\log \cot A = \frac{1.09601}{}$$

$$\log b = 3.90015$$

$$b = 7946.$$

$$\log c = \log a + \text{colog } \sin A.$$

$$\log a = 2.80414$$

$$\text{colog } \sin A = \frac{1.09740}{}$$

$$\log c = 3.90154$$

$$c = 7971.5$$

$$= 7972.$$

$$\log S = \text{colog } 2 + \log a + \log b.$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 2.80414$$

$$\log b = \frac{3.90015}{}$$

$$\log S = 6.40326$$

$$S = 2,531,000.$$

$$13. a = 48.53, A = 36^\circ 44'.$$

$$B = 53^\circ 16'.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog } \sin A.$$

$$\log a = 1.68601$$

$$\text{colog } \sin A = \frac{0.22323}{}$$

$$\log c = 1.90924$$

$$c = 81.14.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 1.90924$$

$$\log \cos A = \frac{9.90386 - 10}{}$$

$$\log b = 1.81310$$

$$b = 65.028$$

$$= 65.03.$$

$$\log S = \text{colog } 2 + \log a + \log b$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 1.68601$$

$$\log b = \frac{1.81310}{}$$

$$\log S = 3.19808$$

$$S = 1577.8$$

$$= 1578.$$

$$14. a = 0.008, A = 86^\circ.$$

$$B = 4^\circ.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\log a = 7.90309 - 10$$

$$\operatorname{colog} \sin A = 0.00106$$

$$\log c = 7.90415 - 10$$

$$= 7.90415.$$

$$c = 0.0080196$$

$$= 0.008020.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 7.90415 - 10$$

$$\log \cos A = 8.84358 - 10$$

$$\log b = 16.74773 - 20$$

$$= 4.74773.$$

$$b = 0.00055941$$

$$= 0.0005594.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b.$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 7.90309 - 10$$

$$\log b = 0.74773 - 4$$

$$\log S = 18.34979 - 24$$

$$= 6.34979.$$

$$S = 0.0000022376$$

$$= 0.000002238.$$

$$15. b = 50.94, B = 43^\circ 48'.$$

$$A = 46^\circ 12'.$$

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

$$\log b = 1.70706$$

$$\log \tan A = 0.01820$$

$$\log a = 1.72526$$

$$a = 53.12.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log a = 1.72526$$

$$\operatorname{colog} \sin A = 0.14161$$

$$\log c = 1.86687$$

$$c = 73.598$$

$$= 73.60.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 1.72526$$

$$\log b = 1.70706$$

$$\log S = 3.13129$$

$$S = 1352.9$$

$$= 1353.$$

$$16. b = 2, B = 3^\circ 38'.$$

$$A = 86^\circ 22'.$$

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

$$\log b = 0.30103$$

$$\log \tan A = 1.19723$$

$$\log a = 1.49826$$

$$a = 31.496$$

$$= 31.50.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log a = 1.49826$$

$$\operatorname{colog} \sin A = 0.00087$$

$$\log c = 1.49913$$

$$c = 31.559$$

$$= 31.56.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 1.49826$$

$$\log b = 0.30103$$

$$\log S = 1.49826$$

$$S = 31.496$$

$$= 31.50.$$

$$17. a = 992, B = 76^\circ 19'.$$

$$A = 13^\circ 41'.$$

$$\sin A = \frac{a}{c}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\log a = 2.99651$$

$$\operatorname{colog} \sin A = 0.62607$$

$$\log c = 3.62258$$

$$c = 4193.5$$

$$= 4194.$$

$$\sin B = \frac{b}{c}.$$

$$b = c \sin B.$$

$$\log b = \log c + \log \sin B.$$

$$\log c = 3.62258$$

$$\log \sin B = 9.98750$$

$$\log b = 3.61008$$

$$b = 4074.5$$

$$= 4075.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b.$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 2.99651$$

$$\log b = 3.61008$$

$$\log S = 6.30556$$

$$S = 2,021,000.$$

$$18. a = 73, B = 68^\circ 52'.$$

$$A = 21^\circ 8'.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\log a = 1.86332$$

$$\operatorname{colog} \sin A = 0.44305$$

$$\log c = 2.30637$$

$$c = 202.47$$

$$= 202.5.$$

$$\sin B = \frac{b}{c}.$$

$$b = c \sin B.$$

$$\log b = \log c + \log \sin B.$$

$$\log c = 2.30637$$

$$\log \sin B = 9.96976 - 10$$

$$\log b = 2.27613$$

$$b = 188.86$$

$$= 188.9.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b.$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 1.86332$$

$$\log b = 2.27613$$

$$\log S = 3.83842$$

$$S = 6893.1$$

$$= 6893.$$

$$19. a = 2.189, B = 45^\circ 25'.$$

$$A = 44^\circ 35'.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\log a = 0.34025$$

$$\operatorname{colog} \sin A = 0.15370$$

$$\log c = 0.49395$$

$$c = 3.1185$$

$$= 3.119.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 0.49395$$

$$\log \cos A = 9.85262 - 10$$

$$\log b = 0.34657$$

$$b = 2.2211$$

$$= 2.221.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 0.34025$$

$$\log b = 0.34657$$

$$\log S = 0.38579$$

$$S = 2.4310$$

$$= 2.431.$$

$$20. \ b = 4, \ A = 37^\circ 56'.$$

$$B = 52^\circ 4'.$$

$$\cos A = \frac{b}{c}.$$

$$c = \frac{b}{\cos A}.$$

$$\log c = \log b + \operatorname{colog} \cos A.$$

$$\log b = 0.60206$$

$$\operatorname{colog} \cos A = \underline{0.10307}$$

$$\log c = 0.70513$$

$$c = 5.0714$$

$$= 5.071.$$

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

$$\log a = \log b + \log \tan A.$$

$$\log b = 0.60206$$

$$\log \tan A = \underline{9.89177 - 10}$$

$$\log a = 0.49383$$

$$a = 3.1176$$

$$= 3.118.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b.$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 0.49383$$

$$\log b = \underline{0.60206}$$

$$\log S = 0.79486$$

$$S = 6.2352$$

$$= 6.235.$$

$$21. \ c = 8590, \ a = 4476.$$

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \operatorname{colog} c.$$

$$\log a = 3.65089$$

$$\operatorname{colog} c = \underline{6.06601 - 10}$$

$$\log \sin A = 9.71690 - 10$$

$$A = 31^\circ 24'.$$

$$B = 58^\circ 36'.$$

$$\cot A = \frac{b}{a}.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\log a = 3.65089$$

$$\log \cot A = \underline{0.21438}$$

$$\log b = 3.86527$$

$$b = 7332.8$$

$$= 7333.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 3.65089$$

$$\log b = \underline{3.86527}$$

$$\log S = 7.21513$$

$$S = 16,410,000.$$

$$22. \ c = 86.53, \ a = 71.78.$$

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \operatorname{colog} c.$$

$$\log a = 1.85600$$

$$\operatorname{colog} c = \underline{8.06283 - 10}$$

$$\log \sin A = 9.91883 - 10$$

$$A = 56^\circ 3'.$$

$$B = 33^\circ 57'.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\log a = 1.85600$$

$$\log \cot A = \underline{9.82817 - 10}$$

$$\log b = 1.68417$$

$$b = 48.324$$

$$= 48.32.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 1.85600$$

$$\log b = \underline{1.68417}$$

$$\log S = 3.23914$$

$$S = 1734.4$$

$$= 1734.$$

23. $c = 9.35$, $a = 8.49$.

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \operatorname{colog} c.$$

$$\log a = 0.92891$$

$$\operatorname{colog} c = \frac{9.02919 - 10}{}$$

$$\log \sin A = 9.95810$$

$$A = 65^\circ 14'.$$

$$B = 24^\circ 46'.$$

$$\cos A = \frac{b}{c}$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 0.97081$$

$$\log \cos A = \frac{9.62214 - 10}{}$$

$$\log b = 0.59295$$

$$b = 3.917.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b.$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 0.92891$$

$$\log b = \frac{0.59295}{}$$

$$\log S = 1.22083$$

$$S = 16.627$$

$$= 16.63.$$

24. $c = 2194$, $b = 1312.7$.

$$\cos A = \frac{b}{c}.$$

$$\log \cos A = \log b + \operatorname{colog} c.$$

$$\log b = 3.11816$$

$$\operatorname{colog} c = \frac{6.65876 - 10}{}$$

$$\log \cos A = 9.77692$$

$$A = 53^\circ 15'.$$

$$B = 36^\circ 45'.$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\log c = 3.34124$$

$$\log \sin A = \frac{9.90377 - 10}{}$$

$$\log a = 3.24501$$

$$a = 1758.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 3.24501$$

$$\log b = \frac{3.11816}{}$$

$$\log S = 6.06214$$

$$S = 1,154,000.$$

25. $c = 30.69$, $b = 18.25$.

$$\cos A = \frac{b}{c}.$$

$$\log \cos A = \log b + \operatorname{colog} c.$$

$$\log b = 1.26126$$

$$\operatorname{colog} c = \frac{8.51300 - 10}{}$$

$$\log \cos A = 9.77426$$

$$A = 53^\circ 31'.$$

$$B = 36^\circ 29'.$$

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

$$\log a = \log \tan A + \log b.$$

$$\log \tan A = 10.13106 - 10$$

$$\log b = \frac{1.26126}{}$$

$$\log a = 1.39232$$

$$a = 24.678$$

$$= 24.68.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 1.39232$$

$$\log b = \frac{1.26126}{}$$

$$\log S = 2.35255$$

$$S = 225.18$$

$$= 225.2.$$

$$26. a = 38.31, b = 19.52.$$

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \operatorname{colog} b.$$

$$\log a = 1.58331$$

$$\operatorname{colog} b = \frac{8.70952 - 10}{}$$

$$\log \tan A = 10.29283$$

$$A = 63^\circ.$$

$$B = 27^\circ.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\log a = 1.58331$$

$$\operatorname{colog} \sin A = \frac{0.05012}{}$$

$$\log c = 1.63343$$

$$c = 42.998$$

$$= 43.00.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b.$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 1.58331$$

$$\log b = \frac{1.29048}{}$$

$$\log S = 2.57276$$

$$S = 373.9.$$

$$27. a = 1.229, b = 14.95.$$

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \operatorname{colog} b$$

$$\log a = 0.08955$$

$$\operatorname{colog} b = \frac{8.82536 - 10}{}$$

$$\log \tan A = 8.91491$$

$$A = 4^\circ 42'.$$

$$B = 85^\circ 18'.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\log a = 0.08955$$

$$\operatorname{colog} \sin A = \frac{1.08651}{}$$

$$\log c = 1.17606$$

$$c = 14.998$$

$$= 15.00.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 0.08955$$

$$\log b = \frac{1.17464}{}$$

$$\log S = 0.96316$$

$$S = 9.1867$$

$$= 9.187.$$

$$28. a = 415.3, b = 62.08.$$

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \operatorname{colog} b.$$

$$\log a = 2.61836$$

$$\operatorname{colog} b = \frac{8.20705 - 10}{}$$

$$\log \tan A = 10.82541$$

$$A = 81^\circ 30'.$$

$$B = 8^\circ 30'.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\log a = 2.61836$$

$$\operatorname{colog} \sin A = \frac{0.00480}{}$$

$$\log c = 2.62316$$

$$c = 419.91$$

$$= 419.9.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 2.61836$$

$$\log b = \frac{1.79295}{}$$

$$\log S = 4.11028$$

$$S = 12,890.$$

29. $a = 13.69$, $b = 16.92$.

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \operatorname{colog} b.$$

$$\log a = 1.13640$$

$$\operatorname{colog} b = \frac{8.77160 - 10}{}$$

$$\log \tan A = 9.90800$$

$$A = 38^\circ 59'.$$

$$B = 51^\circ 1'.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\log a = 1.13640$$

$$\operatorname{colog} \sin A = \frac{0.20128}{}$$

$$\log c = 1.33768$$

$$c = 21.761$$

$$= 21.76.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b.$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 1.13640$$

$$\log b = \frac{1.22840}{}$$

$$\log S = 2.06377$$

$$S = 115.81$$

$$= 115.8.$$

30. $c = 91.92$, $a = 2.19$.

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \operatorname{colog} c.$$

$$\log a = 0.34044$$

$$\operatorname{colog} c = \frac{8.03659 - 10}{}$$

$$\log \sin A = 8.37703$$

$$A = 1^\circ 22'.$$

$$B = 88^\circ 38'.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 1.96341$$

$$\log \cos A = \frac{9.99988 - 10}{}$$

$$\log b = 1.96329$$

$$b = 91.894$$

$$= 91.89.$$

$$\log S = \operatorname{colog} 2 + \log a + \log b.$$

$$\operatorname{colog} 2 = 9.69897 - 10$$

$$\log a = 0.34044$$

$$\log b = \frac{1.96329}{}$$

$$\log S = 2.00270$$

$$S = 100.62$$

$$= 100.6.$$

Compute the unknown parts and also the area, having given:

31. $a = 5$, $b = 6$.

$$\tan A = \frac{a}{b}.$$

$$\log a = 0.69897$$

$$\operatorname{colog} b = \frac{9.22185 - 10}{}$$

$$\log \tan A = 9.92082$$

$$A = 39^\circ 48'.$$

$$B = 50^\circ 12'.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log a = 0.69897$$

$$\operatorname{colog} \sin A = \frac{0.19375}{}$$

$$\log c = 0.89272$$

$$c = 7.8112.$$

$$S = \frac{ab}{2} = \frac{30}{2} = 15.$$

32. $a = 0.615$, $c = 70$.

$$\sin A = \frac{a}{c}.$$

$$\log a = 9.78888 - 10$$

$$\text{colog } c = \frac{8.15490 - 10}{}$$

$$\log \sin A = 7.94378 - 10$$

$$A = 30' 12''.$$

$$B = 89^\circ 29' 48''.$$

$$b = c \cos A.$$

$$\log c = 9.99998 - 10$$

$$\log \cos A = \frac{1.84510}{}$$

$$\log b = 1.84508$$

$$b = 69.997$$

$$= 70.$$

$$S = \frac{1}{2} ab.$$

$$\log a = 9.78888 - 10$$

$$\log b = 1.84508$$

$$\text{colog } 2 = \frac{9.69897 - 10}{}$$

$$\log S = 1.33293$$

$$S = 21.525.$$

33. $b = \sqrt[3]{2}$, $c = \sqrt{3}$.

$$\cos A = \frac{b}{c}.$$

$$\log b = 0.10034$$

$$\text{colog } c = \frac{9.76144 - 10}{}$$

$$\log \cos A = 9.86178$$

$$A = 43^\circ 20'.$$

$$B = 46^\circ 40'.$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log c = 0.23856$$

$$\log \sin A = \frac{9.83648 - 10}{}$$

$$\log a = 0.07504$$

$$a = 1.1886.$$

$$S = \frac{1}{2} ab.$$

$$\log a = 0.07504$$

$$\log b = 0.10034$$

$$\text{colog } 2 = \frac{9.69897 - 10}{}$$

$$\log S = 9.87435 - 10$$

$$S = 0.74876$$

$$= 0.7488.$$

34. $a = 7$, $A = 18^\circ 14'$.

$$B = 71^\circ 46'.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log a = 0.84510$$

$$\text{colog } \sin A = \frac{0.50461}{}$$

$$\log c = 1.34971$$

$$c = 22.372.$$

$$\tan A = \frac{a}{b}.$$

$$b = \frac{a}{\tan A}.$$

$$\log a = 0.84510$$

$$\text{colog } \tan A = \frac{0.48224}{}$$

$$\log b = 1.32734$$

$$b = 21.249$$

$$= 21.25.$$

$$S = \frac{1}{2} ab.$$

$$\log a = 0.84510$$

$$\log b = 1.32734$$

$$\text{colog } 2 = \frac{9.69897 - 10}{}$$

$$\log S = 1.87141$$

$$S = 74.372.$$

35. $b = 12$, $A = 29^\circ 8'$.

$$A = 29^\circ 8'.$$

$$B = 60^\circ 52'.$$

$$\cos A = \frac{b}{c}.$$

$$c = \frac{b}{\cos A}.$$

$$\log b = 1.07918$$

$$\operatorname{colog} \cos A = \underline{0.05874}$$

$$\log c = \underline{1.13792}$$

$$c = 13.738.$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log c = 1.13792$$

$$\log \sin A = \underline{9.68739 - 10}$$

$$\log a = \underline{0.82531}$$

$$a = 6.6882.$$

$$S = \frac{1}{2} ab.$$

$$\log a = 0.82531$$

$$\log b = 1.07918$$

$$\operatorname{colog} 2 = \underline{9.69897 - 10}$$

$$\log S = \underline{1.60346}$$

$$S = 40.129$$

$$= 40.13.$$

36. $c = 68$, $A = 69^\circ 54'$.

$$A = 69^\circ 54'.$$

$$B = 20^\circ 6'.$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log c = 1.83251$$

$$\log \sin A = \underline{9.97271 - 10}$$

$$\log a = \underline{1.80522}$$

$$a = 63.859$$

$$= 63.86.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log c = 1.83251$$

$$\log \cos A = \underline{9.53613 - 10}$$

$$\log b = \underline{1.36864}$$

$$b = 23.369$$

$$= 23.37.$$

$$S = \frac{1}{2} ab.$$

$$\log a = 1.80522$$

$$\log b = 1.36864$$

$$\operatorname{colog} 2 = \underline{9.69897 - 10}$$

$$\log S = \underline{2.87283}$$

$$S = 746.15.$$

37. $c = 27$, $B = 44^\circ 4'$.

$$A = 45^\circ 56'.$$

$$a = c \sin A.$$

$$\log c = 1.43136$$

$$\log \sin A = \underline{9.85645 - 10}$$

$$\log a = \underline{1.28781}$$

$$a = 19.40.$$

$$b = c \cos A.$$

$$\log c = 1.43136$$

$$\log \cos A = \underline{9.84229 - 10}$$

$$\log b = \underline{1.27365}$$

$$b = 18.778$$

$$= 18.78.$$

$$S = \frac{1}{2} ab.$$

$$\log a = 1.28781$$

$$\log b = 1.27365$$

$$\operatorname{colog} 2 = \underline{9.69897 - 10}$$

$$\log S = \underline{2.26043}$$

$$S = 182.15.$$

$$38. a = 47, B = 48^\circ 49'.$$

$$A = 41^\circ 11'.$$

$$b = a \cot A.$$

$$\log a = 1.67210$$

$$\log \cot A = \frac{10.05803 - 10}{}$$

$$\log b = 1.73013$$

$$b = 53.719$$

$$= 53.72.$$

$$c = \frac{a}{\sin A}.$$

$$\log a = 1.67210$$

$$\text{colog} \sin A = 0.18146$$

$$\log c = 1.85356$$

$$c = 71.377$$

$$= 71.38.$$

$$S = \frac{1}{2} ab.$$

$$\log a = 1.67210$$

$$\log b = 1.73013$$

$$\text{colog} 2 = \frac{9.69897 - 10}{}$$

$$\log S = 3.10120$$

$$S = 1262.4.$$

$$39. b = 9, B = 34^\circ 44'.$$

$$A = 55^\circ 16'.$$

$$a = b \tan A.$$

$$\log b = 0.95424$$

$$\log \tan A = \frac{10.15908 - 10}{}$$

$$\log a = 1.11332$$

$$a = 12.981$$

$$= 12.98.$$

$$c = \frac{a}{\sin A}.$$

$$\log a = 1.11332$$

$$\text{colog} \sin A = 0.08523$$

$$\log c = 1.19855$$

$$c = 15.796$$

$$= 15.80.$$

$$S = \frac{1}{2} ab.$$

$$\log a = 1.11332$$

$$\log b = 0.95424$$

$$\text{colog} 2 = \frac{9.69897 - 10}{}$$

$$\log S = 1.76653$$

$$S = 58.416$$

$$= 58.42.$$

$$40. c = 8.462, B = 86^\circ 4'.$$

$$A = 3^\circ 56'.$$

$$a = c \sin A.$$

$$\log c = 0.92747$$

$$\log \sin A = \frac{8.83630 - 10}{}$$

$$\log a = 9.76377 - 10$$

$$a = 0.5805.$$

$$b = c \cos A.$$

$$\log c = 0.92747$$

$$\log \cos A = \frac{9.99898 - 10}{}$$

$$\log b = 0.92645$$

$$b = 8.442.$$

$$S = \frac{1}{2} ab.$$

$$\log a = 9.76377 - 10$$

$$\log b = 0.92645$$

$$\text{colog} 2 = \frac{9.69897 - 10}{}$$

$$\log S = 0.38919$$

$$S = 2.4501.$$

41. Find the value of S in terms of c and A .

$$S = \frac{1}{2} ab.$$

$$\sin A = \frac{a}{c}.$$

$$\therefore a = c \sin A.$$

$$\cos A = \frac{b}{c}.$$

$$\therefore b = c \cos A.$$

$$S = \frac{1}{2} ab$$

$$= \frac{1}{2} c^2 \sin A \cos A.$$

42. Find the value of S in terms of a and A .

$$\begin{aligned} S &= \frac{1}{2} ab. \\ \cot A &= \frac{b}{a}. \\ \therefore b &= a \cot A. \\ S &= \frac{1}{2} ab \\ &= \frac{1}{2} a^2 \cot A. \end{aligned}$$

43. Find the value of S in terms of b and A .

$$\begin{aligned} S &= \frac{1}{2} ab. \\ \tan A &= \frac{a}{b}. \\ \therefore a &= b \tan A. \\ S &= \frac{1}{2} ab \\ &= \frac{1}{2} b^2 \tan A. \end{aligned}$$

44. Find the value of S in terms of a and c .

$$\begin{aligned} S &= \frac{1}{2} ab. \\ c^2 &= a^2 + b^2. \\ b^2 &= c^2 - a^2. \\ \therefore b &= \sqrt{c^2 - a^2}. \\ S &= \frac{1}{2} a \sqrt{c^2 - a^2}. \end{aligned}$$

45. Given $S = 58$ and $a = 10$, solve the right triangle.

$$\begin{aligned} S &= \frac{1}{2} ab. \\ b &= \frac{2S}{a}. \\ \log 2S &= 2.06446 \\ \text{colog } a &= \frac{9.00000 - 10}{\log b = 1.06446} \\ b &= 11.6. \\ \tan A &= \frac{a}{b}. \end{aligned}$$

$$\begin{aligned} \log a &= 1.00000 \\ \text{colog } b &= \frac{8.93554 - 10}{\log \tan A = 9.93554 - 10} \end{aligned}$$

$$A = 40^\circ 45' 48''.$$

$$B = 49^\circ 14' 12''.$$

$$c = \frac{a}{\sin A}.$$

$$\begin{aligned} \log a &= 1.00000 \\ \text{colog } \sin A &= 0.18513 \\ \log c &= 1.18513 \end{aligned}$$

$$c = 15.315.$$

46. Given $S = 18$ and $b = 5$, solve the right triangle.

$$S = \frac{1}{2} ab.$$

$$a = \frac{2S}{b}.$$

$$\begin{aligned} \log 2S &= 1.55630 \\ \text{colog } b &= \frac{9.30103 - 10}{\log a = 0.85733} \end{aligned}$$

$$a = 7.2.$$

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned} \log a &= 0.85733 \\ \text{colog } b &= \frac{9.30103 - 10}{\log \tan A = 10.15836} \end{aligned}$$

$$A = 55^\circ 13' 20''.$$

$$B = 34^\circ 46' 40''.$$

$$c = \frac{a}{\sin A}.$$

$$\begin{aligned} \log a &= 0.85733 \\ \text{colog } \sin A &= 0.08546 \\ \log c &= 0.94279 \end{aligned}$$

$$c = 8.7658$$

$$= 8.766.$$

47. Given $S = 12$ and $A = 29^\circ$,
solve the right triangle.

$$B = 61^\circ.$$

$$S = \frac{1}{2} ab = 12.$$

$$ab = 24.$$

$$a = \frac{24}{b}.$$

$$\tan A = \frac{a}{b}.$$

$$\tan 29^\circ = \frac{24}{b^2}.$$

$$b^2 = \frac{24}{\tan 29^\circ}.$$

$$\log b = \frac{1}{2} (\log 24 + \operatorname{colog} \tan 29^\circ).$$

$$\log 24 = 1.38021$$

$$\operatorname{colog} \tan 29^\circ = 0.25625$$

$$\begin{array}{r} 2 \overline{) 1.63646} \end{array}$$

$$\log b = 0.81823$$

$$b = 6.58.$$

$$\tan 29^\circ = \frac{a}{b}.$$

$$a = b \tan 29^\circ.$$

$$\log b = 0.81823$$

$$\log \tan 29^\circ = \frac{9.74375 - 10}{10}$$

$$\log a = 0.56198$$

$$a = 3.6474.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin 29^\circ}.$$

$$\log a = 0.56198$$

$$\operatorname{colog} \sin 29^\circ = \frac{0.31443}{10}$$

$$\log c = 0.87641$$

$$c = 7.5233.$$

48. Given $S = 98$ and $c = 22$,
solve the right triangle.

$$S = \frac{1}{2} ab = 98.$$

$$ab = 196.$$

$$a = \frac{196}{b}.$$

$$a^2 = \frac{38416}{b^2}.$$

$$a^2 + b^2 = c^2 = 484.$$

Substitute,

$$\frac{38416}{b^2} + b^2 = 484$$

$$38416 + b^4 = 484 b^2.$$

$$b^4 - 484 b^2 + 38416 = 0.$$

$$b^4 - () + (242)^2 = 20148.$$

$$\log \sqrt{20148} = \frac{1}{2} (4.30424)$$

$$= 2.15212.$$

But $2.15212 = \log 141.95.$

$$b^2 - 242 = 141.95.$$

$$b^2 = 383.95.$$

$$\log b = \frac{1}{2} \log 383.95$$

$$= 1.29214.$$

$$b = 19.595.$$

$$\cos A = \frac{b}{c}.$$

$$\log b = 1.29214$$

$$\operatorname{colog} c = \frac{8.65758 - 10}{10}$$

$$\log \cos A = 9.94972$$

$$A = 27^\circ 2' 30''.$$

$$B = 62^\circ 57' 30''.$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log c = 1.34242$$

$$\log \sin A = \frac{9.65767 - 10}{10}$$

$$\log a = 1.00009$$

$$a = 10.002.$$

49. Find the two acute angles of a right triangle if the hypotenuse is equal to three times one of the sides.

Let c = hypotenuse,
and c = three times a , one of the sides.

$$\sin A = \frac{a}{c} = \frac{1}{3}$$

$$\log \sin A = \log a + \text{colog } c.$$

$$\log a = 0.00000$$

$$\text{colog } c = \underline{9.52288 - 10}$$

$$\log \sin A = \underline{9.52288}$$

$$A = 19^\circ 28' 17''.$$

$$B = 70^\circ 31' 43''.$$

50. The latitude of Washington is $38^\circ 55' 15''$ N. Taking the radius of the earth as 4000 mi., what is the radius of the circle of latitude of Washington? What is the circumference of this circle?

$$\cos 38^\circ 55' 15'' = \frac{x}{4000}.$$

$$x = 4000 \cos 38^\circ 55' 15''.$$

$$\log x = \log 4000 + \log \cos 38^\circ 55' 15''.$$

$$\log 4000 = 3.60206$$

$$\log \cos 38^\circ 55' 15'' = \underline{9.89098 - 10}$$

$$\log x = \underline{3.49304}$$

$$x = 3112.$$

$$\text{Circumference} = 2\pi r = 2 \times 3.1416 \times 3112.$$

$$\log 2 = 0.30103$$

$$\log 3.1416 = 0.49715$$

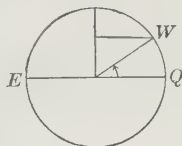
$$\log 3112 = \underline{3.49304}$$

$$\log 2\pi r = \underline{4.29122}$$

$$2\pi r = 19,553.1$$

$$= 19,553.$$

The radius is 3112 mi. and the circumference 19,553 mi.



51. What is the difference between the length of a degree of latitude and the length of a degree of longitude at Washington?

$$19,553 \text{ mi.} \div 360 = \text{length of a degree of longitude.}$$

$$\log 19,553 = 4.29122$$

$$\text{colog } 360 = \underline{7.44370 - 10}$$

$$= \underline{1.73492}$$

$$= \log 54.315.$$

$\therefore$ a degree of longitude = 54.315 mi.

$2\pi 4000 \text{ mi.} \div 360 = \text{length of a degree of latitude.}$

$$\log 2 = 0.30103$$

$$\log 3.1416 = 0.49715$$

$$\log 4000 = 3.60206$$

$$\text{colog } 360 = \frac{7.44370 - 10}{1.84394}$$

$$1.84394$$

$$= 1.84394 = \log 69.813.$$

$\therefore$ a degree of latitude = 69.813 mi.

The required difference is 69.813 mi. — 54.315 mi. = 15.498 mi.

52. From the top of a mountain 1 mi. high, overlooking the sea, an observer looks toward the horizon. What is the angle of depression of the line of sight?

$$\cos x = \frac{4000}{4001}.$$

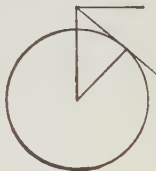
$$\log \cos x = \log 4000 + \text{colog } 4001.$$

$$\log 4000 = 3.60206$$

$$\text{colog } 40,001 = \frac{6.39783 - 10}{9.99989 - 10}$$

$$\log \cos x = 9.99989 - 10$$

The angle of depression lies between $1^\circ 15' 30''$ and $1^\circ 19' 10''$.



53. At a horizontal distance of 120 ft. from the foot of a steeple, the angle of elevation of the top is found to be $60^\circ 30'$. Find the height of the steeple above the instrument.

$$\tan 60^\circ 30' = \frac{x}{120}.$$

$$x = 120 \tan 60^\circ 30'.$$

$$\log x = \log 120 + \log \tan 60^\circ 30'.$$

$$\log 120 = 2.07918$$

$$\log \tan 60^\circ 30' = \frac{10.24736 - 10}{2.32654}$$

$$\log x = 2.32654$$

$$x = 212.1.$$

The height of the steeple is 212.1 ft.

54. From the top of a rock which rises vertically 326 ft. out of the water, the angle of depression of a boat is found to be 24° . Find the distance of the boat from the base of the rock.

$$\cot 24^\circ = \frac{x}{326}.$$

$$x = 326 \cot 24^\circ.$$

$$\begin{aligned}\log 326 &= 2.51322 \\ \log \cot 24^\circ &= \underline{0.35142} \\ \log x &= 2.86464 \\ x &= 732.2.\end{aligned}$$

The distance of the boat from the base of the rock is 732.2 ft.

55. How far from the eye is a monument on a level plain if the height of the monument is 200 ft. and the angle of elevation of the top is $3^\circ 30'$?

$$\begin{aligned}\cot 3^\circ 30' &= \frac{x}{200}, \\ x &= 200 \cot 3^\circ 30'. \\ \log 200 &= 2.30103 \\ \log \cot 3^\circ 30' &= \underline{1.21351} \\ \log x &= 3.51454 \\ x &= 3270.\end{aligned}$$

The monument is 3270 ft. from the eye.

56. A distance AB of 96 ft. is measured along the bank of a river from a point A opposite a tree C on the other bank. The angle ABC is $21^\circ 14'$. Find the breadth of the river.

$$\begin{aligned}\tan 21^\circ 14' &= \frac{x}{96}, \\ x &= 96 \tan 21^\circ 14'. \\ \log x &= \log 96 + \log \tan 21^\circ 14'. \\ \log 96 &= 1.98227 \\ \log \tan 21^\circ 14' &= \underline{9.58944 - 10} \\ \log x &= 1.57171 \\ x &= 37.3.\end{aligned}$$

The breadth of the river is 37.3 ft.

57. What is the angle of elevation of an inclined plane if it rises 1 ft in a horizontal distance of 40 ft.?

$$\begin{aligned}\cot x &= \frac{40}{1}, \\ \log \cot x &= \log 40. \\ \log 40 &= 1.60206. \\ \log \cot x &= 1.60206. \\ x &= 1^\circ 25' 56''.\end{aligned}$$

58. Find the angle of elevation of the sun when a tower 120 ft. high casts a horizontal shadow 70 ft. long.

$$\tan x = \frac{120}{70}.$$

$$\log \tan x = \log 120 + \text{colog } 70.$$

$$\log 120 = 2.07918$$

$$\text{colog } 70 = \frac{8.15490 - 10}{}$$

$$\log \tan x = 10.23408$$

$$x = 59^\circ 44' 35''.$$

59. How high is a tree which casts a horizontal shadow 80 ft. in length when the angle of elevation of the sun is 50° ?

$$\tan 50^\circ = \frac{x}{80}.$$

$$x = 80 \tan 50^\circ.$$

$$\log x = \log 80 + \log \tan 50^\circ.$$

$$\log 80 = 1.90309$$

$$\log \tan 50^\circ = \frac{10.07619 - 10}{}$$

$$\log x = 1.97928$$

$$x = 95.34.$$

The tree is 95.34 ft. high.

60. A rectangle 7.5 in. long has a diagonal 8.2 in. long. What angle does the diagonal make with the base?

$$\cos x = \frac{7.5}{8.2}.$$

$$\log \cos x = \log 7.5 - \log 8.2.$$

$$\log 7.5 = 10.87506 - 10$$

$$\log 8.2 = \frac{0.91381}{}$$

$$\log \cos x = 9.96125$$

$$x = 23^\circ 50' 40''.$$

61. A rectangle $8\frac{1}{4}$ in. long has an area of $49\frac{1}{2}$ sq. in. Find the angle which the diagonal makes with the base.

$$\text{Width} = 49\frac{1}{2} \text{ in.} \div 8\frac{1}{4}.$$

$$\tan x = \frac{49.5 \div 8.25}{8.25}.$$

$$\log 49.5 = 1.69461$$

$$\text{colog } 8.25 = 9.08355 - 10$$

$$\text{colog } 8.25 = 9.08355 - 10$$

$$\log \tan x = 9.86171$$

$$x = 36^\circ 1' 42''.$$

62. The length AB of a rectangular field $ABCD$ is 80 rd. and the width AD is 60 rd. The field is divided into two equal parts by a straight fence PQ starting from a point P on AD which is 15 rd. from A . What angle does PQ make with AD ?

$$AB = 80 \text{ rd.},$$

$$AD = 60 \text{ rd.},$$

$$AP = CQ = 15 \text{ rd.},$$

$$DP = BQ = 45 \text{ rd.}$$

$$\tan x = \frac{30}{45}.$$

$$\log 80 = 1.90309$$

$$\text{colog } 30 = \frac{8.52288 - 10}{}$$

$$\log \tan x = 10.42597$$

$$x = 69^\circ 26' 38''.$$

The line PQ makes an angle of $69^\circ 26' 38''$ with AD .

63. A ship is sailing due north-east at the rate of 10 mi. an hour. Find the rate at which she is moving due north, and also due east.

$$\cos 45^\circ = \frac{x}{10}.$$

$$x = 10 \cos 45^\circ.$$

$$\log x = \log 10 + \log \cos 45^\circ.$$

$$\log 10 = 1.00000$$

$$\log \cos 45^\circ = \frac{9.84949 - 10}{}$$

$$\log x = 0.84949$$

$$x = 7.071.$$

The ship moves 7.071 mi. an hour due north and 7.071 mi. an hour due east.

64. If the foot of a ladder 22 ft. long is 11 ft. from a house, how far up the side of the house does the ladder reach?



$$\begin{aligned}
 a^2 + b^2 &= c^2. \\
 a^2 &= c^2 - b^2 = (c + a)(c - a) = 363. \\
 \log a^2 &= \log 363 \\
 &= 2.55991. \\
 \log a &= \frac{1}{2} \log 363 \\
 &= 1.27996 \\
 a &= 19.05.
 \end{aligned}$$

The ladder extends 19.05 ft. up the side of the house.

65. In front of a window 20 ft. from the ground there is a flower bed 6 ft. wide and close to the house. How long is a ladder which will just reach from the edge of the bed to the window?

$$\begin{aligned}
 \tan x &= \frac{20}{6}. \\
 \log \tan x &= \log 20 + \text{colog } 6. \\
 \log 20 &= 1.30103 \\
 \text{colog } 6 &= \frac{9.22185 - 10}{} \\
 \log \tan x &= 10.52288 \\
 A &= 73^\circ 18'. \\
 c &= \frac{20}{\sin x}. \\
 \log c &= \log 20 + \text{colog } \sin x. \\
 \log 20 &= 1.30103 \\
 \text{colog } \sin x &= 0.01871 \\
 \log c &= 1.31974 \\
 c &= 20.88. \\
 \text{Length of ladder is } 20.88 \text{ ft.}
 \end{aligned}$$

66. A ladder 40 ft. long can be so placed that it will reach a window 33 ft. above the ground on one side of the street, and by tipping it back without moving its foot it will reach a window 21 ft. above the ground on the other side. Find the width of the street.

$$\cos B = \frac{33}{40}.$$

$$\begin{aligned}
 \log 33 &= 1.51851 \\
 \text{colog } 40 &= \frac{8.39794 - 10}{} \\
 \log \cos B &= 9.91645
 \end{aligned}$$

$$B = 34^\circ 24' 45''.$$

$$\tan B = \frac{b}{33}.$$

$$b = 33 \tan B.$$

$$\begin{aligned}
 \log 33 &= 1.51851 \\
 \log \tan B &= 9.83571 \\
 \log b &= \frac{1.35422}{} \\
 b &= 22.61.
 \end{aligned}$$

$$\cos B' = \frac{21}{40}.$$

$$\begin{aligned}
 \log 21 &= 1.32222 \\
 \text{colog } 40 &= \frac{8.39794 - 10}{} \\
 \log \cos B' &= 9.72016
 \end{aligned}$$

$$B' = 58^\circ 19' 54''.$$

$$\tan B' = \frac{b'}{21}.$$

$$b' = 21 \tan B'.$$

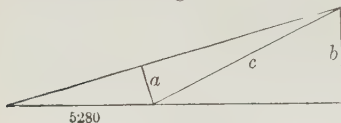
$$\begin{aligned}
 \log 21 &= 1.32222 \\
 \log \tan B' &= 0.20982 \\
 \log b' &= 1.53204
 \end{aligned}$$

$$b' = 34.04.$$

Width of street

$$\begin{aligned}
 &= 22.61 \text{ ft.} + 34.04 \text{ ft.} \\
 &= 56.65 \text{ ft.}
 \end{aligned}$$

67. From the top of a hill the angles of depression of two successive milestones, on a straight, level road leading to the hill, are 5° and 15° . Find the height of the hill.



$$\sin 5^\circ = \frac{a}{5280}.$$

$$a = 5280 \sin 5^\circ.$$

$$\log 5280 = 3.72263$$

$$\log \sin 5^\circ = 8.94030 - 10$$

$$\log a = 2.66293$$

$$\sin 10^\circ = \frac{a}{c}.$$

$$c = \frac{a}{\sin 10^\circ}.$$

$$\log a = 2.66293$$

$$\text{colog } \sin 10^\circ = 0.76033$$

$$\log c = 3.42326$$

$$\cos 75^\circ = \frac{b}{c}.$$

$$b = c \cos 75^\circ.$$

$$\log c = 3.42326$$

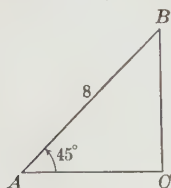
$$\log \cos 75^\circ = 9.41300 - 10$$

$$\log b = 2.83626$$

$$b = 685.9.$$

Height of hill is 685.9 ft.

68. A stick 8 ft. long makes an angle of 45° with the floor of a room, the other end resting against the wall. How far is the foot of the stick from the wall?



$$\cos 45^\circ = \frac{x}{8}.$$

$$x = 8 \cos 45^\circ.$$

$$\log x = \log 8 + \log \cos 45^\circ.$$

$$\log 8 = 0.90309$$

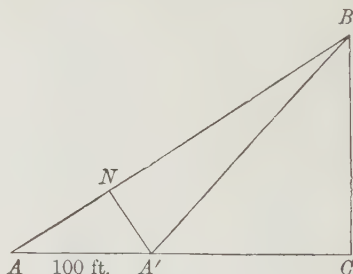
$$\log \cos 45^\circ = 9.84949 - 10$$

$$\log x = 0.75258$$

$$x = 5.657.$$

The stick is 5.657 ft. from the wall.

69. A building stands on a horizontal plain. The angle of elevation at a certain point on the plain is 30° , and at a point 100 ft. nearer the building it is 45° . How high is the building?



Let BC represent the building, AC the horizontal plain.

$BAC =$ angle made by line from eye of observer $= 30^\circ$.

$BA'C = 45^\circ =$ angle of elevation 100 ft. nearer.

From A' draw $A'N \perp$ to AB .

In rt. $\triangle A'A'N$,

$$\angle NAA' = 30^\circ,$$

and $\angle NA'A = 60^\circ.$

$$\therefore NA' = 50 \text{ feet.}$$

$$\begin{aligned} AN &= \sqrt{(100)^2 - (50)^2} \\ &= \sqrt{7500} = 50\sqrt{3} \\ &= 86.602. \end{aligned}$$

In rt. $\triangle BNA'$,

$$\frac{BN}{NA'} = \cot NBA' = \cot 15^\circ,$$

and $BN = NA' \cot 15^\circ$.

$$\log NA' = 1.69897$$

$$\log \cot 15^\circ = 0.57195$$

$$\log BN = 2.27092$$

$$BN = 186.60$$

$$AN = 86.60$$

$$AB = 273.20$$

In rt. $\triangle ABC$,

$$\angle BAC = 30^\circ,$$

and $\angle ABC = 60^\circ$.

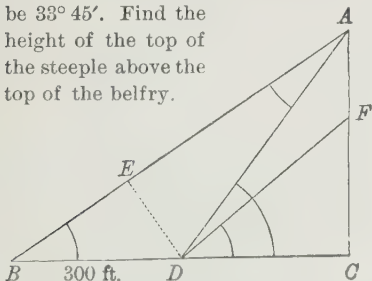
$$\therefore BC = \frac{1}{2} AB$$

$$= \frac{1}{2} \times 273.20 \text{ ft.}$$

$$= 136.6 \text{ ft.}$$

The building is 136.6 ft. high.

70. From a certain point on the ground the angles of elevation of the top of the belfry of a church and of the top of the steeple are found to be 40° and 51° respectively. From a point 300 ft. further off, on a horizontal line, the angle of elevation of the top of the steeple is found to be $33^\circ 45'$. Find the height of the top of the steeple above the top of the belfry.



Draw $DE \perp$ to AB from D .

$$\text{In } \triangle BED, \frac{ED}{BD} = \sin 33^\circ 45'.$$

$$\therefore ED = 300 \times \sin 33^\circ 45'.$$

$$\log 300 = 2.47712$$

$$\log \sin 33^\circ 45' = \frac{9.74474 - 10}{}$$

$$\log ED = 2.22186$$

$$\begin{aligned} \angle EAD &= 180^\circ - 33^\circ 45' - (180^\circ - 51^\circ) \\ &= 17^\circ 15'. \end{aligned}$$

In $\triangle ADE$,

$$\frac{ED}{AD} = \sin 17^\circ 15'.$$

$$\therefore AD = \frac{ED}{\sin 17^\circ 15'}.$$

$$\log ED = 2.22186$$

$$\text{colog } \sin 17^\circ 15' = \frac{0.52791}{}$$

$$\log AD = 2.74977$$

In $\triangle ADC$,

$$\frac{DC}{AD} = \cos 51^\circ.$$

$$\therefore DC = AD \cos 51^\circ.$$

$$\log AD = 2.74977$$

$$\log \cos 51^\circ = \frac{9.79887 - 10}{}$$

$$\log DC = 2.54864$$

In $\triangle ADC$,

$$\frac{AC}{DC} = \tan 51^\circ.$$

$$\therefore AC = DC \tan 51^\circ.$$

$$\log DC = 2.54864$$

$$\log \tan 51^\circ = \frac{10.09163 - 10}{}$$

$$\log AC = 2.64027$$

$$AC = 436.79.$$

In $\triangle FDC$,

$$\frac{FC}{DC} = \tan 40^\circ.$$

$$\therefore FC = DC \tan 40^\circ.$$

$$\log DC = 2.54864$$

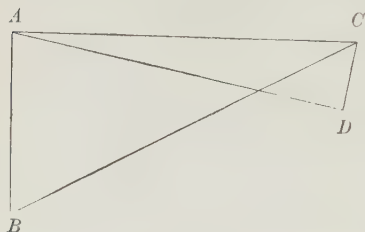
$$\log \tan 40^\circ = \frac{9.92381}{}$$

$$\log FC = 2.47245$$

$$FC = 296.79.$$

Distance from top of belfry to top of steeple is $436.79 \text{ ft.} - 296.79 \text{ ft.} = 140 \text{ ft.}$

71. The angle of elevation of the top C of an inaccessible fort observed from a point A is 12° . At a point B , 219 ft. from A and on a line AB perpendicular to AC , the angle ABC is $61^\circ 45'$. Find the height of the fort.



In rt. $\triangle CAB$, $\frac{AB}{AC} = \cot ABC$.

$$\therefore AC = \frac{AB}{\cot ABC}.$$

$$\log AC = \log AB + \operatorname{colog} \cot ABC.$$

$$\log AB = 2.34044$$

$$\operatorname{colog} \cot ABC = 0.26977$$

$$\log AC = 2.61021$$

In rt. $\triangle ADC$,

$$\frac{CD}{AC} = \sin CAD.$$

$$\therefore CD = AC \sin CAD.$$

$$\log CD = \log AC + \log \sin CAD.$$

$$\log AC = 2.61021$$

$$\log \sin CAD = 9.31788 - 10$$

$$\log CD = 1.92809$$

$$CD = 84.74.$$

Height of fort is 84.74 ft.

Exercise 32. The Isosceles Triangle (Page 71)

Solve the following isosceles triangles:

1. Given a and A , find C , c , and h .

$$C = 180^\circ - 2A$$

$$= 2(90^\circ - A).$$

$$\frac{\frac{1}{2}c}{a} = \cos A.$$

$$c = 2a \cos A.$$

$$\frac{h}{a} = \sin A.$$

$$h = a \sin A.$$

2. Given a and C , find A , c , and h .

$$C + 2A = 180^\circ.$$

$$A = 90^\circ - \frac{1}{2}C.$$

$$\frac{\frac{1}{2}c}{a} = \cos A.$$

$$c = 2a \cos A.$$

$$\frac{h}{a} = \sin A.$$

$$h = a \sin A.$$

3. Given c and A , find C , a , and h .

$$C = 180^\circ - 2A$$

$$= 2(90^\circ - A).$$

$$\frac{\frac{1}{2}c}{a} = \cos A.$$

$$2a = \frac{c}{\cos A}.$$

$$a = \frac{c}{2 \cos A}.$$

$$\frac{h}{a} = \sin A.$$

$$h = a \sin A.$$

4. Given c and C , find A , a , and h .

$$A = 90^\circ - \frac{1}{2}C.$$

$$\frac{\frac{1}{2}c}{a} = \cos A.$$

$$a = \frac{c}{2 \cos A}.$$

$$\frac{h}{a} = \sin A.$$

$$h = a \sin A.$$

5. Given
- h
- and
- A
- , find
- C
- ,
- a
- , and
- c
- .

$$C = 2(90^\circ - A).$$

$$\frac{h}{a} = \sin A.$$

$$a = \frac{h}{\sin A}.$$

$$\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}.$$

$$c = 2a \cos A.$$

6. Given
- h
- and
- C
- , find
- A
- ,
- a
- , and
- c
- .

$$A = 90^\circ - \frac{1}{2}C.$$

$$\frac{h}{a} = \sin A.$$

$$a = \frac{h}{\sin A}.$$

$$\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}.$$

$$c = 2a \cos A.$$

7. Given
- a
- and
- h
- , find
- A
- ,
- C
- , and
- c
- .

$$\frac{h}{a} = \sin A.$$

$$C = 2(90^\circ - A).$$

$$\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}.$$

$$c = 2a \cos A.$$

8. Given
- c
- and
- h
- , find
- A
- ,
- C
- , and
- a
- .

$$\tan A = \frac{h}{\frac{1}{2}c} = \frac{2h}{c}.$$

$$C = 2(90^\circ - A).$$

$$\sin A = \frac{h}{a}.$$

$$a = \frac{h}{\sin A}.$$

9. Given
- $a = 14.3$
- ,
- $c = 11$
- , find
- A
- ,
- C
- , and
- h
- .

$$\cos A = \frac{\frac{1}{2}c}{a}.$$

$$\log \cos A = \log \frac{1}{2}c + \text{colog } a.$$

$$\log \frac{1}{2}c = 0.74036$$

$$\text{colog } a = \frac{8.84466 - 10}{}$$

$$\log \cos A = 9.58502 - 10$$

$$A = 67^\circ 22' 50''.$$

$$\therefore C = 2(90^\circ - A)$$

$$= 45^\circ 14' 20''.$$

$$\sin A = \frac{h}{a}.$$

$$h = a \sin A.$$

$$\log h = \log a + \log \sin A.$$

$$\log a = 1.15534$$

$$\log \sin A = \frac{9.96524 - 10}{}$$

$$\log h = 1.12058$$

$$h = 13.2.$$

10. Given
- $a = 0.295$
- ,
- $A = 68^\circ 10'$
- , find
- c
- ,
- h
- , and
- S
- .

$$\sin A = \frac{h}{a}.$$

$$h = a \sin A.$$

$$\log h = \log a + \log \sin A.$$

$$\log a = 9.46982 - 10$$

$$\log \sin A = \frac{9.96767 - 10}{}$$

$$\log h = 9.43749 - 10$$

$$h = 0.27384.$$

$$\cos A = \frac{\frac{1}{2}c}{a}.$$

$$\frac{1}{2}c = a \cos A.$$

$$\log \frac{1}{2}c = \log a + \log \cos A.$$

$$\log a = 9.46982 - 10$$

$$\log \cos A = \frac{9.57044 - 10}{}$$

$$\log \frac{1}{2}c = 9.04026 - 10$$

$$\frac{1}{2}c = 0.109713.$$

$$c = 0.21943.$$

$$S = \frac{1}{2}ch.$$

$$2S = ch.$$

$$\log 2S = \log c + \log h.$$

$$\log c = 9.34130 - 10$$

$$\log h = 9.43749 - 10$$

$$\log 2S = 8.77879 - 10$$

$$2S = 0.060089.$$

$$S = 0.03004$$

11. Given $c = 2.352$, $C = 69^\circ 49'$,
find a , h , and S .

$$\begin{aligned}\frac{1}{2} C &= 34^\circ 54' 30''. \\ \sin \frac{1}{2} C &= \frac{\frac{1}{2} c}{a}. \\ a &= \frac{\frac{1}{2} c}{\sin \frac{1}{2} C} \\ \log a &= \log \frac{1}{2} c + \text{colog} \sin \frac{1}{2} C. \\ \log \frac{1}{2} c &= 0.07041 \\ \text{colog} \sin \frac{1}{2} C &= 0.24240 \\ \log a &= 0.31281 \\ a &= 2.055. \\ \cos \frac{1}{2} C &= \frac{h}{a}. \\ h &= a \cos \frac{1}{2} C. \\ \log h &= \log a + \log \cos \frac{1}{2} C. \\ \log a &= 0.31281 \\ \log \cos \frac{1}{2} C &= 9.91385 - 10 \\ \log h &= 0.22666 \\ h &= 1.6852. \\ S &= \frac{1}{2} ch. \\ 2S &= ch. \\ \log 2S &= \log c + \log h. \\ \log c &= 0.37144 \\ \log h &= 0.22666 \\ \log 2S &= 0.59810 \\ 2S &= 3.9637. \\ S &= 1.9819.\end{aligned}$$

12. Given $h = 7.4847$, $A = 76^\circ 14'$,
find a , c , and S .

$$\begin{aligned}\sin A &= \frac{h}{a}. \\ a &= \frac{h}{\sin A}. \\ \log a &= \log h + \text{colog} \sin A. \\ \log h &= 0.87417 \\ \text{colog} \sin A &= 0.01266 \\ \log a &= 0.88683 \\ a &= 7.706. \\ \tan A &= \frac{h}{\frac{1}{2} c}.\end{aligned}$$

$$\begin{aligned}\frac{1}{2} c &= \frac{h}{\tan A}. \\ \log \frac{1}{2} c &= \log h + \text{colog} \tan A \\ \log h &= 0.87417 \\ \text{colog} \tan A &= 9.38918 - 10 \\ \log \frac{1}{2} c &= 0.26335 \\ \frac{1}{2} c &= 1.8338. \\ c &= 3.6676. \\ S &= \frac{1}{2} ch. \\ \log S &= \log \frac{1}{2} c + \log h. \\ \log \frac{1}{2} c &= 0.26335 \\ \log h &= 0.87417 \\ \log S &= 1.13752 \\ S &= 13.725.\end{aligned}$$

13. Given $c = 147$, $S = 2572.5$
find A , C , a , and h .

$$\begin{aligned}S &= \frac{1}{2} ch. \\ h &= \frac{2S}{c}. \\ \log h &= \log 2S + \text{colog} c. \\ \log 2S &= 3.71139 \\ \text{colog} c &= 7.83268 - 10 \\ \log h &= 1.54407 \\ h &= 35. \\ \tan A &= \frac{h}{\frac{1}{2} c}. \\ \log \tan A &= \log h + \text{colog} \frac{1}{2} c. \\ \log h &= 1.54407 \\ \text{colog} \frac{1}{2} c &= 8.13371 - 10 \\ \log \tan A &= 9.67778 - 10 \\ A &= 25^\circ 27' 47''. \\ C &= 2(90^\circ - A) \\ &= 129^\circ 4' 26''. \\ a &= \frac{h}{\sin A}. \\ \log a &= \log h + \text{colog} \sin A \\ \log h &= 1.54407 \\ \text{colog} \sin A &= 0.36661 \\ \log a &= 1.91068 \\ a &= 81.41.\end{aligned}$$

14. Given $h = 16.8$, $S = 43.68$,
find A , C , a , and c .

$$S = \frac{1}{2} ch.$$

$$\frac{1}{2} c = \frac{S}{h}.$$

$$\log \frac{1}{2} c = \log S + \text{colog } h.$$

$$\log S = 1.64028$$

$$\text{colog } h = \frac{8.77469 - 10}{}$$

$$\log \frac{1}{2} c = 0.41497$$

$$\frac{1}{2} c = 2.60.$$

$$c = 5.2.$$

$$\tan A = \frac{h}{\frac{1}{2} C}.$$

$$\log \tan A = \log h + \text{colog } \frac{1}{2} c.$$

$$\log h = 1.22531$$

$$\text{colog } \frac{1}{2} c = \frac{9.58503 - 10}{}$$

$$\log \tan A = 10.81034$$

$$A = 81^\circ 12' 9''.$$

$$\frac{1}{2} C = 8^\circ 47' 51''.$$

$$C = 17^\circ 35' 42''.$$

$$\cos A = \frac{\frac{1}{2} c}{a}.$$

$$\log a = \log \frac{1}{2} c + \text{colog } \cos A.$$

$$\log \frac{1}{2} c = 0.41497$$

$$\text{colog } \cos A = 0.81547$$

$$\log a = 1.23044$$

$$a = 17.$$

15. Given $a = 27.56$, $A = 75^\circ 14'$,
find c , h , and S .

$$\sin A = \frac{h}{a}.$$

$$h = a \sin A.$$

$$\log h = \log a + \log \sin A.$$

$$\log a = 1.44028$$

$$\log \sin A = \frac{9.98541 - 10}{}$$

$$\log h = 1.42569$$

$$h = 26.649.$$

$$\cos A = \frac{\frac{1}{2} c}{a}.$$

$$\frac{1}{2} c = a \cos A.$$

$$\log \frac{1}{2} c = \log a + \log \cos A.$$

$$\log a = 1.44028$$

$$\log \cos A = \frac{9.40634 - 10}{}$$

$$\log \frac{1}{2} c = 0.84662$$

$$\frac{1}{2} c = 7.0246.$$

$$c = 14.049.$$

$$S = \frac{1}{2} ch.$$

$$\log S = \log \frac{1}{2} c + \log h.$$

$$\log \frac{1}{2} c = 0.84662$$

$$\log h = 1.42569$$

$$\log S = 2.27231$$

$$S = 187.2.$$

Given an isosceles triangle, ABC :

16. Find the value of S in terms
of a and C .

$$S = \frac{1}{2} ch.$$

$$\frac{1}{2} c = a \sin \frac{1}{2} C.$$

$$h = a \cos \frac{1}{2} C.$$

$$S = a \sin \frac{1}{2} C \times a \cos \frac{1}{2} C$$

$$= a^2 \sin \frac{1}{2} C \cos \frac{1}{2} C.$$

18. Find the value of S in terms of h and C .

$$S = \frac{1}{2} ch.$$

$$\frac{1}{2} c = h \tan \frac{1}{2} C.$$

$$S = h (h \tan \frac{1}{2} C)$$

$$= h^2 \tan \frac{1}{2} C.$$

17. Find the value of S in terms
of a and A .

$$S = \frac{1}{2} ch.$$

$$\frac{1}{2} c = a \cos A.$$

$$h = a \sin A.$$

$$S = a \cos A \times a \sin A$$

$$= a^2 \sin A \cos A.$$

19. A barn is 40 ft. by 80 ft., the pitch of the roof is 45° ; find the length of the rafters and the area of the whole roof.

$$40 \div 2 = 20 = \frac{1}{2}c.$$

$$\cos A = \frac{\frac{1}{2}c}{a} = \frac{20}{a}.$$

$$20 = a \cos A.$$

$$\therefore a = \frac{20}{\cos A}.$$

$$\log a = \log 20 + \operatorname{colog} \cos A.$$

$$\log 20 = 1.30103$$

$$\operatorname{colog} \cos A = \underline{0.15051}$$

$$\log a = 1.45154$$

$$a = 28.284.$$

$$2 \times 28.284 \times 80 = 4525.44.$$

Length of rafters is 28.284 ft.;
area of roof is 4525.44 sq. ft.

20. In a unit circle what is the length of the chord subtending the angle 45° at the center?

$$\sin \frac{1}{2}C = \frac{\frac{1}{2}c}{a}.$$

$$\log \frac{1}{2}C = \log a + \log \sin \frac{1}{2}C.$$

$$\log a = 0.00000$$

$$\log \sin \frac{1}{2}C = \underline{9.58284 - 10}$$

$$\log \frac{1}{2}c = 9.58284 - 10$$

$$\frac{1}{2}c = 0.38268.$$

$$c = 0.76536.$$

21. The radius of a circle is 30 in., and the length of a chord is 44 in.; find the angle subtended at the center.

$$\sin \frac{1}{2}C = \frac{\frac{1}{2}c}{a}.$$

$$\log \sin \frac{1}{2}C = \log \frac{1}{2}c + \operatorname{colog} a.$$

$$\log \frac{1}{2}c = 1.34242$$

$$\operatorname{colog} a = \underline{8.52288 - 10}$$

$$\log \sin \frac{1}{2}C = 9.86530$$

$$\frac{1}{2}C = 47^\circ 10'.$$

$$C = 94^\circ 20'.$$

22. Find the radius of a circle if a chord whose length is 5 in. subtends at the center an angle of 133° .

$$\sin \frac{1}{2}C = \frac{\frac{1}{2}c}{a}.$$

$$\log a = \log \frac{1}{2}c + \operatorname{colog} \sin \frac{1}{2}C.$$

$$\log \frac{1}{2}c = 0.39794$$

$$\operatorname{colog} \sin \frac{1}{2}C = \underline{0.03760}$$

$$\log a = 0.43554$$

$$a = 2.7261.$$

Radius is 2.7261 in.

23. What is the angle at the center of a circle if the subtending chord is equal to $\frac{2}{3}$ of the radius?

Let the circle be a unit circle with radius = 1.

$$\sin \frac{1}{2}C = \frac{\frac{1}{2}c}{a}$$

$$\frac{1}{2} = \frac{1}{3}.$$

$$\log \sin \frac{1}{2}C = \log 1 + \operatorname{colog} 3.$$

$$\log 1 = 0.00000$$

$$\operatorname{colog} 3 = \underline{9.52288 - 10}$$

$$\log \sin \frac{1}{2}C = 9.52288$$

$$\frac{1}{2}C = 19^\circ 28' 16\frac{1}{2}''.$$

$$C = 38^\circ 56' 33''.$$

24. Find the area of a circular sector if the radius of the circle is 12 in., and the angle of the sector is 30° .

$$\text{Area } \odot = \pi R^2.$$

$$\text{Area sector} = \frac{30 \pi R^2}{360}.$$

$$\begin{aligned} \log \text{area sector} &= \log 30 + \operatorname{colog} 360 \\ &\quad + \log \pi + 2 \log R. \end{aligned}$$

$$\log 30 = 1.47712$$

$$\operatorname{colog} 360 = 7.44370 - 10$$

$$\log \pi = 0.49715$$

$$2 \log R = \underline{2.15836}$$

$$\log \text{area} = 1.57633$$

$$\text{Area} = 37.699 \text{ sq. in.}$$

25. Find the tangent of the angle of the slope of an A-roof of a building which is 24 ft. 6 in. wide at the eaves, the ridgepole being 10 ft. 9 in. above the eaves.

$$\tan A = \frac{h}{\frac{1}{2}c}.$$

$$\log \tan A = \log h + \operatorname{colog} \frac{1}{2}c.$$

$$\log h = 1.03141$$

$$\operatorname{colog} \frac{1}{2}c = \frac{8.91186 - 10}{}$$

$$\log \tan A = \frac{9.94327 - 10}{}$$

$$\tan A = 0.8775.$$

Exercise 33. The Regular Polygon (Page 72)

Find the remaining parts of a regular polygon, given:

1. $n = 10, c = 1.$

$$\frac{1}{2}C = \frac{180^\circ}{10} = 18^\circ.$$

$$\frac{1}{2}c = 0.5.$$

$$A = 72^\circ.$$

$$h = \frac{1}{2}c \tan A.$$

$$\log h = \log \frac{1}{2}c + \log \tan A.$$

$$\log \frac{1}{2}c = \frac{9.69897 - 10}{}$$

$$\log \tan A = \frac{10.48822 - 10}{}$$

$$\log h = \frac{0.18719}{}$$

$$h = 1.5388.$$

$$\log r = \log \frac{1}{2}c + \operatorname{colog} \cos A.$$

$$\log \frac{1}{2}c = \frac{9.69897 - 10}{}$$

$$\operatorname{colog} \cos A = \frac{0.51002}{}$$

$$\log r = \frac{0.20899}{}$$

$$r = 1.618.$$

$$S = \frac{1}{2}hp.$$

$$\log h = 0.18719$$

$$\log p = \frac{1.00000}{}$$

$$\log 2S = \frac{1.18719}{}$$

$$2S = 15.388.$$

$$S = 7.694.$$

2. $n = 18, r = 1.$

$$\frac{1}{2}C = 10^\circ.$$

$$A = 80^\circ.$$

$$h = r \sin A.$$

$$\log r = 0.00000$$

$$\log \sin A = \frac{9.99335 - 10}{}$$

$$\log h = \frac{9.99335 - 10}{}$$

$$h = 0.9848.$$

$$\frac{1}{2}c = r \cos A.$$

$$\log r = 0.00000$$

$$\log \cos A = \frac{9.23967 - 10}{}$$

$$\log \frac{1}{2}c = \frac{9.23967 - 10}{}$$

$$\frac{1}{2}c = 0.17365.$$

$$p = 6.2514.$$

$$S = \frac{1}{2}hp.$$

$$\log h = 9.99335 - 10$$

$$\log p = \frac{0.79598}{}$$

$$\log 2S = \frac{0.78933}{}$$

$$2S = 6.1564.$$

$$S = 3.0782.$$

$$3. \quad n = 20, r = 20.$$

$$\frac{1}{2}C = 9^\circ.$$

$$A = 81^\circ.$$

$$h = r \sin A.$$

$$\log r = 1.30103$$

$$\log \sin A = \frac{9.99462 - 10}{}$$

$$\log h = 1.29565$$

$$h = 19.754.$$

$$\frac{1}{2}c = r \cos A.$$

$$\log r = 1.30103$$

$$\log \cos A = \frac{9.19433 - 10}{}$$

$$\log \frac{1}{2}c = 0.49536$$

$$\frac{1}{2}c = 3.1286.$$

$$c = 6.257.$$

$$p = 125.14.$$

$$S = \frac{1}{2}hp.$$

$$\log h = 1.29565$$

$$\log p = \frac{2.09740}{}$$

$$\log 2S = \frac{3.39305}{}$$

$$2S = 2472.$$

$$S = 1236.$$

$$4. \quad n = 8, h = 1.$$

$$\frac{1}{2}C = 22^\circ 30'.$$

$$\tan \frac{1}{2}C = \frac{\frac{1}{2}c}{h}.$$

$$\log \frac{1}{2}c = \log h + \log \tan \frac{1}{2}C.$$

$$\log h = 0.00000$$

$$\log \tan \frac{1}{2}C = \frac{9.61722 - 10}{}$$

$$\log \frac{1}{2}c = 9.61722 - 10$$

$$\frac{1}{2}c = 0.41421.$$

$$c = 0.82842.$$

$$\cos \frac{1}{2}C = \frac{h}{r}.$$

$$\log r = \log h + \text{colog} \cos \frac{1}{2}C.$$

$$\log h = 0.00000$$

$$\text{colog} \cos \frac{1}{2}C = \frac{0.03438}{}$$

$$\log r = 0.03438$$

$$r = 1.0824.$$

$$S = \frac{1}{2}hp \\ = 3.3137.$$

$$5. \quad n = 11, S = 20.$$

$$2S = hp.$$

$$40 = hp.$$

$$c = \frac{p}{11}.$$

$$h = \frac{40}{p}.$$

$$\frac{1}{2}C = 16^\circ 22'.$$

$$\tan \frac{1}{2}C = \frac{\frac{1}{2}c}{h}.$$

Substitute values of h and c ,

$$\tan \frac{1}{2}C = \frac{p}{22} \div \frac{40}{p} = \frac{p^2}{880}.$$

$$\log p = \frac{1}{2}(\log 880 + \log \tan \frac{1}{2}C)$$

$$\log 880 = 2.94448$$

$$\log \tan \frac{1}{2}C = \frac{9.46788 - 10}{2 \overline{)2.41236}}$$

$$\log p = 1.20618$$

$$p = 16.076.$$

$$c = 1.461.$$

$$\sin \frac{1}{2}C = \frac{\frac{1}{2}c}{r}.$$

$$\log r = \log \frac{1}{2}c + \text{colog} \sin \frac{1}{2}C$$

$$\log \frac{1}{2}c = 9.86392 - 10$$

$$\text{colog} \sin \frac{1}{2}C = \frac{0.55008}{}$$

$$\log r = 0.41400$$

$$r = 2.5942.$$

$$\cos \frac{1}{2}C = \frac{h}{r}.$$

$$\log h = \log r + \log \cos \frac{1}{2}C.$$

$$\log r = 0.41400$$

$$\log \cos \frac{1}{2}C = \frac{9.98204 - 10}{}$$

$$\log h = 0.39604$$

$$h = 2.4891.$$

6. $n = 7$, $S = 7$.

$$S = \frac{1}{2} hp.$$

$$14 = hp.$$

$$h = \frac{14}{p}.$$

$$c = \frac{p}{7}.$$

$$\frac{1}{2} C = 25^\circ 43'.$$

$$\tan \frac{1}{2} C = \frac{\frac{1}{2} c}{h}.$$

$$\tan \frac{1}{2} C = \frac{p}{14} \div \frac{14}{p} = \frac{p^2}{196}.$$

$$\log p = \frac{1}{2} (\log 196 + \log \tan \frac{1}{2} C).$$

$$\log 196 = 2.29226$$

$$\log \tan \frac{1}{2} C = \frac{9.68271 - 10}{2}$$

$$\log p = \frac{2 \overline{)1.97497}}{0.98749}$$

$$p = 9.716.$$

$$\frac{1}{2} c = 0.694.$$

$$\tan \frac{1}{2} C = \frac{\frac{1}{2} c}{h}.$$

$$\log h = \log \frac{1}{2} c + \text{colog} \tan \frac{1}{2} C.$$

$$\log \frac{1}{2} c = 9.84136 - 10$$

$$\text{colog} \tan \frac{1}{2} C = \frac{0.31729}{\log h = 0.15865}$$

$$\log h = 0.15865$$

$$h = 1.441.$$

$$\sin \frac{1}{2} C = \frac{\frac{1}{2} c}{r}.$$

$$\log r = \log \frac{1}{2} c + \text{colog} \sin \frac{1}{2} C.$$

$$\log \frac{1}{2} c = 9.84136 - 10$$

$$\text{colog} \sin \frac{1}{2} C = \frac{0.36259}{\log r = 0.20395}$$

$$\log r = 0.20395$$

$$r = 1.5994.$$

7. The side of a regular inscribed hexagon is 1 in.; find the side of a regular inscribed dodecagon.

$$\frac{1}{2} C = 15^\circ.$$

$$\sin \frac{1}{2} C = \frac{\frac{1}{2} c}{r}.$$

$$\log c = \log 2 + \log \sin \frac{1}{2} C.$$

$$\log 2 = 0.30103$$

$$\log \sin \frac{1}{2} C = \frac{9.41300 - 10}{\log c = 9.71403 - 10}$$

$$\log c = 9.71403 - 10$$

$$c = 0.51764 \text{ in.}$$

8. Given n and c , and represent by b the side of the regular inscribed polygon having $2n$ sides, find b in terms of n and c .

Let O be the center of the circle, BC the side of the polygon having n sides, BA the side of the polygon having $2n$ sides. Then OA is $\perp$ to BC at its middle point D .

$$\angle BOA = \frac{360^\circ}{2n} = \frac{180^\circ}{n}.$$

$$\angle OBC = 90^\circ - \frac{180^\circ}{n}.$$

The $\triangle BOA$ is isosceles.

$$\begin{aligned} \therefore \angle OBA &= \frac{1}{2} \left(180^\circ - \frac{180^\circ}{n} \right) \\ &= 90^\circ - \frac{90^\circ}{n}, \end{aligned}$$

$$\begin{aligned} \angle ABC &= \angle OBA - \angle OBC \\ &= \left(90^\circ - \frac{90^\circ}{n} \right) - \left(90^\circ - \frac{180^\circ}{n} \right) = \frac{90^\circ}{n} \end{aligned}$$

$$\frac{\frac{1}{2} c}{b} = \cos \frac{90^\circ}{n}.$$

$$\therefore \frac{1}{2} c = b \cos \frac{90^\circ}{n}.$$

$$\therefore b = \frac{\frac{1}{2} c}{\cos \frac{90^\circ}{n}} = \frac{c}{2 \cos \frac{90^\circ}{n}}.$$

9. Compute the difference between the areas of a regular octagon and a regular nonagon if the perimeter of each is 16 in.

$$\frac{1}{2}c = \frac{p}{2n} = \frac{16}{16} = 1.$$

$$A = \frac{180^\circ}{n} = 22^\circ 30'.$$

$$h = \frac{1}{2}c \cot A.$$

$$\log h = \log \frac{1}{2}c + \log \cot A.$$

$$\log \frac{1}{2}c = 0.00000$$

$$\log \cot A = \frac{10.38278 - 10}{}$$

$$\log h = 0.38278$$

$$S = \frac{1}{2}ch.$$

$$\log S = \log h + \log \frac{1}{2}p.$$

$$\log h = 0.38278$$

$$\log \frac{1}{2}p = \frac{0.90309}{}$$

$$\log S = 1.28587$$

$$S = 19.3139.$$

$$\frac{1}{2}c' = \frac{p}{2n'} = \frac{16}{18} = 0.8889.$$

$$A' = \frac{180^\circ}{n'} = 20^\circ.$$

$$\log h' = \log \frac{1}{2}c' + \log \cot A'.$$

$$\log \frac{1}{2}c' = 9.94885 - 10$$

$$\log \cot A' = \frac{10.43893 - 10}{}$$

$$\log h' = 0.38778$$

$$\log S' = \log h' + \log \frac{1}{2}p.$$

$$\log h' = 0.38778$$

$$\log \frac{1}{2}p = \frac{0.90309}{}$$

$$\log S' = 1.29087$$

$$S' = 19.5377.$$

$$S' - S = 19.5377 - 19.3139 \\ = 0.2238 \text{ sq. in.}$$

10. Compute the difference between the perimeters of a regular pentagon and a regular hexagon if the area of each is 12 sq. in.

$$S = 12, \quad n = 5.$$

$$\frac{1}{2}C = \frac{180^\circ}{5} = 36^\circ.$$

$$S = \frac{1}{2}hp.$$

$$h = \frac{24}{p}.$$

$$\frac{1}{2}c = \frac{p}{2n} = \frac{p}{10}.$$

$$\tan \frac{1}{2}C = \frac{\frac{1}{2}c}{h} = \frac{\frac{p}{10}}{\frac{24}{p}} = \frac{p^2}{240}$$

$$p^2 = 240 \tan \frac{1}{2}C.$$

$$\log 240 = 2.38021$$

$$\log \tan \frac{1}{2}C = \frac{9.86126 - 10}{}$$

$$2 \overline{) 2.24147}$$

$$\log p = 1.12074$$

$$p = 13.205.$$

$$n' = 6, \quad \frac{1}{2}C' = 30^\circ.$$

$$\tan \frac{1}{2}C' = \frac{\frac{p'}{12}}{\frac{24}{p'}} = \frac{p'^2}{288}.$$

$$p^2 = 288 \tan \frac{1}{2}C'.$$

$$\log 288 = 2.45939$$

$$\log \tan \frac{1}{2}C' = \frac{9.76144 - 10}{}$$

$$2 \overline{) 2.22083}$$

$$\log p' = 1.11042$$

$$p' = 12.895.$$

$$p - p' = 0.310 \text{ in.}$$

11. Find the perimeter of a regular dodecagon circumscribed about a circle the circumference of which is 1 in.

Given circumference of inscribed circle = 1, $n = 12$; find p .

$$2\pi r = \text{circumference.}$$

$$r = \frac{\text{circ.}}{2\pi}.$$

$$\frac{1}{2}C = \frac{360^\circ}{24} = 15^\circ.$$

$$\tan 15^\circ = \frac{\frac{1}{2}c}{r} = \pi c.$$

$$c = \frac{\tan 15^\circ}{3.1416}.$$

$$p = c \times 12.$$

$$\log \tan 15^\circ = 9.42805$$

$$\text{colog } 3.1416 = 9.50285 - 10$$

$$\log c = 8.93090 - 10$$

$$\log 12 = \frac{1.07918}{}$$

$$\log p = 9.01008$$

$$p = 1.0235 \text{ in.}$$

12. What is the side of the regular inscribed polygon of 100 sides, the radius of the circle being unity? What is the perimeter?

$$\frac{1}{2}C = 1^\circ 48'.$$

$$\sin \frac{1}{2}C = \frac{\frac{1}{2}c}{r}.$$

$$\log c = \log 2 + \log \sin \frac{1}{2}C.$$

$$\log 2 = 0.30103$$

$$\log \sin \frac{1}{2}C = \frac{8.49708 - 10}{}$$

$$\log c = 8.79811 - 10$$

$$= 8.79811.$$

$$c = 0.062821.$$

$$\text{Perimeter} = 100 \times 0.062821$$

$$= 6.2821.$$

13. What is the perimeter of the regular inscribed polygon of 360 sides, the radius of the circle being unity?

$$\frac{1}{2}C = 30'.$$

$$\sin \frac{1}{2}C = \frac{\frac{1}{2}c}{r}.$$

$$p = c \times 360.$$

$$\log c = \log 2 + \log \sin \frac{1}{2}C.$$

$$\log 2 = 0.30103$$

$$\log \sin \frac{1}{2}C = \frac{7.94084 - 10}{}$$

$$\log c = 8.24187 - 10$$

$$\log 360 = \frac{2.55630}{}$$

$$\log p = 0.79817$$

$$p = 6.283.$$

14. The area of a regular polygon of twenty-five sides is 40 sq. in.; find the area of the ring included between the circumferences of the inscribed and circumscribed circles.

$$\frac{1}{2}ch = \frac{40}{2} = 1.6.$$

$$\frac{1}{2}C = 7^\circ 12'.$$

$$\frac{\frac{1}{2}c}{h} = \tan \frac{1}{2}C,$$

$$\text{or } \frac{\frac{1}{2}ch}{h^2} = \tan \frac{1}{2}C.$$

$$\therefore h^2 = \frac{1.6}{\tan \frac{1}{2}C}.$$

$$\log 1.6 = 0.20412$$

$$\text{colog } \tan \frac{1}{2}C = 0.89850$$

$$\log h^2 = 1.10262$$

$$\log h = 0.55131.$$

$$\frac{h}{r} = \cos \frac{1}{2}C.$$

$$\therefore r = \frac{h}{\cos \frac{1}{2}C}.$$

$$\log h = 0.55131$$

$$\text{colog } \cos \frac{1}{2}C = 0.00344$$

$$\log r = 0.55475$$

$$\log r^2 = 1.10950.$$

$$\pi r^2 = \text{area of circumscribed } \odot.$$

$$\log \pi = 0.49715$$

$$\log r^2 = 1.10950$$

$$\log \pi r^2 = 1.60665$$

$$\pi r^2 = 40.425.$$

$$\pi h^2 = \text{area of inscribed } \odot.$$

$$\log \pi = 0.49715$$

$$\log h^2 = 1.10262$$

$$\log \pi h^2 = 1.59977$$

$$\pi h^2 = 39.790.$$

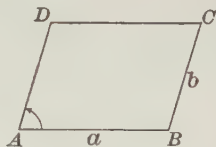
Area of inscribed circle is 39.790 sq. in.; area of ring is 40.425 sq. in. - 39.790 sq. in. = 0.635 sq. in.

Exercise 34. Review Problems (Page 73)

1. Prove that the area of the parallelogram here shown is equal to $ab \sin A$.

From D drop a perpendicular h upon AB .

$$\begin{aligned}\text{Area} &= ah. \\ \sin A &= \frac{h}{b}. \\ h &= b \sin A. \\ \therefore \text{area} &= ab \sin A.\end{aligned}$$



2. Two sides of a parallelogram are 5 in. and 6 in. respectively, and their included angle is $82^\circ 45'$. What is the area?

$$\begin{aligned}\text{Area} &= ab \sin A \\ &= 5 \cdot 6 \sin A.\end{aligned}$$

$$\begin{aligned}\log 5 &= 0.69897 \\ \log 6 &= 0.77815 \\ \log \sin A &= 9.99651 - 10 \\ \hline \log \text{area} &= 1.47363\end{aligned}$$

$$\text{Area} = 29.76.$$

The area is 29.76 sq. in.

3. Two sides of a parallelogram are 9 ft. and 12 ft. respectively, and their included angle is 74.5° . What is the area?

$$\begin{aligned}\text{Area} &= ab \sin A \\ &= 9 \cdot 12 \sin A.\end{aligned}$$

$$\begin{aligned}\log 9 &= 0.95424 \\ \log 12 &= 1.07918 \\ \log \sin A &= 9.98391 - 10 \\ \hline \log \text{area} &= 2.01733\end{aligned}$$

$$\text{Area} = 104.07.$$

The area is 104.07 sq. ft.

4. Each side of a rhombus is 7.35 in., and one angle is $42^\circ 27'$. What is the area?

$$\begin{aligned}\text{Area} &= ab \sin A \\ &= 7.35^2 \sin A.\end{aligned}$$

$$\begin{aligned}\log 7.35^2 &= 1.73258 \\ \log \sin A &= 9.82927 - 10 \\ \hline \log \text{area} &= 1.56185\end{aligned}$$

$$\text{Area} = 36.463.$$

The area is 36.463 sq. in.

5. The area of a rhombus is 250 sq. in., and one of the angles is $37^\circ 25'$. What is the length of each side?

$$\text{area} = ab \sin A.$$

$$250 = a^2 \sin A.$$

$$a^2 = \frac{250}{\sin A}.$$

$$\log 250 = 2.39794$$

$$\text{colog } \sin A = 0.21638$$

$$\log a^2 = 2.61432$$

$$\log a = 1.30716.$$

$$a = 20.284.$$

The length of each side is 20.284 in.

6. A pole BD stands on the top of a mound BC . From a point A the angles of elevation of the top and foot of the pole are 60° and 30° respectively. Prove that the height of the pole is twice the height of the mound.

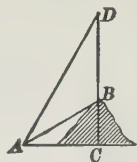
$$\tan 30^\circ = \frac{BC}{AC} \text{ and } \tan 60^\circ = \frac{DC}{AC}.$$

$$BC = AC \tan 30^\circ = AC \cdot \frac{1}{3}\sqrt{3},$$

$$DC = AC \tan 60^\circ = AC \cdot \sqrt{3}.$$

$$\frac{DC}{BC} = \frac{AC \cdot \sqrt{3}}{AC \cdot \frac{1}{3}\sqrt{3}} = \frac{3}{1}.$$

$$\therefore BC = \frac{1}{3} DC, \text{ or } BD = 2 BC.$$



and

7. A ladder 38 ft. long is resting against a wall. The foot of the ladder is 7 ft. 2 in. from the wall. What is the height of the top of the ladder above the ground?

$$a^2 = c^2 - b^2 = (c + b)(c - b).$$

$$\log (c + b) = 1.65482$$

$$\log (c - b) = 1.48901$$

$$\log a^2 = 3.14383$$

$$\log a = 1.57192.$$

$$a = 37.319.$$

The top of the ladder is 37.319 ft. above the ground.

8. From a boat 1325 ft. from the base of a vertical cliff the angle of elevation of the top of the cliff is observed to be $14^\circ 30'$. Find the height of the cliff.

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

$$\log b = 3.12222$$

$$\log \tan A = 9.41266 - 10$$

$$\log a = 2.53488$$

$$a = 342.67.$$

The height of the cliff is 342.67 ft.

9. On the top of a building 50 ft. high there is a flagstaff BD . From a point A on the ground the angles of elevation of B and D are 30° and 45° respectively. Find the length of the flagstaff and the distance AC of the observer from the building, as shown in the annexed figure.

$$\tan 30^\circ = \frac{50}{x}.$$

$$x = \frac{50}{\tan 30^\circ}.$$

$$\log 50 = 1.69897$$

$$\text{colog } \tan 30^\circ = 0.23856$$

$$\log x = 1.93753$$

$$x = 86.602.$$

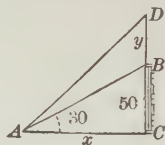
$$\begin{aligned} \text{Since } \triangle ACD \text{ is a rt. } \triangle, \quad \angle D &= 90^\circ - \angle CAD \\ &= 90^\circ - 45^\circ = 45^\circ. \end{aligned}$$

$$\text{Hence} \quad \angle D = \angle CAD,$$

$$\text{and} \quad x = DC,$$

$$\text{or} \quad DC = 86.602.$$

$$\begin{aligned} \text{Then} \quad y &= DC - BC \\ &= 86.602 - 50 \\ &= 36.602. \end{aligned}$$



The flagstaff is 36.602 ft. long; the distance of the observer is 86.602 ft. from the building.

10. A man whose eye is 5 ft. 8 in. above the ground stands midway between two telegraph poles which are 200 ft. apart. The elevation of the top of each pole is $48^\circ 50'$. What is the height of each?

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned} a &= b \tan A \\ &= 100 \tan A. \end{aligned}$$

$$\log b = 2.00000$$

$$\log \tan A = 10.05829 - 10$$

$$\log a = 2.05829$$

$$a = 114.36.$$

$$114.36 \text{ ft.} + 5.66\frac{2}{3} \text{ ft.} = 120.02\frac{2}{3} \text{ ft.}$$

The height of each pole is 120.03 ft.

11. The captain of a ship observed a lighthouse directly to the east. After sailing north 2 mi. he observed it to lie $55^{\circ} 30'$ east of south. How far was the ship from the lighthouse at the time of each observation?

$$\tan B = \frac{b}{a}.$$

$$b = a \tan B.$$

$$\log a = 0.30103$$

$$\log \tan B = \frac{10.16287 - 10}{}$$

$$\log b = 0.46390$$

$$b = 2.9101.$$

$$\cos B = \frac{a}{c}.$$

$$c = \frac{a}{\cos B}.$$

$$\log a = 0.30103$$

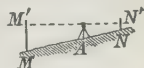
$$\text{colog } \cos B = 0.24687$$

$$\log c = 0.54790$$

$$c = 3.531.$$

The distances from the lighthouse were 2.9101 mi. and 3.531 mi.

12. A leveling instrument is placed at A on the slope MN , and the line $M'N'$ is sighted to two upright rods. By measurement MM' is found to be 12.8 ft., NN' to be 3.4 ft., and $M'N'$ to be 48.3 ft. Required the angle of the slope of MN and the distance MN .



$$\tan M = \frac{48.3}{12.8 - 3.4} = \frac{48.3}{9.4}.$$

$$\sin M = \frac{48.3}{MN}.$$

$$\log 48.3 = 1.68395$$

$$\text{colog } 9.4 = \frac{9.02687 - 10}{}$$

$$\log \tan M = 10.71082$$

$$M = 78^{\circ} 59' 13''.$$

$$90^{\circ} - M = 11^{\circ} 47''.$$

$$MN = \frac{48.3}{\sin M}.$$

$$\log 48.3 = 1.68395$$

$$\text{colog } \sin M = 0.00807$$

$$\log MN = 1.69202$$

$$MN = 49.206.$$

The angle of slope is $11^{\circ} 47''$ and the distance MN is 49.206 ft.

13. A wire stay is fastened to a telegraph pole 6.8 ft. from the ground and is stretched tightly so as to reach the ground 5.2 ft. from the foot of the pole. What angle does the wire stay make with the ground?

$$a = 6.8; b = 5.2.$$

$$\tan A = \frac{a}{b}.$$

$$\log a = 0.83251$$

$$\text{colog } b = \frac{9.28400 - 10}{}$$

$$\log \tan A = 10.11651$$

$$A = 52^{\circ} 35' 42''.$$

The wire makes an angle of $52^{\circ} 35' 42''$ with the ground.

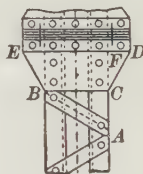
14. The top of a conical tent is 8 ft. 7 in. above the ground, and the diameter of the base is 9 ft. 8 in. Find the inclination of the side of the tent to the horizontal. Check the result by drawing the figure to scale and measuring the angle with a protractor.

$$\begin{aligned}\tan A &= \frac{h}{\frac{1}{2}c} & h &= 8 \text{ ft. } 7 \text{ in.} = 103 \text{ in.} \\ & & c &= 9 \text{ ft. } 8 \text{ in.} = 116 \text{ in.} \\ \log h &= 2.01284 \\ \text{colog } \frac{1}{2}c &= 8.23657 - 10 \\ \log \tan A &= 10.24941 \\ A &= 60^\circ 36' 58''.\end{aligned}$$

The inclination of the side of the tent to the horizontal is $60^\circ 36' 58''$.

15. In this piece of iron construction work $BC = 11$ in. and AB makes an angle of 30° with BC . What is the length of AC ?

$$\begin{aligned}\tan B &= \frac{AC}{BC} \\ AC &= BC \tan B. \\ \log BC &= 1.04139 \\ \log \tan B &= 9.76144 - 10 \\ \log AC &= 0.80283 \\ AC &= 6.3509.\end{aligned}$$



The length of AC is 6.3509 in.

16. In Ex. 15 it is also known that BE and CD are each 9 in. long and make angles of 60° with BC produced. What is the length of ED ?

$$\begin{aligned}ED &= 2x + BC. \\ \cos E &= \frac{x}{BE}. \\ x &= BE \cos E. \\ &= 9 \times \frac{1}{2} \\ &= 4.50. \\ 2x &= 9. \\ ED &= 9 \text{ in.} + 11 \text{ in.} \\ &= 20 \text{ in.}\end{aligned}$$

The length of ED is 20 in.

17. From the conditions given in Ex. 16, find the length of CF .

$$\sin D = \frac{CF}{CD}$$

$$CF = CD \sin D.$$

$$\log CD = 0.95424$$

$$\log \sin D = \underline{9.93753 - 10}$$

$$\log CF = 0.89177$$

$$CF = 7.7942.$$

The length of CF is 7.7942 in.

19. In constructing the spire represented in the figure below it is planned to have $AB = 42$ ft. and $PM = 92$ ft. What angle of slope must the builders give to AP ?

$$\tan A = \frac{PM}{AM}$$

$$\log PM = 1.96379$$

$$\text{colog } AM = \underline{8.67778 - 10}$$

$$\log \tan A = 10.64157$$

$$A = 77^\circ 8' 31''.$$

The angle of slope is $77^\circ 8' 31''$.

20. In Ex. 19 find the length of AP and find the angle P .

$$\frac{PM}{AP} = \sin A.$$

$$AP = \frac{PM}{\sin A}.$$

$$\log AP = \log PM + \text{colog } \sin A.$$

$$\log PM = 1.96379$$

$$\text{colog } \sin A = \underline{0.01103}$$

$$\log AP = 1.97482$$

$$AP = 94.368.$$

$$\frac{1}{2} P = (90^\circ - A).$$

$$P = 2(90^\circ - A).$$

$$P = 25^\circ 42' 58''.$$

The length of AP is 94.368 ft., and angle P is $25^\circ 42' 58''$.

18. The base of a rectangle is $14\frac{5}{8}$ in., and the diagonal is $19\frac{1}{8}$ in. What angle does the diagonal make with the base? Check the result by drawing the figure to scale and measuring the angle with a protractor.

$$\cos A = \frac{b}{c}.$$

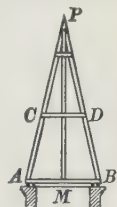
$$\log b = 1.16510$$

$$\text{colog } c = \underline{8.71840 - 10}$$

$$\log \cos A = 9.88350$$

$$A = 40^\circ 7' 6''.$$

The diagonal makes an angle of $40^\circ 7' 6''$ with the base.



21. In the figure of Ex. 19 the brace CD is put in 38 ft. above AB . What is its length?

$$PO = 92 \text{ ft.} - 38 \text{ ft.} = 54 \text{ ft.}$$

$$\frac{1}{2} P = 12^\circ 51' 29''.$$

$$CO = \frac{1}{2} CD.$$

$$\tan \frac{1}{2} P = \frac{CO}{PO}.$$

$$CO = PO \tan \frac{1}{2} P.$$

$$\log PO = 1.73239$$

$$\log \tan \frac{1}{2} P = \underline{9.35843 - 10}$$

$$\log CO = 1.09082$$

$$CO = 12.326.$$

The length of CD is 24.652 ft.

22. The spire of Ex. 19 rests on a tower. A man standing on the ground at a distance of 400 ft. from the base of the tower observes the angle of elevation of P to be $25^\circ 38'$, the instrument being 5 ft. above the ground. What is the height of P above the ground?

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

$$\log b = 2.60206$$

$$\log \tan A = 9.68109 - 10$$

$$\log a = 2.28315$$

$$a = 191.93.$$

$$191.93 \text{ ft.} + 5 \text{ ft.} = 196.93 \text{ ft.}$$

The height of P above the ground is 196.93 ft.

23. When the angle of elevation of the sun is 38.4° , what is the length of the shadow of a tower 175 ft. high?

$$\tan A = \frac{a}{b}.$$

$$b = \frac{a}{\tan A}.$$

$$\log a = 2.24304$$

$$\text{colog } \tan A = 0.10095$$

$$\log b = 2.34399$$

$$b = 220.8.$$

The length of the shadow is 220.8 ft.

24. Two men, M and N , 3200 ft. apart, observe an aeroplane A at the same instant, and at a time when the plane MNA is vertical. The angle of elevation at M is $41^\circ 27'$ and the angle at N is $61^\circ 42'$. Required AB , the height of the aeroplane.

$$\cot M = \frac{MB}{h}.$$

$$\cot N = \frac{BN}{h}.$$

$$h \cot M = MB.$$

$$h \cot N = BN.$$

$$h \cot M + h \cot N = MB + BN = MN.$$

$$h = \frac{MN}{\cot M + \cot N}.$$

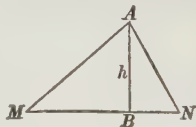
$$\log MN = 3.50515$$

$$\text{colog } (\cot M + \cot N) = 9.77710 - 10$$

$$\log h = 3.28225$$

$$h = 1915.3.$$

The height of the aeroplane is 1915.3 ft.



25. A kite string 475 ft. long makes an angle of elevation of $49^{\circ} 40'$. Assuming the string to be straight, what is the altitude of the kite?

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log c = 2.67669$$

$$\log \sin A = \underline{9.88212 - 10}$$

$$\log a = 2.55881$$

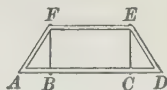
$$a = 362.09.$$

The altitude of the kite is 362.09 ft.

26. A steel bridge has a truss $ADEF$ in which it is given that $AD = 20$ ft., $BF = 6$ ft. 8 in., and $FE = 12$ ft., as shown in the figure. Required the angle of slope which AF makes with AD .

$$\frac{1}{2}(20 - 12) = 4 = AB.$$

$$\tan A = \frac{BF}{AB}.$$



$$\log BF = 0.82391$$

$$\text{colog } AB = \underline{9.39794 - 10}$$

$$\log \tan A = 10.22185$$

$$A = 59^{\circ} 2' 10''.$$

The angle of slope is $59^{\circ} 2' 10''$.

27. Two tangents are drawn from a point P to a circle and contain an angle of 37.4° . The radius of the circle is 5 in. Find the length of each tangent and the distance of P from the center.

$$\angle P = 37^{\circ} 24'.$$

$$\frac{1}{2}P = 18^{\circ} 42'.$$

$$\sin \frac{1}{2}P = \frac{r}{PO}.$$

$$PO = \frac{r}{\sin \frac{1}{2}P}.$$

$$\log r = 0.69897$$

$$\text{colog } \sin \frac{1}{2}P = \underline{0.49402}$$

$$\log PO = 1.19299$$

$$PO = 15.595.$$

Let PA = the length of the tangent.

$$\tan \frac{1}{2}P = \frac{r}{PA}.$$

$$PA = \frac{r}{\tan \frac{1}{2}P}.$$

$$\log r = 0.69897$$

$$\text{colog } \tan \frac{1}{2}P = \underline{0.47047}$$

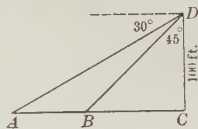
$$\log PA = 1.16944$$

$$PA = 14.772.$$

$$PA = PB.$$

The length of each tangent is 14.772 in. and the distance of P from the center is 15.595 in.

28. From the top of a cliff 95 ft. high, the angles of depression of two boats at sea are observed, by the aid of an instrument 5 ft. above the ground, to be 45° and 30° respectively. The boats are in a straight line with a point at the foot of the cliff directly beneath the observer. What is the distance between the boats?



$$\begin{aligned}\tan 45^\circ &= \frac{BC}{100} \\ BC &= 100. \\ \tan 60^\circ &= \frac{AC}{100} \\ AC &= 100 \tan 60^\circ. \\ \log AC &= \log 100 + \log \tan 60^\circ. \\ \log 100 &= 2.00000 \\ \log \tan 60^\circ &= \frac{10.23856 - 10}{10} \\ \log AC &= \frac{2.23856}{10} \\ AC &= 173.21. \\ AB &= 173.21 - BC \\ &= 173.21 - 100 \\ &= 73.21.\end{aligned}$$

The distance between the boats is 73.21 ft.

29. A carpenter's square BCA is held against the vertical stick BD resting on a sloping roof AD , as in the figure. It is found that $AC = 24$ in. and $CD = 11.5$ in. Find the angle of slope of the roof with the horizontal.

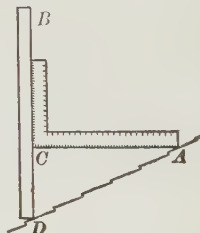
$$\begin{aligned}\tan A &= \frac{CD}{AC} \\ \log CD &= 1.06070 \\ \text{colog } AC &= \frac{8.61979 - 10}{10} \\ \log \tan A &= \frac{9.68049 - 10}{10} \\ A &= 25^\circ 36' 9''.\end{aligned}$$

The angle of slope is $25^\circ 36' 9''$.

30. In Ex. 29 find the length of AD .

$$\begin{aligned}\sin A &= \frac{CD}{AD} \\ AD &= \frac{CD}{\sin A} \\ \log CD &= 1.06070 \\ \text{colog } \sin A &= \frac{0.36439}{10} \\ \log AD &= \frac{1.42509}{10} \\ AD &= 26.613.\end{aligned}$$

The length of AD is 26.613 in.



31. A man 6 ft. tall stands 4 ft. 9 in. from a street lamp. If the length of his shadow is 19 ft., how high is the light above the street?

$$\begin{aligned}\tan A &= \frac{6}{19}. & a &= 23\frac{3}{4} \tan A. \\ \log 6 &= 0.77815 & \log 23\frac{3}{4} &= 1.37566 \\ \text{colog } 19 &= 8.72125 - 10 & \log \tan A &= 9.49940 - 10 \\ \log \tan A &= 9.49940 & \log a &= 0.87506 \\ A &= 17^\circ 31' 33''. & a &= 7.5. \\ \tan A &= \frac{a}{b} = \frac{a}{19 + 4\frac{3}{4}}. & \text{The lamp is } 7.5 \text{ ft. above the street.}\end{aligned}$$

32. The shadow of a city building is observed to be 100 ft. long, and at the same time the shadow of a lamp-post 9 ft. high is observed to be 5.2 ft. long. Find the angle of elevation of the sun and the height of the building.

$$\begin{aligned}\tan A &= \frac{a}{b} = \frac{9}{5.2}. & \tan A &= \frac{a'}{b'}. \\ \log a &= 0.95424 & a' &= b' \tan A. \\ \text{colog } b &= 9.28400 - 10 & \log b' &= 2.00000 \\ \log \tan A &= 10.23824 & \log \tan A &= \frac{10.23824 - 10}{2.23824} \\ A &= 59^\circ 58' 54''. & \log a' &= 2.23824 \\ & & a' &= 173.08.\end{aligned}$$

The angle of elevation is $59^\circ 58' 54''$ and the height of the building is 173.08 ft.

33. A man 5 ft. 10 in. tall walks along a straight line that passes at a distance of 2 ft. 9 in. from a street light. If the light is 9 ft. 6 in. above the ground, find the length of the man's shadow when his distance from the point on his path that is nearest to the lamp is 3 ft. 8 in.

In the figure, DE represents the man, CB the height of the light, MN the line of walk.

$$CD = \sqrt{(2\frac{3}{4})^2 + (3\frac{2}{3})^2} = 4\frac{7}{12}.$$

$$\tan A = \frac{5\frac{5}{6}}{AD}.$$

But

$$\tan A = \frac{9\frac{1}{2}}{4\frac{7}{12} + AD}.$$

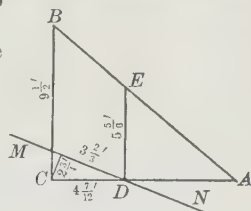
$$\therefore \frac{5\frac{5}{6}}{AD} = \frac{9\frac{1}{2}}{4\frac{7}{12} + AD} \text{ and } \frac{35}{6AD} = \frac{114}{55 + 12AD}.$$

$$AD = \frac{1925}{264}.$$

$$\begin{aligned}\log 1925 &= 3.28443 \\ \text{colog } 264 &= 7.57840 - 10 \\ \log AD &= 0.86283\end{aligned}$$

$$AD = 7.2917.$$

The length of the man's shadow is 7.2917 ft.



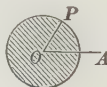
34. A man on a bridge 35 ft. above the stream, using an instrument 5 ft. high, sees a rowboat at an angle of depression of $27^{\circ}30'$. If the boat is approaching at the rate of $2\frac{3}{4}$ mi. an hour, in how many seconds will it reach the bridge?

$$\begin{aligned}\text{time} &= \frac{\text{distance}}{\text{rate per second}}. \\ \text{rate per second} &= \frac{2.75 \times 5280}{3600}. \\ \text{distance} &= \frac{40}{\tan 27^{\circ}30'}. \\ \text{time} &= \frac{40}{\tan 27^{\circ}30'} \times \frac{3600}{2.75 \times 5280}. \\ \log 40 &= 1.60206 \\ \log 3600 &= 3.55630 \\ \text{colog } \tan 27^{\circ}30' &= 0.28352 \\ \text{colog } 2.75 &= 9.56067 - 10 \\ \text{colog } 5280 &= 6.27737 - 10 \\ \log \text{ time} &= 1.27992 \\ \text{time} &= 19.051 \text{ sec.}\end{aligned}$$

The time required to reach the bridge is 19.051 sec.

35. A shaft O , of diameter 4 in., makes 480 revolutions per minute. If the point P starts on the horizontal line OA , how far is it above OA after $\frac{1}{8}$ of a second?

$$\begin{aligned}\frac{480}{60} &= 8, \text{ number of revolutions per second.} \\ \frac{1}{8} \times 8 &= \frac{1}{6}, \text{ number of revolutions in } \frac{1}{8} \text{ second.} \\ \frac{1}{6} \times 360^{\circ} &= 60^{\circ}. \\ \sin 60^{\circ} &= \frac{x}{2}. \\ x &= 2 \sin 60^{\circ}. \\ \log 2 &= 0.30103 \\ \log \sin 60^{\circ} &= 9.93753 - 10 \\ \log x &= 0.23856 \\ x &= 1.732.\end{aligned}$$



The point P is 1.732 in. above OA .

36. Assuming the earth to be a sphere with radius 3957 mi., find the radius of the circle of latitude which passes through a place in latitude $47^{\circ}27'10''$ N.

$$\begin{aligned}\cos A &= \frac{b}{c}. \\ b &= c \cos A.\end{aligned}$$

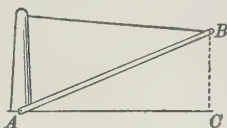
$$\begin{aligned}\log c &= 3.59737 \\ \log \cos A &= \frac{9.83008 - 10}{} \\ \log b &= \frac{3.42745}{} \\ b &= 2675.8.\end{aligned}$$

The radius of the required circle is 2675.8 mi.

37. When a hoisting crane AB , 28 ft. long, makes an angle of 23° with the horizontal AC , what is the length of AC ? Suppose that the angle CAB is doubled, what is then the length of AC ?

$$\begin{aligned}\cos A &= \frac{AC}{AB} & A' &= 2 \times 23^\circ = 46^\circ. \\ AC &= AB \cos A. & \cos A' &= \frac{AC}{AB}.\end{aligned}$$

$$\begin{aligned}\log AB &= 1.44716 & \log AB &= 1.44716 \\ \log \cos A &= \frac{9.96403 - 10}{} & \log \cos A' &= \frac{9.84177 - 10}{} \\ \log AC &= 1.41119 & \log AC &= 1.28893 \\ AC &= 25.775. & AC &= 19.45.\end{aligned}$$



The length of AC is 25.775 ft. and 19.45 ft.

38. In Ex. 37 find the length of BC in each of the two cases.

$$\begin{aligned}\sin A &= \frac{BC}{AB} & \sin A' &= \frac{BC}{AB} \\ BC &= AB \sin A. & \log AB &= 1.44716 \\ \log AB &= 1.44716 & \log \sin A' &= \frac{9.85693 - 10}{} \\ \log \sin A &= \frac{9.59188 - 10}{} & \log BC &= 1.30409 \\ \log BC &= 1.03904 & BC &= 20.141. \\ BC &= 10.941.\end{aligned}$$

The length of BC is 10.941 ft. and 20.141 ft.

39. Wishing to measure the distance AB , a man swings a 100-foot tape line about B , describing an arc on the ground, and then does the same about A . The arcs intersect at C , and the angle ACB is found to be $32^\circ 10'$. What is the length of AB ?

$$\begin{aligned}\sin \frac{1}{2} C &= \frac{\frac{1}{2} c}{a} \\ \frac{1}{2} c &= a \sin \frac{1}{2} C. \\ \log a &= 2.00000 \\ \log \sin \frac{1}{2} C &= \frac{9.44253 - 10}{} \\ \log \frac{1}{2} c &= 1.44253 \\ \frac{1}{2} c &= 27.703. \\ c &= 55.406\end{aligned}$$



The length of AB is 55 406 ft.

40. From the top of a mountain 15,250 ft. high, overlooking the sea to the south, over how many minutes of latitude can a person see if he looks southward? Use the assumption stated in Ex. 36.

Let A be the top of the mountain, B the center of the earth. $BC = 3957$ mi.

$$1^{\circ} 52' 50'' = \text{height of mt. in mi.}$$

$$\log 15250 = 4.18327$$

$$\text{colog } 5280 = 6.27737 - 10$$

$$0.46064$$

$$h = 2.8883 \text{ miles,}$$

$$BA = 3957 + 2.9 = 3959.9 \text{ mi.}$$

$$\cos B = \frac{BC}{BA}.$$

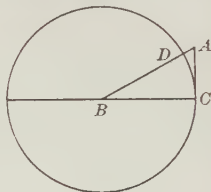
$$\log 3957 = 3.59737$$

$$\text{colog } 3959.9 = 6.40231 - 10$$

$$\log \cos B = 9.99968 - 10$$

$$B = 2^{\circ} 11' \text{ to } 2^{\circ} 12'$$

$$= 131' \text{ to } 132'.$$



41. The length of each blade of a pair of shears, from the screw to the points, is $5\frac{1}{4}$ in. When the points of the open shears are $3\frac{7}{8}$ in. apart, what angle do the blades make with each other?

$$\sin \frac{1}{2} C = \frac{\frac{1}{2} c}{a}.$$

$$\log \frac{1}{2} c = 0.28724$$

$$\text{colog } a = 9.27984 - 10$$

$$\log \sin \frac{1}{2} c = 9.56708 - 10$$

$$\frac{1}{2} c = 21^{\circ} 39' 24''.$$

$$c = 43^{\circ} 18' 48''.$$

The blades make an angle of $43^{\circ} 18' 48''$.

42. In Ex. 41 how far apart are the points when the blades make an angle of $28^{\circ} 45'$ with each other?

$$\frac{1}{2} C = 14^{\circ} 22' 30''.$$

$$\sin \frac{1}{2} C = \frac{\frac{1}{2} c}{a}.$$

$$\frac{1}{2} c = a \sin \frac{1}{2} C.$$

$$\log a = 0.72016$$

$$\log \sin \frac{1}{2} C = 9.39492 - 10$$

$$\log \frac{1}{2} c = 0.11508.$$

$$\frac{1}{2} c = 1.3034.$$

$$c = 2.6068.$$

The points are 2.6068 in. apart.

43. The wheel here represented has eight spokes, each being 19 in. long. How far is it from A to B ? from B to D ?

$$\frac{360^\circ}{8} = 45^\circ. \quad \frac{1}{2} AOB = 22^\circ 30'.$$

$$\sin \frac{1}{2} AOB = \frac{\frac{1}{2} AB}{OA}.$$

$$\frac{1}{2} AB = OA \sin \frac{1}{2} AOB.$$

$$\log OA = 1.27875$$

$$\log \sin \frac{1}{2} AOB = 9.58284 - 10$$

$$\log \frac{1}{2} AB = 0.86159$$

$$\frac{1}{2} AB = 7.271.$$

$$AB = 14.542.$$

$$\angle DOB = 90^\circ.$$

$$\sin 45^\circ = \frac{\frac{1}{2} DB}{OD}.$$

$$\frac{1}{2} DB = OD \sin 45^\circ.$$

$$\log \frac{1}{2} DB = \log 19 + \log \sin 45^\circ.$$

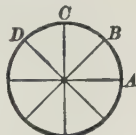
$$\log OD = 1.27875$$

$$\log \sin 45^\circ = 9.84949$$

$$\log \frac{1}{2} DB = 1.12824$$

$$\frac{1}{2} DB = 13.435.$$

$$DB = 26.870.$$



The distance from A to B is 14.542 in. and from B to D 26.87 in.

44. The angle of elevation of a balloon from a station directly south of it is 60° . From a second station lying 5280 ft. directly west of the first one the angle of elevation is 45° . The instrument being 5 ft. above the level of the ground, what is the height of the balloon?

Let $AC + 5$ ft. be the height of the balloon, and B and B' the positions of the observers. $BB' = 1$ mile.

$$\frac{b}{a} = \tan 60^\circ = \sqrt{3}, \quad a = \frac{b}{\sqrt{3}}. \quad b = \sqrt{\frac{b^2}{3} + 1^2}.$$

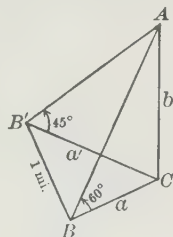
$$a' = \sqrt{\frac{b^2}{3} + 1^2}.$$

$$2b^2 = 3.$$

$$b = \frac{1}{2} \sqrt{6}.$$

$$\tan 45^\circ = \frac{b}{a'}$$

$$= \frac{b}{\sqrt{\frac{b^2}{3} + 1^2}} = 1.$$



$$b + 5 \text{ ft.} = \frac{1}{2} \sqrt{6} \times 5280 \text{ ft.} + 5 \text{ ft.}$$

$$= (2640 \times 2.4495) \text{ ft.} + 5 \text{ ft.}$$

$$= 6466.7 \text{ ft.} + 5 \text{ ft.}$$

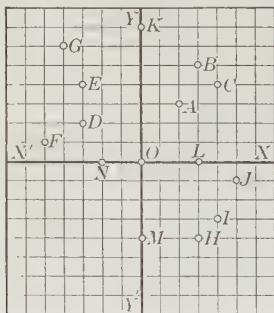
$$= 6471.7 \text{ ft.}$$

The height of the balloon is 6471.7 ft.

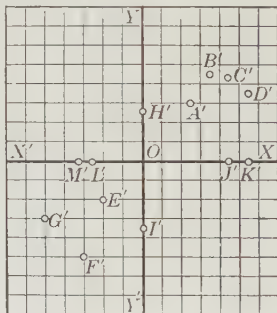
Exercise 35. Distances from the Origin (Page 80)

Using squared paper, or measuring with a ruler, plot the following points.

1. (2, 3) A is the point.
2. (3, 5) B is the point.
3. (4, 4) C is the point.
4. $(2\frac{1}{2}, 3)$ A' is the point.
5. $(3\frac{1}{2}, 4\frac{1}{2})$ B' is the point.
6. $(4\frac{1}{4}, 4\frac{1}{4})$ C' is the point.
7. $(5\frac{1}{2}, 3\frac{1}{2})$ D' is the point.



15. (3, -4) H is the point.
16. (4, -3) I is the point.
17. (5, -1) J is the point.
18. (0, 7) K is the point.
19. (3, 0) L is the point.
20. (0, -4) M is the point.
21. (-2, 0) N is the point.



8. (-3, 2) D is the point.
9. (-3, 4) E is the point.
10. (-5, 1) F is the point.
11. (-4, 6) G is the point.
12. (-2, -2) E' is the point.
13. (-3, -5) F' is the point.
14. (-5, -3) G' is the point.

22. (0, 0) O is the point.
23. $(0, 2\frac{1}{2})$ H' is the point.
24. $(0, -3\frac{1}{2})$ I' is the point.
25. $(4\frac{1}{2}, 0)$ J' is the point.
26. $(5\frac{1}{2}, 0)$ K' is the point.
27. $(-2\frac{1}{2}, 0)$ L' is the point.
28. $(-3\frac{1}{4}, 0)$ M' is the point.

Find the distance of each of the following points from the origin :

29. (6, 8).

$$r = \sqrt{6^2 + 8^2} = \sqrt{100} = 10.$$

30. (9, 12).

$$r = \sqrt{9^2 + 12^2} = \sqrt{225} = 15.$$

31. (5, 12).

$$r = \sqrt{5^2 + 12^2} = \sqrt{169} = 13.$$

32. $(1\frac{1}{2}, 2)$.

$$r = \sqrt{(1\frac{1}{2})^2 + 2^2} = \sqrt{\frac{25}{4}} = 2\frac{1}{2}.$$

33. $(\frac{3}{4}, 1)$.

$$r = \sqrt{(\frac{3}{4})^2 + 1^2} = \sqrt{\frac{25}{16}} = 1\frac{1}{4}.$$

34. $(2\frac{1}{4}, 3)$.

$$r = \sqrt{(2\frac{1}{4})^2 + 3^2} = \sqrt{\frac{25}{4}} = 3\frac{3}{4}.$$

35. $(2, \sqrt{5})$.

$$r = \sqrt{2^2 + (\sqrt{5})^2} = \sqrt{9} = 3.$$

38. $(0, 7)$.

$$r = \sqrt{0^2 + 7^2} = \sqrt{49} = 7.$$

36. $(-3, 4)$.

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$$

39. $(5, 0)$.

$$r = \sqrt{5^2 + 0^2} = \sqrt{25} = 5.$$

37. $(0, 0)$.

$$r = \sqrt{0^2 + 0^2} = \sqrt{0} = 0.$$

40. $(-12, -9)$.

$$r = \sqrt{(-12)^2 + (-9)^2} = \sqrt{225} = 15$$

41. Find the distance from $(3, 2)$ to $(-2, 3)$.

$$r = \sqrt{3^2 + 2^2} = \sqrt{13}; \quad r' = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

$$\text{Distance} = \sqrt{(\sqrt{13})^2 + (\sqrt{13})^2} = \sqrt{26} = 5.10.$$

42. Find the distance from $(-3, -2)$ to $(2, -3)$.

$$r = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}; \quad r' = \sqrt{2^2 + (-3)^2} = \sqrt{13}.$$

$$\text{Distance} = \sqrt{(\sqrt{13})^2 + (\sqrt{13})^2} = \sqrt{26} = 5.10.$$

43. Find the distance from $(4, 1)$ to $(-4, -1)$.

$$r = \sqrt{4^2 + 1^2} = \sqrt{17} = 4.12; \quad r' = \sqrt{(-4)^2 + (-1)^2} = \sqrt{17} = 4.12.$$

$$\text{Distance} = 2 \times 4.12 = 8.24.$$

44. Find the distance from $(0, 3)$ to $(-3, 0)$.

$$r = \sqrt{0^2 + 3^2} = \sqrt{9} = 3; \quad r' = \sqrt{(-3)^2 + 0^2} = \sqrt{9} = 3.$$

$$\text{Distance} = \sqrt{3^2 + 3^2} = \sqrt{18} = 4.24.$$

45. A point moves to the right 7 in., up 4 in., to the right 10 in., and up $18\frac{2}{3}$ in. How far is it then from the starting point?

$$18\frac{2}{3} + 4 = 22\frac{2}{3}; \quad 10 + 7 = 17.$$

$$r = \sqrt{17^2 + (22\frac{2}{3})^2} = \sqrt{\frac{7225}{9}} = \frac{85}{3} = 28\frac{1}{3}.$$

The point is $28\frac{1}{3}$ in. from the starting point.

46. A point moves to the right 9 in., up 5 in., to the left 4 in., and up 3 in. How far is it then from the starting point?

$$9 - 4 = 5; \quad 5 + 3 = 8.$$

$$r = \sqrt{5^2 + 8^2} = \sqrt{89} = 9.43.$$

The point is 9.43 in. from the starting point.

47. Find the distance from $(-\frac{1}{2}, \frac{1}{2}\sqrt{3})$ to $(\frac{1}{2}, -\frac{1}{2}\sqrt{3})$.

$$r = \sqrt{(-\frac{1}{2})^2 + (\frac{1}{2}\sqrt{3})^2} = \sqrt{1} = 1; \quad r' = \sqrt{(\frac{1}{2})^2 + (-\frac{1}{2}\sqrt{3})^2} = \sqrt{1} = 1$$

$$\text{Distance} = 2 \times 1 = 2.$$

48. A triangle is formed by joining the points $(1, 0)$, $(-\frac{1}{2}, \frac{1}{2}\sqrt{3})$, and $(-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$. Find the perimeter of the triangle. Draw the figure to scale.

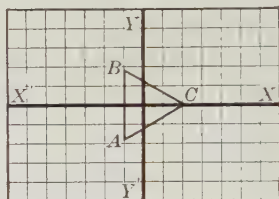
Let 2 spaces = 1 unit.

$$AB = 2 \times \frac{1}{2} \sqrt{3} = \sqrt{3}.$$

$$AC = \sqrt{(\frac{1}{2}\sqrt{3})^2 + (1\frac{1}{2})^2} = \sqrt{\frac{1^2}{4} + \frac{9}{4}} = \sqrt{3}.$$

$$BC = \sqrt{3}.$$

The perimeter of the triangle is $3\sqrt{3}$.



49. Find the area of the triangle in Ex. 48.

$$S = \frac{1}{2}ch = \frac{1}{2} \times \sqrt{3} \times (1 + \frac{1}{2}) = \frac{3}{4}\sqrt{3}.$$

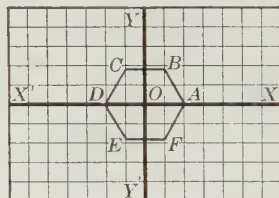
50. A hexagon is formed by joining in order the points $(1, 0)$, $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $(-\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $(-1, 0)$, $(-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$, $(\frac{1}{2}, -\frac{1}{2}\sqrt{3})$, and $(1, 0)$. Is the figure a regular hexagon? Prove it.

Let 2 spaces = 1 unit.

$$BC = \frac{1}{2} + \frac{1}{2} = 1 = EF.$$

$$BA = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2}\sqrt{3})^2} = \sqrt{1} = 1 = AF.$$

$$DE = \sqrt{(-\frac{1}{2})^2 + (-\frac{1}{2}\sqrt{3})^2} = \sqrt{1} = 1 = CD.$$



Draw OB.

$$OB = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2}\sqrt{3})^2} = 1.$$

$$\therefore OB = AB, \text{ but } OA = 1.$$

$$\therefore OB = OA.$$

In the same way it may be proved that all the lines from O to the six angles are equal to each other. They are then the radii of a circumscribed circle and the hexagon must be regular.

Each side of the hexagon is equal to 1, and each angle to 120° . Therefore the hexagon is regular.

51. A polygon is formed by joining in order the points $(1, 0)$, $(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$, $(0, 1)$, $(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$, $(-1, 0)$, $(-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$, $(0, -1)$, $(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$, and $(1, 0)$. Draw the figure, state the kind of polygon, and find its area.

Let 4 spaces = 1 unit.

The figure is a regular octagon.

$$AB = \sqrt{\left(1 - \frac{1}{2}\sqrt{2}\right)^2 + \left(\frac{1}{2}\sqrt{2}\right)^2}$$

$$= \sqrt{2} - \sqrt{2} = \sqrt{0.586} = BC.$$

Each angle is equal to 135° .

$$\angle OAB = 67^\circ 30'.$$

$$\sin 67^\circ 30' = \frac{h}{OA} = \frac{h}{1}.$$

$$h = 0.9239.$$

$$S = \frac{1}{2}hp$$

$$= \frac{1}{2} \times 0.9239 \times 8\sqrt{0.586}$$

$$= 0.9239 \times 4\sqrt{0.586}$$

$$= 0.9239 \times 3.062.$$

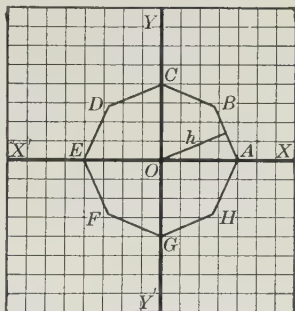
$$\log S = \log 0.9239 + \log 3.062.$$

$$\log 0.9239 = 9.96562 - 10$$

$$\log 3.062 = 0.48601$$

$$\log S = 0.45163$$

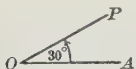
$$S = 2.829.$$



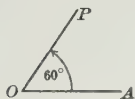
Exercise 36. Construction of Angles and Functions (Page 84)

Using the protractor, construct the following angles :

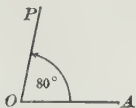
1. 30° .



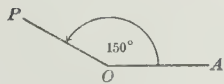
2. 60° .



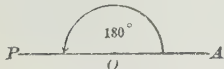
3. 80° .



4. 150° .



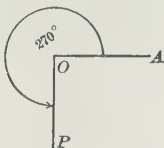
5. 180° .



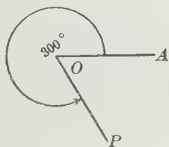
6. 200° .



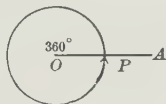
7. 270° .



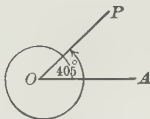
8. 300° .



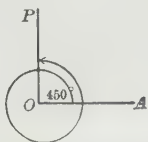
9. 360° .



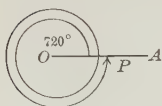
10. 405° .



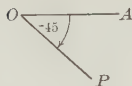
11. 450° .



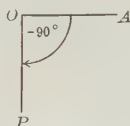
12. 720° .



13. -45° .



14. -90° .



15. -180° .



State the quadrants in which the terminal sides of the following angles lie :

16. 45° .

Quadrant I.

21. 315° .

Quadrant IV.

26. 765° .

Quadrant I.

17. 75° .

Quadrant I.

22. 390° .

Quadrant I.

27. 820° .

Quadrant II.

18. 120° .

Quadrant II.

23. 495° .

Quadrant II.

28. 930° .

Quadrant III.

19. 150° .

Quadrant II.

24. 570° .

Quadrant III.

29. 990° .

On OY' .

20. 210° .

Quadrant III.

25. 660° .

Quadrant IV.

30. 1080° .

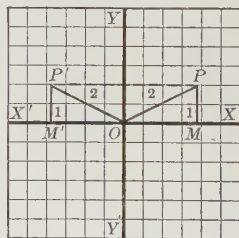
On OX .

Construct two angles A , given the following :

31. $\sin A = \frac{1}{2}$.

Let 2 spaces = 1 unit.

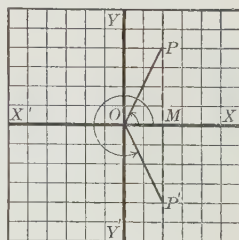
Draw a line parallel to XX' and 1 unit above it, and then rotate a line OP , 2 units long, about O until P rests upon this parallel. Angles XOP and XOP' are the required angles.



32. $\cos A = \frac{1}{2}$.

Let 2 spaces = 1 unit.

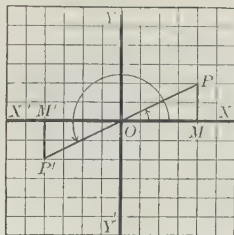
Draw a line parallel to YY' and 1 unit to the right of it, and then rotate a line OP , 2 units long, about O until P rests upon this parallel. Angles XOP and XOP' are the required angles.



33. $\tan A = \frac{1}{2}$.

Let 2 spaces = 1 unit.

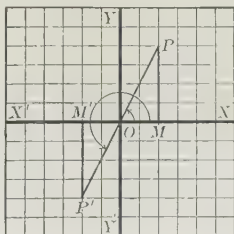
Draw an abscissa 2 and an ordinate 1. Then $\angle XOP$ has $\frac{1}{2}$ for its tangent. Similarly locate the point $(-2, -1)$ in quadrant III, and we have $\angle XOP'$ whose tangent is $\frac{1}{2}$.



34. $\cot A = \frac{1}{2}$.

Let 2 spaces = 1 unit.

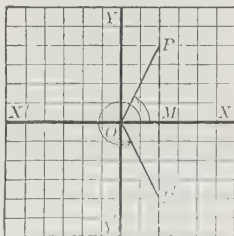
Draw an abscissa 1 and an ordinate 2. Then $\angle XOP$ has $\frac{1}{2}$ for its cotangent. Similarly locate the point $(-1, -2)$ in quadrant III, and we have $\angle XOP'$ whose cotangent is $\frac{1}{2}$.



35. $\sec A = 2$.

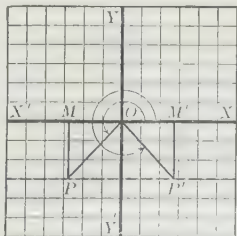
Let 2 spaces = 1 unit.

Draw a line parallel to YY' and 1 unit to the right of it, and then rotate a line OP , 2 units long, about O until P rests upon this parallel. Angles XOP and XOP' are the required angles.



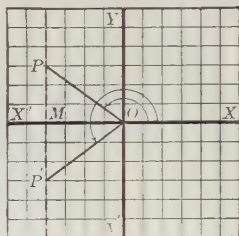
36. $\sin A = -\frac{3}{4}$.

Draw a line parallel to XX' and 3 units below it, and then rotate a line OP , 4 units long, about O until P rests upon this parallel. Angles XOP and XOP' are the required angles.



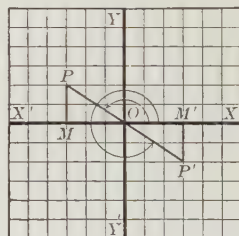
37. $\cos A = -\frac{4}{5}$.

Draw a line parallel to YY' and 4 units to the left of it, and then rotate a line OP , 5 units long, about O until P rests upon this parallel. Angles XOP and XOP' are the required angles.



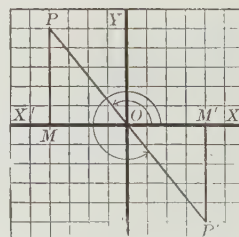
38. $\tan A = -\frac{2}{3}$.

Draw an abscissa -3 and an ordinate 2 . The $\angle XOP$ has $-\frac{2}{3}$ for its tangent. Similarly locate the point $(3, -2)$ in quadrant IV, and we have $\angle XOP'$ whose tangent is $-\frac{2}{3}$.



39. $\cot A = -\frac{4}{5}$.

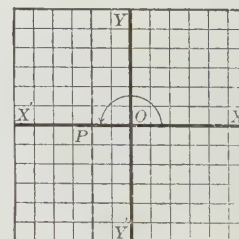
Draw an abscissa -4 and an ordinate 5 . The angle XOP has $-\frac{4}{5}$ for its cotangent. Similarly locate the point $(4, -5)$ in quadrant IV, and we have $\angle XOP'$ whose tangent is $-\frac{2}{3}$.



40. $\sec A = -1$.

Let 2 spaces = 1 unit.

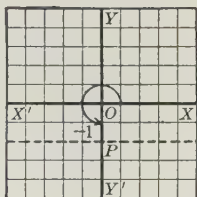
Draw a line parallel to YY' and 1 unit to the left of it, and then rotate a line OP , 1 unit long, about O until P rests upon this parallel. Angle XOP is the required angle.



41. $\sin A = -1$.

Let 2 spaces = 1 unit.

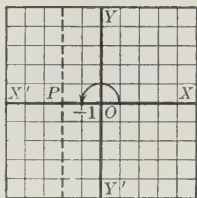
Draw a line parallel to XX' and 1 unit below it, and then rotate a line OP , 1 unit long, about O until P rests upon this parallel. Angle XOP is the required angle.



42. $\cos A = -1$.

Let 2 spaces = 1 unit.

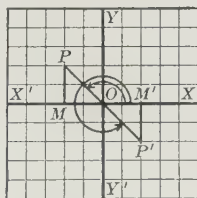
Draw a line parallel to YY' and 1 unit to the left of it, and then rotate a line OP , 1 unit long, about O until P rests upon this parallel. Angle XOP is the required angle.



43. $\tan A = -1$.

Let 2 spaces = 1 unit.

Draw an abscissa -1 and an ordinate 1 . The angle XOP has -1 for its tangent. Similarly locate the point $(1, -1)$ in quadrant IV, and we have $\angle XOP'$ whose tangent is -1 .



44. $\cot A = -1$.

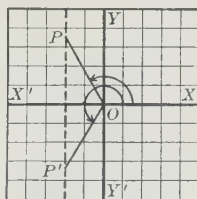
Draw an abscissa -1 and an ordinate 1 . The angle XOP has -1 for its cotangent. Similarly locate the point $(1, -1)$ in quadrant IV, and we have $\angle XOP'$, whose cotangent is -1 .

(See diagram, Prob. 43).

45. $\sec A = -2$.

Let 2 spaces = 1 unit.

Draw a line parallel to YY' and 1 unit to the left of it, and then rotate a line OP , 2 units long, about O until P rests upon this parallel. Angles XOP and XOP' are the required angles.



Given the following functions of angle A , construct the other functions:

46. $\sin A = \frac{3}{5}$.

$\sin A = \frac{3}{5}$.

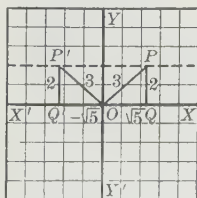
$\csc A = \frac{5}{3}$.

$\cos A = \pm \frac{\sqrt{5}}{5}$.

$\sec A = \pm \frac{5}{\sqrt{5}}$.

$\tan A = \pm \frac{2}{\sqrt{5}}$.

$\cot A = \pm \frac{\sqrt{5}}{2}$.



47. $\cos A = \frac{3}{4}$.

$$\sin A = \pm \frac{\sqrt{7}}{4}.$$

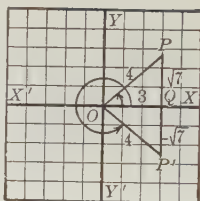
$$\cos A = \frac{3}{4}.$$

$$\tan A = \pm \frac{\sqrt{7}}{3}.$$

$$\csc A = \pm \frac{4}{\sqrt{7}}.$$

$$\sec A = \frac{4}{3}.$$

$$\cot A = \pm \frac{3}{\sqrt{7}}.$$



48. $\tan A = \frac{4}{5}$.

$$\sin A = \pm \frac{4}{\sqrt{41}}.$$

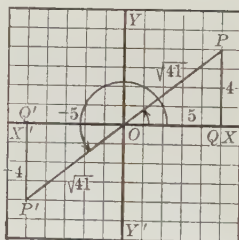
$$\cos A = \pm \frac{5}{\sqrt{41}}.$$

$$\tan A = \frac{4}{5}.$$

$$\csc A = \pm \frac{\sqrt{41}}{4}.$$

$$\sec A = \pm \frac{\sqrt{41}}{5}.$$

$$\cot A = \frac{5}{4}.$$



49. $\cot A = \frac{3}{8}$.

$$\sin A = \pm \frac{8}{\sqrt{73}}.$$

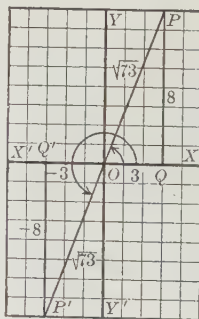
$$\cos A = \pm \frac{3}{\sqrt{73}}.$$

$$\tan A = \frac{8}{3}.$$

$$\csc A = \pm \frac{\sqrt{73}}{8}.$$

$$\sec A = \pm \frac{\sqrt{73}}{3}.$$

$$\cot A = \frac{3}{8}.$$



50. $\csc A = 3$.

Let 2 spaces = 1 unit.

$$\sin A = \frac{1}{3}.$$

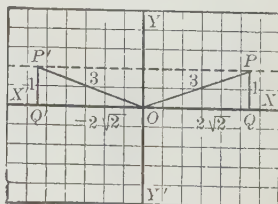
$$\cos A = \pm \frac{2}{3} \sqrt{2}.$$

$$\tan A = \pm \frac{1}{2} \sqrt{2}.$$

$$\csc A = 3.$$

$$\sec A = \pm \frac{3}{2} \sqrt{2}.$$

$$\cot A = \pm 2 \sqrt{2}.$$



51. $\sin A = -\frac{4}{5}$.

$\sin A = -\frac{4}{5}$.

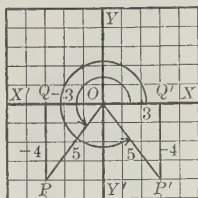
$\cos A = \mp \frac{3}{5}$.

$\tan A = \pm \frac{4}{3}$.

$\csc A = -\frac{5}{4}$.

$\sec A = \mp \frac{5}{3}$.

$\cot A = \pm \frac{3}{4}$.



52. $\cos A = -1$.

Let 2 spaces = 1 unit.

$\sin A = \pm 0$.

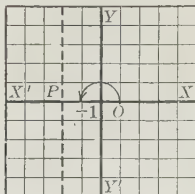
$\cos A = -1$.

$\tan A = \mp 0$.

$\csc A = \pm \infty$.

$\sec A = -1$.

$\cot A = \mp \infty$.

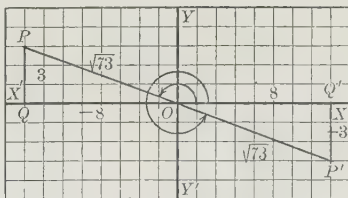


53. $\tan A = -\frac{3}{8}$.

$\sin A = \pm \frac{3}{\sqrt{73}}$ $\csc A = \pm \frac{\sqrt{73}}{3}$.

$\cos A = \mp \frac{8}{\sqrt{73}}$ $\sec A = \mp \frac{\sqrt{73}}{8}$.

$\tan A = -\frac{3}{8}$ $\cot A = -\frac{8}{3}$.



54. $\sec A = -2$.

Let 2 spaces = 1 unit.

$\sin A = \pm \frac{\sqrt{3}}{2}$.

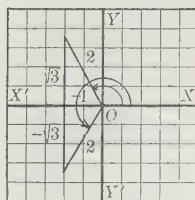
$\cos A = -\frac{1}{2}$.

$\tan A = \mp \sqrt{3}$.

$\csc A = \pm \frac{2}{\sqrt{3}}$.

$\sec A = -2$.

$\cot A = \mp \frac{1}{\sqrt{3}}$.



55. $\csc A = -1$.

Let 2 spaces = 1 unit.

$\sin A = -1$.

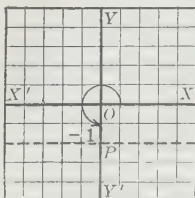
$\cos A = \mp 0$.

$\tan A = \pm \infty$.

$\csc A = -1$.

$\sec A = \mp \infty$.

$\cot A = \pm 0$.



56. $\sin A = -\frac{1}{2}$.

Let 2 spaces = 1 unit.

$$\sin A = -\frac{1}{2}.$$

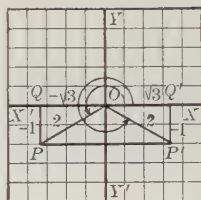
$$\cos A = \mp \frac{\sqrt{3}}{2}.$$

$$\tan A = \pm \frac{1}{\sqrt{3}}.$$

$$\csc A = -2.$$

$$\sec A = \mp \frac{2}{\sqrt{3}}.$$

$$\cot A = \pm \sqrt{3}.$$



57. $\cos A = -\frac{1}{2}$.

Let 2 spaces = 1 unit.

$$\sin A = \pm \frac{\sqrt{3}}{2}.$$

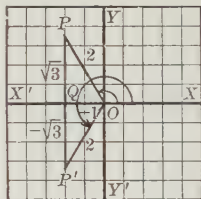
$$\cos A = -\frac{1}{2}.$$

$$\tan A = \mp \sqrt{3}.$$

$$\csc A = \pm \frac{2}{\sqrt{3}}.$$

$$\sec A = -2.$$

$$\cot A = \mp \sqrt{3}.$$



58. $\tan A = -\frac{1}{2}$.

Let 2 spaces = 1 unit.

$$\sin A = \pm \frac{1}{\sqrt{5}}.$$

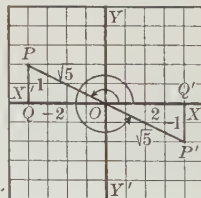
$$\cos A = \mp \frac{2}{\sqrt{5}}.$$

$$\tan A = -\frac{1}{2}.$$

$$\csc A = \pm \sqrt{5}.$$

$$\sec A = \mp \frac{\sqrt{5}}{2}.$$

$$\cot A = -2.$$



59. $\cot A = -\frac{1}{2}$.

Let 2 spaces = 1 unit.

$$\sin A = \pm \frac{2}{\sqrt{5}}.$$

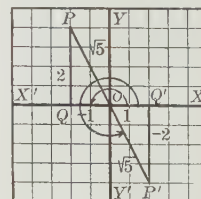
$$\cos A = \mp \frac{1}{\sqrt{5}}.$$

$$\tan A = -2.$$

$$\csc A = \pm \frac{\sqrt{5}}{2}.$$

$$\sec A = \mp \sqrt{5}.$$

$$\cot A = -\frac{1}{2}.$$



60. $\sec A = -2\frac{1}{2}$.

$$\sin A = \pm \frac{\sqrt{21}}{5}.$$

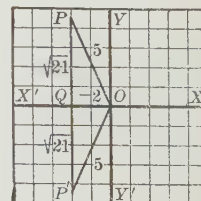
$$\cos A = -\frac{5}{2}.$$

$$\tan A = \mp \frac{\sqrt{21}}{2}.$$

$$\csc A = \pm \frac{5}{\sqrt{21}}.$$

$$\sec A = -2\frac{1}{2}.$$

$$\cot A = \mp \frac{2}{\sqrt{21}}.$$



61. If $\tan A = \sqrt{2}$, show that $\cot A$ is half as large. What are the values of $\sin A$, $\cos A$, $\sec A$, and $\csc A$?

$$\begin{aligned}\tan A &= \sqrt{2}. & \csc A &= \sqrt{1 + \cot^2 A} = \sqrt{1 + \left(\frac{1}{2}\sqrt{2}\right)^2} \\ \cot A &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2}. & &= \frac{1}{2}\sqrt{6}. \\ \sec A &= \sqrt{1 + \tan^2 A} = \sqrt{1 + 2} = \sqrt{3}. & \sin A &= \frac{1}{\csc A} = \frac{1}{\frac{\sqrt{6}}{2}} = \frac{1}{3}\sqrt{6}. \\ \cos A &= \frac{1}{\sec A} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.\end{aligned}$$

62. The product $2 \sin 45^\circ \cos 45^\circ$ is equal to the sine of what angle?

$$\begin{aligned}2 \sin 45^\circ \cos 45^\circ &= 2 \times \frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{2} = 1. \\ \sin 90^\circ &= 1.\end{aligned}$$

63. The product $2 \sin 30^\circ \cos 30^\circ$ is equal to the sine of what angle?

$$\begin{aligned}2 \sin 30^\circ \cos 30^\circ &= 2 \times \frac{1}{2} \times \frac{1}{2}\sqrt{3} = \frac{1}{2}\sqrt{3}. \\ \sin 60^\circ &= \frac{1}{2}\sqrt{3}.\end{aligned}$$

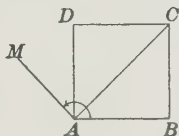
64. To the diagonal AC of a square $ABCD$ a perpendicular AM is drawn. Find the values of the six functions of angle BAM .

$$AC = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

$$\sin BAM = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}. \quad \cot BAM = \frac{-1}{1} = -1.$$

$$\cos BAM = \frac{-1}{\sqrt{2}} = -\frac{1}{2}\sqrt{2}. \quad \sec BAM = -\sqrt{2}.$$

$$\tan BAM = \frac{1}{-1} = -1. \quad \csc BAM = \sqrt{2}.$$



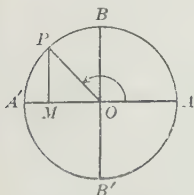
65. In the figure of Ex. 64, suppose AM rotates further, until it is in line with BA . What are then the six functions of angle BAM ?

$$\begin{aligned}\sin BAM &= 0. & \tan BAM &= 0. & \sec BAM &= -1. \\ \cos BAM &= -1. & \cot BAM &= \infty. & \csc BAM &= \infty.\end{aligned}$$

Exercise 37. Variations in the Functions (Page 88)

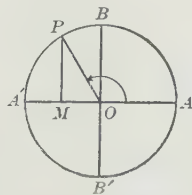
Represent the following functions by lines in a unit circle:

1. $\sin 135^\circ$.



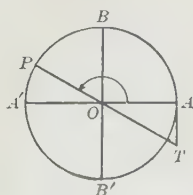
$$\sin 135^\circ = MP.$$

2. $\cos 120^\circ$.



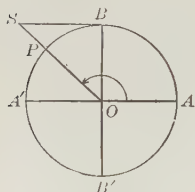
$$\cos 120^\circ = OM.$$

3. $\tan 150^\circ$.



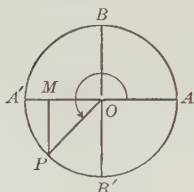
$$\tan 150^\circ = AT.$$

4. $\cot 135^\circ$.



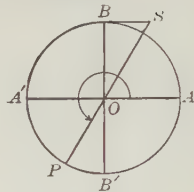
$\cot 135^\circ = BS.$

8. $\cos 225^\circ$.



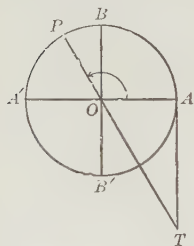
$\cos 225^\circ = OM.$

12. $\csc 240^\circ$.



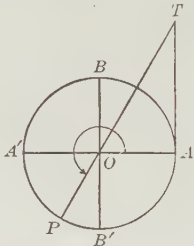
$\csc 240^\circ = OS.$

5. $\sec 120^\circ$.



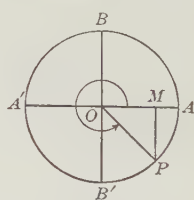
$\sec 120^\circ = OT.$

9. $\tan 240^\circ$.



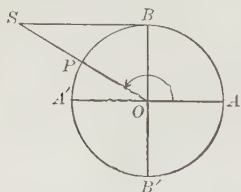
$\tan 240^\circ = AT.$

13. $\sin 300^\circ$.



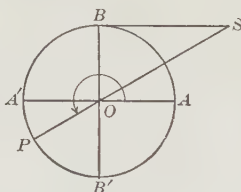
$\sin 300^\circ = MP.$

6. $\csc 150^\circ$.



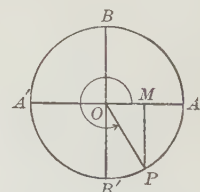
$\csc 150^\circ = OS.$

10. $\cot 210^\circ$.



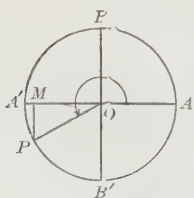
$\cot 210^\circ = BS.$

14. $\cos 315^\circ$.



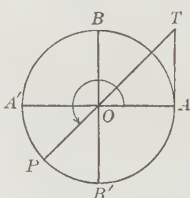
$\cos 315^\circ = OM$

7. $\sin 210^\circ$.



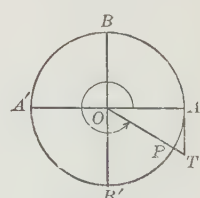
$\sin 210^\circ = MP.$

11. $\sec 225^\circ$.



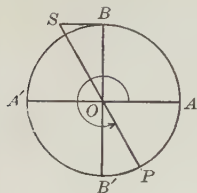
$\sec 225^\circ = OT.$

15. $\tan 330^\circ$.



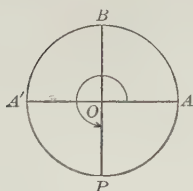
$\tan 330^\circ = AT.$

16. $\cot 300^\circ$.



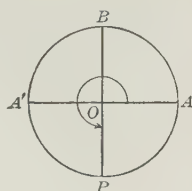
$\cot 300^\circ = BS.$

19. $\sin 270^\circ$.



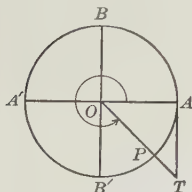
$\sin 270^\circ = OP.$

22. $\cot 270^\circ$.



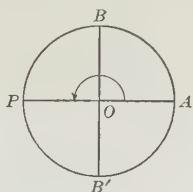
$\cot 270^\circ = 0.$

17. $\sec 315^\circ$.



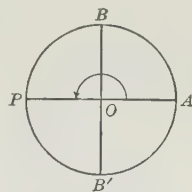
$\sec 315^\circ = OT.$

20. $\cos 180^\circ$.



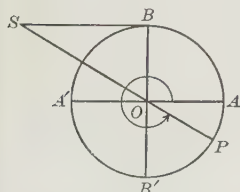
$\cos 180^\circ = OP.$

23. $\sec 180^\circ$.



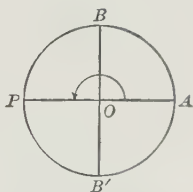
$\sec 180^\circ = OP.$

18. $\csc 330^\circ$.



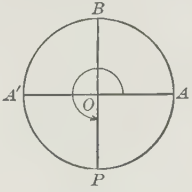
$\csc 330^\circ = OS.$

21. $\tan 180^\circ$.



$\tan 180^\circ = 0.$

24. $\csc 270^\circ$.



$\csc 270^\circ = OP.$

25. Prepare a table showing the signs of all the functions in each of the four quadrants.

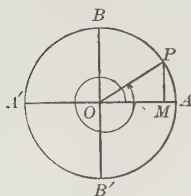
Functions	1st	2d	3d	4th
Sine	+	+	-	-
Cosine	+	-	-	+
Tangent	+	-	+	-
Cotangent	+	-	+	-
Secant	+	-	-	+
Cosecant	+	+	-	-

26. Prepare a table showing which functions always have the minus sign in each of the four quadrants.

Quadrant	sin	cos	tan	cot	sec	csc
I						
II		—	—	—	—	—
III	—	—			—	—
IV	—		—	—		—

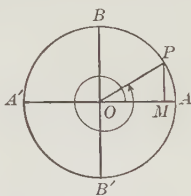
Represent the following functions by lines in a unit circle:

27. $\sin 390^\circ$.



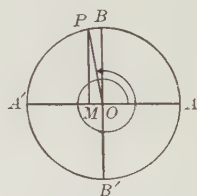
$\sin 390^\circ = MP$.

30. $\cos 390^\circ$.



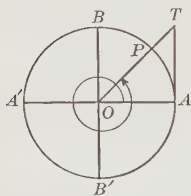
$\cos 390^\circ = OM$.

33. $\sin 460^\circ$.



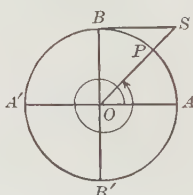
$\sin 460^\circ = MP$.

28. $\tan 405^\circ$.



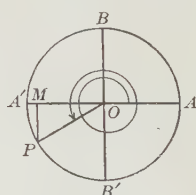
$\tan 405^\circ = AT$.

31. $\cot 405^\circ$.



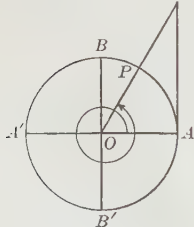
$\cot 405^\circ = BS$.

34. $\sin 570^\circ$.



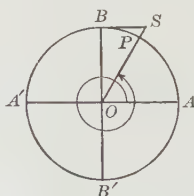
$\sin 570^\circ = MP$.

29. $\sec 420^\circ$.



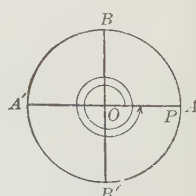
$\sec 420^\circ = OT$.

32. $\csc 420^\circ$.



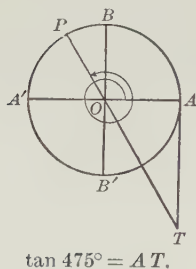
$\csc 420^\circ = OS$.

35. $\sin 720^\circ$.

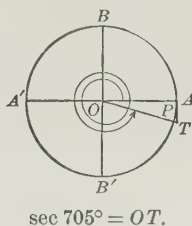


$\sin 720^\circ = 0$.

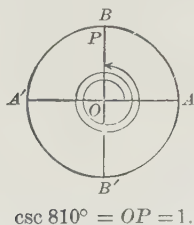
36. $\tan 475^\circ$.



37. $\sec 705^\circ$.



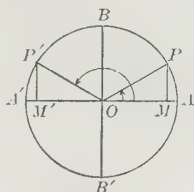
38. $\csc 810^\circ$.



Show by lines in a unit circle that:

39. $\sin 150^\circ = \sin 30^\circ$.

$MP = M'P'$.



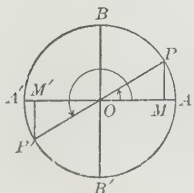
40. $\cos 150^\circ = -\cos 30^\circ$.

$OM = -OM'$.

41. $\sin 210^\circ = -\sin 30^\circ$.

$-M'P' = MP$.

$M'P' = -MP$.



42. $\cos 210^\circ = -\cos 30^\circ$.

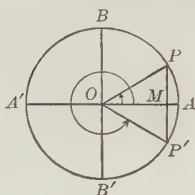
$-OM' = OM$.

$OM' = -OM$.

43. $\sin 330^\circ = -\sin 30^\circ$.

$-MP' = MP$.

$MP' = -MP$.



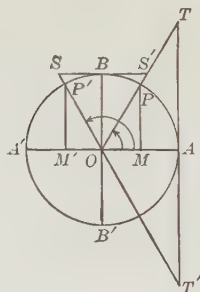
44. $\cos 330^\circ = \cos 30^\circ$.

$OM = OM$.

45. $\tan 120^\circ = -\tan 60^\circ$.

$-AT' = AT$.

$AT' = -AT$.



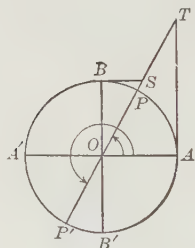
46. $\cot 120^\circ = -\cot 60^\circ$.

$-BS' = BS$.

$BS' = -BS$.

47. $\tan 240^\circ = \tan 60^\circ$.

$AT = AT$.



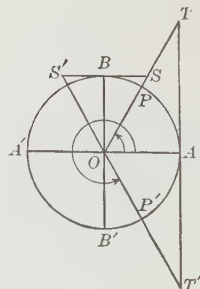
48. $\cot 240^\circ = \cot 60^\circ$.

$BS = BS$.

49. $\tan 300^\circ = -\tan 60^\circ$.

$-AT' = AT$.

$AT' = -AT$.



50. $\cot 300^\circ = -\cot 60^\circ$.

$-BS' = BS$.

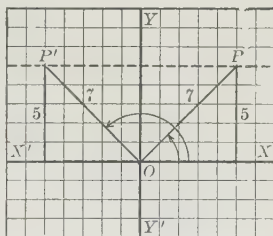
$BS' = -BS$.

51. Write the signs of the functions of the following angles : 340° , 239° , 145° , 400° , 700° , 1200° , 3800° .

Functions	340°	239°	145°	400°	700°	1200°	3800°
Sine	—	—	+	+	—	+	—
Cosine	+	—	—	+	+	—	—
Tangent	—	+	—	+	—	—	+
Cotangent	—	+	—	+	—	—	+
Secant	+	—	—	+	+	—	—
Cosecant	—	—	+	+	—	+	—

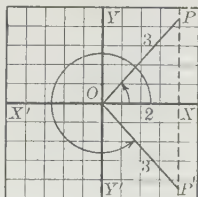
52. How many values less than 360° can the angle x have if $\sin x = +\frac{5}{7}$, and in what quadrants do the angles lie? Draw a figure.

The angle x can have two values less than 360° ; in the first and second quadrants.



53. How many values less than 720° can the angle x have if $\cos x = +\frac{2}{3}$, and in what quadrants do the angles lie? Draw a figure.

The angle x can have four values less than 720° ; two in the first quadrant and two in the fourth quadrant.



54. If we take into account only angles less than 180° , how many values can x have if $\sin x = \frac{5}{7}$? if $\cos x = \frac{1}{5}$? if $\cos x = -\frac{4}{5}$? if $\tan x = \frac{2}{3}$? if $\cot x = -7$?

If $\sin x = \frac{5}{7}$, x has two values.

If $\cos x = \frac{1}{5}$, x has one value.

If $\cos x = -\frac{4}{5}$, x has one value.

If $\tan x = \frac{2}{3}$, x has one value.

If $\cot x = -7$, x has one value.

55. Within what limits between 0° and 360° must the angle x lie if $\cos x = -\frac{2}{3}$? if $\cot x = 4$? if $\sec x = 80$? if $\csc x = -3$?

If $\cos x = -\frac{2}{3}$, x must lie between 90° and 270° .

If $\cot x = 4$, x must lie between 0° and 90° , or between 180° and 270° .

If $\sec x = 80$, x must lie between 0° and 90° , or between 270° and 360° .

If $\csc x = -3$, x must lie between 180° and 360° .

56. Why may $\cot 360^\circ$ be considered as either $+\infty$ or $-\infty$?

The nearer an acute angle is to 0° , the greater the positive value of its cotangent; and the nearer an angle is to 360° , the greater the negative value of its cotangent. When the angle is 0° or 360° , the cotangent is parallel to the horizontal diameter and cannot meet it. But the cotangent of 360° may be regarded as extending either in the positive or in the negative direction and hence as either $+\infty$ or $-\infty$.

57. Find the values of $\sin 450^\circ$, $\tan 540^\circ$, $\cos 630^\circ$, $\cot 720^\circ$, $\sin 810^\circ$, $\csc 900^\circ$, $\cos 1800^\circ$, $\sin 3600^\circ$.

$$\sin 450^\circ = \sin (360^\circ + 90^\circ) = \sin 90^\circ = 1.$$

$$\tan 540^\circ = \tan (360^\circ + 180^\circ) = \tan 180^\circ = 0.$$

$$\cos 630^\circ = \cos (360^\circ + 270^\circ) = \cos 270^\circ = 0.$$

$$\cot 720^\circ = \cot (360^\circ + 360^\circ) = \cot 360^\circ = \infty.$$

$$\sin 810^\circ = \sin (2 \times 360^\circ + 90^\circ) = \sin 90^\circ = 1.$$

$$\csc 900^\circ = \csc (2 \times 360^\circ + 180^\circ) = \csc 180^\circ = \infty.$$

$$\cos 1800^\circ = \cos (5 \times 360^\circ) = \cos 360^\circ = 1.$$

$$\sin 3600^\circ = \sin (10 \times 360^\circ) = \sin 360^\circ = 0.$$

58. What functions of an angle of a triangle may be negative? In what cases are they negative?

When an angle of a triangle is acute, its functions are all positive. When an angle is obtuse, its functions are those of an angle in Quadrant II. Hence sines and cosecant are always positive, and cosine, tangent, cotangent, and secant may be negative.

59. In what quadrant does an angle lie if sine and cosine are both negative? if cosine and tangent are both negative?

If the sine and cosine are negative, the angle lies in the third quadrant.

If the cosine and tangent are negative, the angle lies in the second quadrant.

60. Between 0° and 3600° how many angles are there whose sines have the absolute value $\frac{3}{5}$? Of these sines how many are positive?

Between 0° and 3600° there are 10 revolutions, and in each there are 4 angles whose sines have the absolute value $\frac{3}{5}$. $10 \times 4 = 40$, angles whose sines have the absolute value $\frac{3}{5}$. Of these sines 20 are positive.

Compute the values of the following expressions :

61. $a \sin 0^\circ + b \cos 90^\circ - c \tan 180^\circ$. **62.** $a \cos 90^\circ - b \tan 180^\circ + c \cot 90^\circ$.
 $\sin 0^\circ = 0$; $\cos 90^\circ = 0$; $\tan 180^\circ = 0$. $\cos 90^\circ = 0$, $\tan 180^\circ = 0$, $\cot 90^\circ = 0$.

Substituting, $a \cdot 0 + b \cdot 0 - c \cdot 0 = 0$. Substituting, $a \cdot 0 - b \cdot 0 + c \cdot 0 = 0$.

63. $a \sin 90^\circ - b \cos 360^\circ + (a - b) \cos 180^\circ$.

$$\sin 90^\circ = 1, \cos 360^\circ = 1, \cos 180^\circ = -1.$$

Substituting, $a \cdot 1 - b \cdot 1 + (a - b)(-1) = a - b - a + b = 0$.

64. $(a^2 - b^2) \cos 360^\circ - 4ab \sin 270^\circ + \sin 360^\circ$.

$$\cos 360^\circ = 1, \sin 270^\circ = -1, \sin 360^\circ = 0.$$

Substituting, $(a^2 - b^2)1 - 4ab(-1) + 0 = a^2 - b^2 + 4ab$.

65. $(a^2 + b^2) \cos 180^\circ + (a^2 + b^2) \sin 180^\circ + (a^2 + b^2) \tan 135^\circ$.

$$\cos 180^\circ = -1, \sin 180^\circ = 0, \tan 135^\circ = -1.$$

Substituting, $(a^2 + b^2)(-1) + (a^2 + b^2)0 + (a^2 + b^2)(-1)$
 $= -a^2 - b^2 - a^2 - b^2 = -2(a^2 + b^2)$.

66. $(a^2 + 2ab + b^2) \sin 90^\circ + (a^2 - 2ab + b^2) \cos 180^\circ - 4ab \tan 225^\circ$.

$$\sin 90^\circ = 1, \cos 180^\circ = -1, \tan 225^\circ = 1.$$

Substituting, $(a^2 + 2ab + b^2)1 + (a^2 - 2ab + b^2)(-1) - 4ab \cdot 1$
 $= a^2 + 2ab + b^2 - a^2 + 2ab - b^2 - 4ab = 0$.

67. $(a - b + c - d) \sin 270^\circ - (a - b + c - d) \cos 180^\circ + a \tan 360^\circ$.

$$\sin 270^\circ = -1, \cos 180^\circ = -1, \tan 360^\circ = 0.$$

Substituting, $(a - b + c - d)(-1) - (a - b + c - d)(-1) + a \cdot 0$
 $= -a + b - c + d + a - b + c - d = 0$.

State the sign of each of the six functions of the following angles :

		sin	cos	tan	cot	sec	csc
68.	75	+	+	+	+	+	+
69.	125	+	-	-	-	-	+
70.	155	+	-	-	-	-	+
71.	185	-	-	+	+	-	-
72.	275°	-	+	+	+	-	-
73.	325°	-	+	+	+	-	-
74.	355°	-	+	+	+	-	-
75.	-65°	-	+	+	+	-	-

Find the four smallest angles that satisfy the following conditions :

76. $\sin A = \frac{1}{2}$.

$$\begin{aligned} A &= 30^\circ \\ &= 180^\circ - 30^\circ = 150^\circ \\ &= 360^\circ + 30^\circ = 390^\circ \\ &= 360^\circ + 150^\circ = 510^\circ. \end{aligned}$$

79. $\cos A = \frac{1}{2}$.

$$\begin{aligned} A &= 60^\circ \\ &= 360^\circ - 60^\circ = 300^\circ \\ &= 360^\circ + 60^\circ = 420^\circ \\ &= 2 \times 360^\circ - 60^\circ = 660^\circ. \end{aligned}$$

77. $\cos A = \frac{1}{2}\sqrt{3}$.

$$\begin{aligned} A &= 30^\circ \\ &= 360^\circ - 30^\circ = 330^\circ \\ &= 360^\circ + 30^\circ = 390^\circ \\ &= 2 \times 360^\circ - 30^\circ = 690^\circ. \end{aligned}$$

80. $\tan A = \frac{1}{3}\sqrt{3}$.

$$\begin{aligned} A &= 30^\circ \\ &= 180^\circ + 30^\circ = 210^\circ \\ &= 360^\circ + 30^\circ = 390^\circ \\ &= 360^\circ + 210^\circ = 570^\circ. \end{aligned}$$

78. $\sin A = \frac{1}{2}\sqrt{3}$.

$$\begin{aligned} A &= 60^\circ \\ &= 180^\circ - 60^\circ = 120^\circ \\ &= 360^\circ + 60^\circ = 420^\circ \\ &= 360^\circ + 120^\circ = 480^\circ. \end{aligned}$$

81. $\tan A = \sqrt{3}$.

$$\begin{aligned} A &= 60^\circ \\ &= 180^\circ + 60^\circ = 240^\circ \\ &= 360^\circ + 60^\circ = 420^\circ \\ &= 360^\circ + 240^\circ = 600^\circ. \end{aligned}$$

Find two angles less than 360° that satisfy the following conditions :

82. $\sin A = -\frac{1}{2}$.

$$\begin{aligned} A &= 180^\circ + 30^\circ = 210^\circ \\ &= 360^\circ - 30^\circ = 330^\circ. \end{aligned}$$

85. $\cos A = -\frac{1}{2}\sqrt{2}$.

$$\begin{aligned} A &= 180^\circ - 45^\circ = 135^\circ \\ &= 180^\circ + 45^\circ = 225^\circ. \end{aligned}$$

83. $\cos A = -\frac{1}{2}$.

$$\begin{aligned} A &= 180^\circ - 60^\circ = 120^\circ \\ &= 180^\circ + 60^\circ = 240^\circ. \end{aligned}$$

86. $\tan A = -1$.

$$\begin{aligned} A &= 180^\circ - 45^\circ = 135^\circ \\ &= 270^\circ + 45^\circ = 315^\circ. \end{aligned}$$

84. $\sin A = -\frac{1}{2}\sqrt{2}$.

$$\begin{aligned} A &= 180^\circ + 45^\circ = 225^\circ \\ &= 360^\circ - 45^\circ = 315^\circ. \end{aligned}$$

87. $\cot A = -1$.

$$\begin{aligned} A &= 180^\circ - 45^\circ = 135^\circ \\ &= 270^\circ + 45^\circ = 315^\circ. \end{aligned}$$

If A , B , and C are the angles of any triangle ABC , prove that :

88. $\cos \frac{1}{2}A = \sin \frac{1}{2}(B+C)$.

$$A+B+C = 180^\circ.$$

$$\frac{1}{2}(A+B+C) = 90^\circ.$$

$$\frac{1}{2}A = 90^\circ - \frac{1}{2}(B+C).$$

$$\begin{aligned} \cos \frac{1}{2}A &= \sin [90^\circ - [90^\circ - \frac{1}{2}(B+C)]] \\ &= \sin \frac{1}{2}(B+C). \end{aligned}$$

89. $\sin \frac{1}{2}C = \cos \frac{1}{2}(A+B)$.

$$A+B+C = 180^\circ.$$

$$\frac{1}{2}(A+B+C) = 90^\circ.$$

$$\frac{1}{2}C = 90^\circ - \frac{1}{2}(A+B).$$

$$\begin{aligned} \sin \frac{1}{2}C &= \cos [90^\circ - [90^\circ - \frac{1}{2}(A+B)]] \\ &= \cos \frac{1}{2}(A+B). \end{aligned}$$

$$90. \cos \frac{1}{2} B = \sin \frac{1}{2} (A + C).$$

$$A + B + C = 180^\circ.$$

$$\frac{1}{2} (A + B + C) = 90^\circ.$$

$$\frac{1}{2} B = 90^\circ - \frac{1}{2} (A + C).$$

$$\cos \frac{1}{2} B = \sin \{90^\circ - [90^\circ - \frac{1}{2} (A + C)]\} \\ = \sin \frac{1}{2} (A + C).$$

$$91. \sin \frac{1}{2} A = \cos \frac{1}{2} (B + C).$$

$$A + B + C = 180^\circ.$$

$$\frac{1}{2} (A + B + C) = 90^\circ.$$

$$\frac{1}{2} A = 90^\circ - \frac{1}{2} (B + C).$$

$$\sin \frac{1}{2} A = \cos \{90^\circ - [90^\circ - \frac{1}{2} (B + C)]\} \\ = \cos \frac{1}{2} (B + C).$$

As angle A increases from 0° to 360° , trace the changes in sign and magnitude of the following :

$$92. \sin A \cos A.$$

$$A = 0^\circ, \quad \sin A \cos A = 0 \cdot 1 = 0.$$

$$A = 45^\circ, \quad \sin A \cos A = (\frac{1}{2} \sqrt{2})^2 = \frac{1}{2}.$$

$$A = 90^\circ, \quad \sin A \cos A = 1 \cdot 0 = 0.$$

$$A = 135^\circ, \quad \sin A \cos A = (\frac{1}{2} \sqrt{2}) (-\frac{1}{2} \sqrt{2}) = -\frac{1}{2}.$$

$$A = 180^\circ, \quad \sin A \cos A = 0 (-1) = 0.$$

$$A = 225^\circ, \quad \sin A \cos A = (-\frac{1}{2} \sqrt{2})^2 = \frac{1}{2}.$$

$$A = 270^\circ, \quad \sin A \cos A = -1 \cdot 0 = 0.$$

$$A = 315^\circ, \quad \sin A \cos A = (-\frac{1}{2} \sqrt{2}) (\frac{1}{2} \sqrt{2}) = -\frac{1}{2}.$$

$$A = 360^\circ, \quad \sin A \cos A = 0 \cdot 1 = 0.$$

In the first quadrant $\sin A \cos A$ increases from 0 to $\frac{1}{2}$ and decreases from $\frac{1}{2}$ to 0 and is positive; in the second it is negative and increases from 0 to $\frac{1}{2}$ and decreases from $\frac{1}{2}$ to 0 in absolute value; in the third it is positive and increases from 0 to $\frac{1}{2}$ and decreases from $\frac{1}{2}$ to 0; in the fourth it is negative and increases from 0 to $\frac{1}{2}$ and decreases from $\frac{1}{2}$ to 0 in absolute value.

$$93. \sin A + \cos A.$$

$$A = 0^\circ, \quad \sin A + \cos A = 1. \quad A = 225^\circ, \quad \sin A + \cos A = -\sqrt{2}.$$

$$A = 45^\circ, \quad \sin A + \cos A = \sqrt{2}. \quad A = 270^\circ, \quad \sin A + \cos A = -1.$$

$$A = 90^\circ, \quad \sin A + \cos A = 1. \quad A = 315^\circ, \quad \sin A + \cos A = 0.$$

$$A = 135^\circ, \quad \sin A + \cos A = 0. \quad A = 360^\circ, \quad \sin A + \cos A = 1.$$

$$A = 180^\circ, \quad \sin A + \cos A = -1.$$

If A is in Quadrant I, $\sin A + \cos A$ must be positive, since both the sine and the cosine are positive. In Quadrant II the sine is positive and the cosine negative; hence, so long as the sine is greater than, or equal to, the cosine, the expression $\sin A + \cos A$ is positive; but after passing the middle of Quadrant II, viz., 135° , the cosine of A is greater than the sine, and the expression is negative. In Quadrant III both sine and cosine are negative, and hence their sum must be negative. In Quadrant IV the sine is negative and the cosine positive. The sine and the cosine are equal at 315° , after which the cosine is greater than the sine. Hence the expression $\sin A + \cos A$ is negative from 135° to 315° , and positive between 0° and 135° , and 315° and 360° .

94. $\sin A - \cos A$.

$$A = 0^\circ, \quad \sin A - \cos A = -1. \quad A = 225^\circ, \quad \sin A - \cos A = 0.$$

$$A = 45^\circ, \quad \sin A - \cos A = 0. \quad A = 270^\circ, \quad \sin A - \cos A = -1.$$

$$A = 90^\circ, \quad \sin A - \cos A = 1. \quad A = 315^\circ, \quad \sin A - \cos A = -\sqrt{2}.$$

$$A = 135^\circ, \quad \sin A - \cos A = \sqrt{2}. \quad A = 360^\circ, \quad \sin A - \cos A = -1.$$

$$A = 180^\circ, \quad \sin A - \cos A = 1.$$

As A increases from 0° to 45° , the sine increases in value, and the cosine decreases, until at 45° sine = cosine. Hence, up to this point $\sin A - \cos A$ is negative. For the remainder of Quadrant I the sine is greater than the cosine, and consequently the expression $\sin A - \cos A$ is positive. In Quadrant II the sine is positive and the cosine negative, so the expression $\sin A - \cos A$ is uniformly positive. In Quadrant III the sine is negative and the cosine negative; hence, so long as the sine is less than the cosine, the expression is positive, viz., to 225° ; after that point the sine is greater than the cosine, and $\sin A - \cos A$ is negative. In Quadrant IV the sine is negative and the cosine positive; therefore $\sin A - \cos A$ is uniformly negative. The expression is, then, negative between 0° and 45° , and 225° and 360° ; positive between 45° and 225° .

95. $\sin A \div \cos A$.

$$A = 0^\circ, \quad \sin A \div \cos A = 0. \quad A = 225^\circ, \quad \sin A \div \cos A = 1.$$

$$A = 45^\circ, \quad \sin A \div \cos A = 1. \quad A = 270^\circ, \quad \sin A \div \cos A = \pm \infty.$$

$$A = 90^\circ, \quad \sin A \div \cos A = \pm \infty. \quad A = 315^\circ, \quad \sin A \div \cos A = -1.$$

$$A = 135^\circ, \quad \sin A \div \cos A = -1. \quad A = 360^\circ, \quad \sin A \div \cos A = 0.$$

$$A = 180^\circ, \quad \sin A \div \cos A = 0.$$

In the first quadrant $\sin A \div \cos A$ increases from 0 to 1 to ∞ and is positive; in the second it decreases from ∞ to 1 to 0 in absolute value and is negative; in the third it increases from 0 to 1 to ∞ and is positive; in the fourth it decreases from ∞ to 1 to 0 in absolute value and is negative.

96. $\tan A + \cot A$.

$$A = 0^\circ, \quad \tan A + \cot A = \mp \infty. \quad A = 225^\circ, \quad \tan A + \cot A = 2.$$

$$A = 45^\circ, \quad \tan A + \cot A = 2. \quad A = 270^\circ, \quad \tan A + \cot A = \pm \infty.$$

$$A = 90^\circ, \quad \tan A + \cot A = \pm \infty. \quad A = 315^\circ, \quad \tan A + \cot A = -2.$$

$$A = 135^\circ, \quad \tan A + \cot A = -2. \quad A = 360^\circ, \quad \tan A + \cot A = \pm \infty.$$

$$A = 180^\circ, \quad \tan A + \cot A = \mp \infty.$$

In the first quadrant $\tan A + \cot A$ decreases from ∞ to 2 and increases from 2 to ∞ and is positive; in the second it decreases from ∞ to 2 and increases from 2 to ∞ in absolute value and is negative; in the third it decreases from ∞ to 2 and increases from 2 to ∞ and is positive; in the fourth it decreases from ∞ to 2 and increases from 2 to ∞ in absolute value and is negative.

97. $\tan A - \cot A$.

$$A = 0^\circ, \quad \tan A - \cot A = \pm \infty. \quad A = 225^\circ, \tan A - \cot A = 0.$$

$$A = 45^\circ, \quad \tan A - \cot A = 0. \quad A = 270^\circ, \tan A - \cot A = \pm \infty.$$

$$A = 90^\circ, \quad \tan A - \cot A = \pm \infty. \quad A = 315^\circ, \tan A - \cot A = 0.$$

$$A = 135^\circ, \tan A - \cot A = 0. \quad A = 360^\circ, \tan A - \cot A = \pm \infty.$$

$$A = 180^\circ, \tan A - \cot A = \pm \infty.$$

In Quadrant I, as A increases from 0° to 45° $\tan A$ increases from 0 to 1 and $\cot A$ decreases from ∞ to 1. As $\cot A$ is greater than $\tan A$ and both are positive, $\tan A - \cot A$ is negative. From 45° to 90° $\tan A$ increases from 1 to ∞ and $\cot A$ decreases from 1 to 0; therefore $\tan A - \cot A$ increases in value and is positive. The same conditions are found in Quadrant III. From 90° to 135° in Quadrant II $\tan A$ decreases in absolute value from ∞ to 1 and is negative, while $\cot A$ increases from 0 to 1, also negative. As $\tan A$ is greater than $\cot A$, $\tan A - \cot A$ is negative. From 135° to 180° $\cot A$ increases to ∞ and is negative, while $\tan A$ decreases from 1 to 0, also negative. Therefore $\tan A - \cot A$ is positive. The same conditions are found in Quadrant IV.

Exercise 38. Reduction to the First Quadrant (Page 91)

Express the following as functions of angles less than 90° :

1. $\sin 170^\circ = \sin (180^\circ - 10^\circ) = \sin 10^\circ.$
2. $\cos 160^\circ = \cos (180^\circ - 20^\circ) = -\cos 20^\circ.$
3. $\tan 148^\circ = \tan (180^\circ - 32^\circ) = -\tan 32^\circ.$
4. $\cot 156^\circ = \cot (180^\circ - 24^\circ) = -\cot 24^\circ.$
5. $\sin 180^\circ = \sin (180^\circ - 0^\circ) = \sin 0^\circ.$
6. $\tan 180^\circ = \tan (180^\circ - 0^\circ) = -\tan 0^\circ.$
7. $\sin 200^\circ = \sin (180^\circ + 20^\circ) = -\sin 20^\circ.$
8. $\cos 225^\circ = \cos (180^\circ + 45^\circ) = -\cos 45^\circ.$
9. $\tan 258^\circ = \tan (180^\circ + 78^\circ) = \tan 78^\circ.$
10. $\cot 262^\circ = \cot (180^\circ + 82^\circ) = \cot 82^\circ.$
11. $\sin 275^\circ = \sin (360^\circ - 85^\circ) = -\sin 85^\circ.$
12. $\sin 345^\circ = \sin (360^\circ - 15^\circ) = -\sin 15^\circ.$
13. $\tan 282^\circ = \tan (360^\circ - 78^\circ) = -\tan 78^\circ.$
14. $\tan 325^\circ = \tan (360^\circ - 35^\circ) = -\tan 35^\circ.$
15. $\cos 290^\circ = \cos (360^\circ - 70^\circ) = \cos 70^\circ.$
16. $\cos 350^\circ = \cos (360^\circ - 10^\circ) = \cos 10^\circ.$
17. $\cot 295^\circ = \cot (360^\circ - 65^\circ) = -\cot 65^\circ.$
18. $\cot 347^\circ = \cot (360^\circ - 13^\circ) = -\cot 13^\circ.$
19. $\sin 360^\circ = \sin (360^\circ - 0^\circ) = -\sin 0^\circ.$
20. $\cos 360^\circ = \cos (360^\circ - 0^\circ) = \cos 0^\circ.$

21. $\sin 148^\circ 10' = \sin(180^\circ - 31^\circ 50') = \sin 31^\circ 50'.$
22. $\cos 192^\circ 20' = \cos(180^\circ + 12^\circ 20') = -\cos 12^\circ 20'.$
23. $\tan 265^\circ 30' = \tan(180^\circ + 85^\circ 30') = \tan 85^\circ 30'.$
24. $\cot 287^\circ 40' = \cot(360^\circ - 72^\circ 20') = -\cot 72^\circ 20'.$
25. $\sin 187^\circ 10' 3'' = \sin(180^\circ + 7^\circ 10' 3'') = -\sin 7^\circ 10' 3''.$
26. $\cos 274^\circ 5' 14'' = \cos(360^\circ - 85^\circ 54' 46'') = \cos 85^\circ 54' 46''.$
27. $\tan 322^\circ 8' 15'' = \tan(360^\circ - 37^\circ 51' 45'') = -\tan 37^\circ 51' 45''.$
28. $\cot 375^\circ 10' 3'' = \cot(360^\circ + 15^\circ 10' 3'') = \cot 15^\circ 10' 3''.$
29. $\sin 147.75^\circ = \sin(180^\circ - 32.25^\circ) = \sin 32.25^\circ.$
30. $\cos 232.25^\circ = \cos(180^\circ + 52.25^\circ) = -\cos 52.25^\circ.$

Exercise 39. Reduction of Functions (Page 93)

Express the following as functions of angles less than 45° :

1. $\sin 100^\circ = \sin(90^\circ + 10^\circ) = \cos 10^\circ.$
2. $\sin 120^\circ = \sin(90^\circ + 30^\circ) = \cos 30^\circ.$
3. $\sin 110^\circ = \sin(90^\circ + 20^\circ) = \cos 20^\circ.$
4. $\sin 130^\circ = \sin(90^\circ + 40^\circ) = \cos 40^\circ.$
5. $\cos 95^\circ = \cos(90^\circ + 5^\circ) = -\sin 5^\circ.$
6. $\cos 97^\circ = \cos(90^\circ + 7^\circ) = -\sin 7^\circ.$
7. $\cos 111^\circ = \cos(90^\circ + 21^\circ) = -\sin 21^\circ.$
8. $\cos 127^\circ = \cos(90^\circ + 37^\circ) = -\sin 37^\circ.$
9. $\tan 91^\circ = \tan(90^\circ + 1^\circ) = -\cot 1^\circ.$
10. $\tan 99^\circ = \tan(90^\circ + 9^\circ) = -\cot 9^\circ.$
11. $\tan 119^\circ = \tan(90^\circ + 29^\circ) = -\cot 29^\circ.$
12. $\tan 129^\circ = \tan(90^\circ + 39^\circ) = -\cot 39^\circ.$
13. $\cot 94^\circ 1' = \cot(90^\circ + 4^\circ 1') = -\tan 4^\circ 1'.$
14. $\cot 97^\circ 2' = \cot(90^\circ + 7^\circ 2') = -\tan 7^\circ 2'.$
15. $\cot 98^\circ 3' = \cot(90^\circ + 8^\circ 3') = -\tan 8^\circ 3'.$
16. $\cot 99^\circ 9' = \cot(90^\circ + 9^\circ 9') = -\tan 9^\circ 9'.$

Express the following as functions of positive angles:

17. $\sin(-3^\circ) = -\sin 3^\circ.$
18. $\sin(-9^\circ) = -\sin 9^\circ.$
19. $\sin(-86^\circ) = -\sin 86^\circ.$
20. $\cos(-75^\circ) = \cos 75^\circ.$
21. $\cos(-87^\circ) = \cos 87^\circ.$
22. $\cos(-95^\circ) = \cos 95^\circ = -\sin 5^\circ.$
23. $\tan(-100^\circ) = -\tan 100^\circ = \tan(180^\circ + 80^\circ) = \tan 80^\circ.$
24. $\tan(-150^\circ) = -\tan 150^\circ = \tan(180^\circ + 30^\circ) = \tan 30^\circ.$
25. $\tan(-200^\circ) = -\tan 200^\circ = \tan(180^\circ - 20^\circ) = -\tan 20^\circ.$
26. $\cot(-1.5^\circ) = -\cot 1.5^\circ.$
27. $\cot(-7.8^\circ) = -\cot 7.8^\circ.$
28. $\cot(-9.1^\circ) = -\cot 9.1^\circ.$

Find the following by aid of the tables :

$$29. \sin 178^\circ 30' = \sin (180^\circ - 1^\circ 30') = \sin 1^\circ 30' = 0.0262.$$

$$30. \cos 236^\circ 45' = \cos (180^\circ + 56^\circ 45') = -\cos 56^\circ 45' = -0.5483.$$

$$31. \tan 322^\circ 18' = \tan (360^\circ - 37^\circ 42') = -\tan 37^\circ 42' = -0.7729.$$

$$32. \cot 423^\circ 15' = \cot (360^\circ + 63^\circ 15') = \cot 63^\circ 15' = 0.5040.$$

$$33. \sin (-7^\circ 29' 30'') = -\sin 7^\circ 29' 30'' = -0.1304.$$

$$34. \cos (-29^\circ 42' 19'') = \cos 29^\circ 42' 19'' = 0.8686.$$

$$35. \tan (-172^\circ 16' 14'') = -\tan 172^\circ 16' 14'' = -\tan (180^\circ - 7^\circ 43' 46'') \\ = \tan 7^\circ 43' 46'' = 0.1357.$$

$$36. \cot (-262^\circ 17' 15'') = -\cot 262^\circ 17' 15'' = -\cot (180^\circ + 82^\circ 17' 15'') \\ = -\cot 82^\circ 17' 15'' = -0.1354.$$

$$37. \log \sin 127.5^\circ = \log \sin 127^\circ 30' = \log \sin (180^\circ - 52^\circ 30') \\ = \log \sin 52^\circ 30' = 9.89947 - 10.$$

$$38. \log \cos 226.4^\circ = \log \cos 226^\circ 24' = \log \cos (180^\circ + 46^\circ 24') \\ = -\log \cos 46^\circ 24' = - (9.83861 - 10).$$

$$39. \log \tan 327.8^\circ = \log \tan 327^\circ 48' = \log \tan (360^\circ - 32^\circ 12') \\ = -\log \tan 32^\circ 12' = - (9.79916 - 10).$$

$$40. \log \cot 343.3^\circ = \log \cot 343^\circ 18' = \log \cot (360^\circ - 16^\circ 42') \\ = -\log \cot 16^\circ 42' = - (10.52286 - 10).$$

$$41. \log \sin 236^\circ 13' 5'' = \log \sin (180^\circ + 56^\circ 13' 5'') \\ = -\log \sin 56^\circ 13' 5'' = - (9.91969 - 10).$$

$$42. \log \cos 327^\circ 5' 11'' = \log \cos (360^\circ - 32^\circ 54' 49'') \\ = \log \cos 32^\circ 54' 49'' = 9.92401 - 10.$$

$$43. \log \tan (-125^\circ 27') = -\log \tan 125^\circ 27' = -\log \tan (180^\circ - 54^\circ 33') \\ = \log \tan 54^\circ 33' = 10.14753 - 10.$$

$$44. \log \cot (-236^\circ 15') = -\log \cot 236^\circ 15' = -\log \cot (180^\circ + 56^\circ 15') \\ = -\log \cot 56^\circ 15' = - (9.82489 - 10).$$

45. Show that the angles 42° , 138° , -318° , 402° , and -222° all have the same sine.

$$\sin 42^\circ = 0.6691$$

$$\sin 138^\circ = \sin (180^\circ - 42^\circ) = \sin 42^\circ = 0.6691.$$

$$\sin (-318^\circ) = -\sin 318^\circ = -\sin (360^\circ - 42^\circ) = \sin 42^\circ = 0.6691.$$

$$\sin 402^\circ = \sin (360^\circ + 42^\circ) = \sin 42^\circ = 0.6691.$$

$$\sin (-222^\circ) = -\sin 222^\circ = -\sin (180^\circ + 42^\circ) = \sin 42^\circ = 0.6691$$

46. Find four angles between 0° and 720° which satisfy the equation $\sin x = -\frac{1}{2}\sqrt{2}$.

$$\sin 45^\circ = \frac{1}{2}\sqrt{2}.$$

$$\sin(180^\circ + 45^\circ) = \sin 225^\circ = -\frac{1}{2}\sqrt{2}.$$

$$\sin(360^\circ - 45^\circ) = \sin 315^\circ = -\frac{1}{2}\sqrt{2}.$$

$$\sin(360^\circ + 225^\circ) = \sin 585^\circ = -\frac{1}{2}\sqrt{2}.$$

$$\sin(360^\circ + 315^\circ) = \sin 675^\circ = -\frac{1}{2}\sqrt{2}.$$

$\therefore$ the angles are 225° , 315° , 585° , and 675° .

47. Draw a circle with unit radius, and represent by lines the sine, cosine, tangent, and cotangent of -325° .

$$\sin(-325^\circ) = -\sin 325^\circ = -\sin(360^\circ - 35^\circ)$$

$$= \sin 35^\circ = MP.$$

$$\cos(-325^\circ) = \cos 325^\circ = \cos(360^\circ - 35^\circ)$$

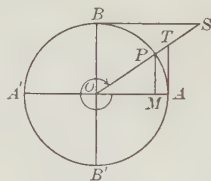
$$= \cos 35^\circ = OM.$$

$$\tan(-325^\circ) = -\tan 325^\circ = -\tan(360^\circ - 35^\circ)$$

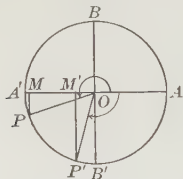
$$= \tan 35^\circ = AT.$$

$$\cot(-325^\circ) = -\cot 325^\circ = -\cot(360^\circ - 35^\circ)$$

$$= \cot 35^\circ = BS.$$



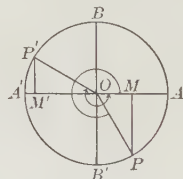
48. Show by drawing a figure that $\sin 195^\circ = \cos(-105^\circ)$, and that $\cos 300^\circ = \sin(-210^\circ)$.



$$\sin 195^\circ = -MP.$$

$$\cos(-105^\circ) = -OM'.$$

$$MP = OM'.$$

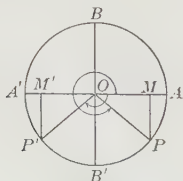


$$\cos 300^\circ = OM.$$

$$\sin(-210^\circ) = M'P'.$$

$$OM = M'P'.$$

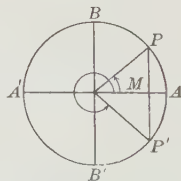
49. Show by drawing a figure that $\cos 320^\circ = -\cos(-140^\circ)$, and that $\sin 320^\circ = -\sin 40^\circ$.



$$\cos 320^\circ = OM.$$

$$-\cos(-140^\circ) = OM'.$$

$$OM = OM'.$$

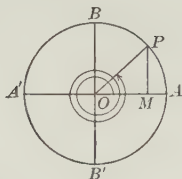


$$\sin 320^\circ = -MP'.$$

$$-\sin 40^\circ = -MP.$$

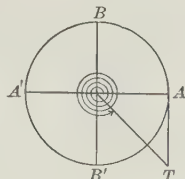
$$MP' = MP.$$

50. Show by drawing a figure that $\sin 765^\circ = \frac{1}{2} \sqrt{2}$, and that $\tan 1395^\circ = -1$.



$$\sin 765^\circ = MP = \frac{1}{2} \sqrt{2}.$$

$$\begin{aligned} \sin 765^\circ &= \sin(720^\circ + 45^\circ) \\ &= \sin 45^\circ. \end{aligned}$$



$$\tan 1395^\circ = AT = -1.$$

$$1395^\circ = 3 \times 360^\circ + 315^\circ.$$

$$\begin{aligned} \tan 315^\circ &= \tan(360^\circ - 45^\circ) \\ &= -\tan 45^\circ. \end{aligned}$$

$$\begin{aligned} \tan 1395^\circ &= \tan(1080^\circ + 315^\circ) \\ &= -\tan 45^\circ. \end{aligned}$$

51. In the triangle ABC show that $\cos A = -\cos(B + C)$, and that $\cos B = -\cos(A + C)$.

$$\begin{aligned} A &= 180^\circ - (B + C), \\ \cos A &= \cos[180^\circ - (B + C)] \\ &= -\cos(B + C). \end{aligned}$$

$$\begin{aligned} B &= 180^\circ - (A + C), \\ \cos B &= \cos[180^\circ - (A + C)] \\ &= -\cos(A + C). \end{aligned}$$

Exercise 40. Relations of the Functions (Page 95)

1. Prove each of the formulas given in § 89.

$$\sin x = \frac{1}{\csc x}.$$

$$\sin x = \frac{a}{c}, \csc x = \frac{c}{a}.$$

$$\frac{a}{c} = \frac{1}{\frac{c}{a}}.$$

$$\cos x = \frac{1}{\sec x}.$$

$$\cos x = \frac{b}{c}, \sec x = \frac{c}{b}.$$

$$\frac{b}{c} = \frac{1}{\frac{c}{b}}.$$

$$\sin x = \pm \sqrt{1 - \cos^2 x}.$$

$$\frac{a}{c} = \pm \sqrt{1 - \frac{b^2}{c^2}}.$$

$$\frac{a^2}{c^2} = 1 - \frac{b^2}{c^2}.$$

$$\frac{a^2}{c^2} = \frac{c^2 - b^2}{c^2}.$$

$$a^2 = c^2 - b^2.$$

$$\cos x = \pm \sqrt{1 - \sin^2 x}.$$

$$\frac{b}{c} = \pm \sqrt{1 - \frac{a^2}{c^2}}.$$

$$\frac{b^2}{c^2} = 1 - \frac{a^2}{c^2}.$$

$$\frac{b^2}{c^2} = \frac{c^2 - a^2}{c^2}.$$

$$b^2 = c^2 - a^2.$$

$$\tan x = \frac{1}{\cot x}.$$

$$\tan x = \frac{a}{b}, \cot x = \frac{b}{a}.$$

$$\frac{a}{b} = \frac{1}{\frac{b}{a}}.$$

$$\tan x = \pm \frac{\sin x}{\sqrt{1 - \sin^2 x}}.$$

$$\frac{a}{b} = \pm \frac{\frac{a}{c}}{\sqrt{1 - \frac{a^2}{c^2}}}.$$

$$\frac{a^2}{b^2} = \frac{a^2}{c^2} \div \frac{c^2 - a^2}{c^2}$$

$$= \frac{a^2}{c^2 - a^2}.$$

$$\frac{a^2}{b^2} = \frac{a^2}{b^2}.$$

$$\tan x = \pm \sqrt{\sec^2 x - 1}.$$

$$\frac{a}{b} = \pm \sqrt{\frac{c^2}{b^2} - 1}.$$

$$\frac{a^2}{b^2} = \frac{c^2 - b^2}{b^2}.$$

$$a^2 = c^2 - b^2.$$

$$\cot x = \frac{1}{\tan x}.$$

$$\frac{b}{a} = \frac{1}{\frac{a}{b}}.$$

$$\tan x = \frac{\sin x}{\cos x}.$$

$$\tan x = \frac{a}{b}, \sin x = \frac{a}{c},$$

$$\cos x = \frac{b}{c}.$$

$$\frac{a}{b} = \frac{a}{c} \div \frac{b}{c}.$$

$$\tan x = \pm \frac{\sqrt{1 - \cos^2 x}}{\cos x}$$

$$\frac{a}{b} = \pm \frac{\sqrt{1 - \frac{b^2}{c^2}}}{\frac{b}{c}}.$$

$$\frac{a^2}{b^2} = \frac{c^2 - b^2}{c^2} \div \frac{b^2}{c^2}$$

$$= \frac{c^2 - b^2}{b^2}.$$

$$a^2 = c^2 - b^2.$$

$$\tan x = \sin x \sec x$$

$$\frac{a}{b} = \frac{a}{c} \times \frac{c}{b}.$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\frac{b}{a} = \frac{\frac{b}{c}}{\frac{a}{c}}.$$

$$\frac{b}{a} = \frac{b}{a}.$$

$$\cot x = \pm \frac{\cos x}{\sqrt{1 - \cos^2 x}}.$$

$$\frac{b}{a} = \pm \frac{\frac{b}{c}}{\sqrt{1 - \frac{b^2}{c^2}}}.$$

$$\frac{b^2}{a^2} = \frac{b^2}{c^2} \div \frac{c^2 - b^2}{c^2}.$$

$$\frac{b^2}{a^2} = \frac{b^2}{c^2 - b^2}.$$

$$a^2 = c^2 - b^2.$$

$$\cot x = \pm \sqrt{\csc^2 x - 1}.$$

$$\frac{b}{a} = \pm \sqrt{\frac{c^2}{a^2} - 1}.$$

$$\frac{b^2}{a^2} = \frac{c^2 - a^2}{a^2}.$$

$$b^2 = c^2 - a^2$$

$$\sec x = \frac{1}{\cos x}$$

$$\frac{c}{b} = \frac{1}{\frac{b}{c}}.$$

$$\csc x = \frac{1}{\sin x}.$$

$$\frac{c}{a} = \frac{1}{\frac{a}{c}}.$$

Prove the following relations:

$$2. \sin x = \pm \frac{\tan x}{\sqrt{1 + \tan^2 x}}.$$

$$\pm \sqrt{1 + \tan^2 x} = \sec x.$$

$$\sin x = \frac{\tan x}{\sec x}.$$

$$\sin x \sec x = \tan x.$$

$$\tan x = \tan x.$$

$$\cot x = \pm \frac{\sqrt{1 - \sin^2 x}}{\sin x}.$$

$$\frac{b}{a} = \pm \frac{\sqrt{1 - \frac{a^2}{c^2}}}{\frac{a}{c}}.$$

$$\frac{b^2}{a^2} = \frac{c^2 - a^2}{c^2} \div \frac{a^2}{c^2}.$$

$$b^2 = c^2 - a^2.$$

$$\cot x = \cos x \csc x.$$

$$\frac{b}{a} = \frac{b}{c} \times \frac{c}{a}.$$

$$\frac{b}{a} = \frac{b}{a}.$$

$$\sec x = \pm \sqrt{1 + \tan^2 x}$$

$$\frac{c}{b} = \pm \sqrt{1 + \frac{a^2}{b^2}}.$$

$$\frac{c^2}{b^2} = \frac{b^2 + a^2}{b^2}.$$

$$c^2 = a^2 + b^2.$$

$$\csc x = \pm \sqrt{1 + \cot^2 x}.$$

$$\frac{c}{a} = \pm \sqrt{1 + \frac{b^2}{a^2}}.$$

$$\frac{c^2}{a^2} = \frac{a^2 + b^2}{a^2}.$$

$$c^2 = a^2 + b^2.$$

$$3. \cos x = \pm \frac{\cot x}{\sqrt{1 + \cot^2 x}}.$$

$$\pm \sqrt{1 + \cot^2 x} = \csc x.$$

$$\cos x = \frac{\cot x}{\csc x}$$

$$\cos x \csc x = \cot x.$$

$$\cot x = \cot x.$$

$$4. \tan x = \pm \frac{1}{\sqrt{\csc^2 x - 1}}.$$

$$\pm \sqrt{\csc^2 x - 1} = \cot x.$$

$$\tan x = \frac{1}{\cot x}.$$

$$\tan x = \tan x.$$

$$5. \cot x = \pm \frac{1}{\sqrt{\sec^2 x - 1}}.$$

$$\pm \sqrt{\sec^2 x - 1} = \tan x.$$

$$\cot x = \frac{1}{\tan x}.$$

$$\cot x = \cot x.$$

6. Find $\sin x$ in terms of $\cot x$.

$$\cot x = \pm \frac{\sqrt{1 - \sin^2 x}}{\sin x}.$$

$$\cot^2 x \sin^2 x = 1 - \sin^2 x.$$

$$\sin^2 x (\cot^2 x + 1) = 1.$$

$$\sin^2 x = \frac{1}{\cot^2 x + 1}.$$

$$\sin x = \pm \frac{1}{\sqrt{\cot^2 x + 1}}.$$

9. Find $\csc x$ in terms of $\cos x$.

$$\cos x = \pm \sqrt{1 - \sin^2 x}.$$

$$\cos^2 x = 1 - \sin^2 x$$

$$= 1 - \frac{1}{\csc^2 x}.$$

$$\cos^2 x \csc^2 x = \csc^2 x - 1.$$

$$\csc^2 x (\cos^2 x - 1) = -1.$$

$$\csc^2 x = \frac{1}{1 - \cos^2 x}.$$

$$\csc x = \pm \frac{1}{\sqrt{1 - \cos^2 x}}.$$

Prove the following relations:

10. $\tan x \cos x = \sin x.$

$$\frac{\sin x}{\cos x} \cdot \cos x = \sin x.$$

$$\sin x = \sin x.$$

7. Find $\cos x$ in terms of $\tan x$.

$$\tan x = \pm \frac{\sqrt{1 - \cos^2 x}}{\cos x}$$

$$\tan^2 x \cos^2 x = 1 - \cos^2 x.$$

$$\cos^2 x (\tan^2 x + 1) = 1.$$

$$\cos^2 x = \frac{1}{\tan^2 x + 1}.$$

$$\cos x = \pm \frac{1}{\sqrt{\tan^2 x + 1}}$$

8. Find $\sec x$ in terms of $\sin x$.

$$\sin x = \pm \sqrt{1 - \cos^2 x}.$$

$$\sin^2 x = 1 - \cos^2 x.$$

$$= 1 - \frac{1}{\sec^2 x}.$$

$$\sin^2 x \sec^2 x = \sec^2 x - 1.$$

$$\sec^2 x (\sin^2 x - 1) = -1.$$

$$\sec^2 x = \frac{1}{1 - \sin^2 x}.$$

$$\sec x = \pm \frac{1}{\sqrt{1 - \sin^2 x}}.$$

11. $\cos^2 x = \cot^2 x - \cot^2 x \cos^2 x.$

$$\cos^2 x = \cot^2 x (1 - \cos^2 x)$$

$$= \frac{\cos^2 x}{\sin^2 x} \cdot \sin^2 x.$$

$$\cos^2 x = \cos^2 x.$$

$$12. \tan^2 x = \sin^2 x + \sin^2 x \tan^2 x.$$

$$\tan^2 x = \sin^2 x (1 + \tan^2 x)$$

$$= \sin^2 x \cdot \sec^2 x$$

$$= \sin^2 x \cdot \frac{1}{\cos^2 x}.$$

$$\tan^2 x = \tan^2 x.$$

$$13. \cos^2 x + 2\sin^2 x = 1 + \sin^2 x.$$

$$\cos^2 x = 1 - \sin^2 x.$$

$$\cos^2 x = \cos^2 x.$$

$$14. \cot^2 x = \cos^2 x + \cos^2 x \cot^2 x.$$

$$\cot^2 x = \cos^2 x (1 + \cot^2 x)$$

$$= \cos^2 x \cdot \csc^2 x$$

$$= \cos^2 x \cdot \frac{1}{\sin^2 x}.$$

$$\cot^2 x = \cot^2 x.$$

$$15. \cot^2 x \sec^2 x = 1 + \cot^2 x.$$

$$\frac{\cos^2 x}{\sin^2 x} \cdot \frac{1}{\cos^2 x} = 1 + \cot^2 x.$$

$$\frac{1}{\sin^2 x} = \csc^2 x.$$

$$\csc^2 x = \csc^2 x.$$

$$16. \csc^2 x - \cot^2 x = 1.$$

$$1 + \cot^2 x - \cot^2 x = 1.$$

$$1 = 1.$$

$$17. \sec^2 x + \csc^2 x = \sec^2 x \csc^2 x.$$

$$\frac{\sec^2 x}{\csc^2 x} + 1 = \sec^2 x.$$

$$\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}.$$

$$\sin^2 x + \cos^2 x = 1.$$

$$1 = 1.$$

18. Show that the sum of the tangent and cotangent of an angle is equal to the product of the secant and cosecant of the angle.

$$\tan x + \cot x = \sec x \csc x.$$

$$\tan x + \frac{1}{\tan x} = \sec x \csc x.$$

$$\frac{\tan^2 x + 1}{\tan x} = \frac{1}{\sin x \cos x}.$$

$$\tan^2 x + 1 = \tan x \cdot \frac{1}{\sin x \cos x}$$

$$\sec^2 x = \frac{1}{\cos^2 x}.$$

$$\sec^2 x = \sec^2 x.$$

Recalling the values given on page 8, find the value of x when:

$$19. 2 \cos x = \sec x.$$

$$2 \cos x = \frac{1}{\cos x}.$$

$$2 \cos^2 x = 1.$$

$$\cos^2 x = \frac{1}{2}.$$

$$\cos x = \frac{1}{2} \sqrt{2}.$$

$$x = 45^\circ.$$

$$20. 4 \sin x = \csc x.$$

$$4 \sin x = \frac{1}{\sin x}.$$

$$4 \sin^2 x = 1.$$

$$\sin^2 x = \frac{1}{4}.$$

$$\sin x = \frac{1}{2}.$$

$$x = 30^\circ.$$

$$21. \sin^2 x = 3 \cos^2 x$$

$$\frac{\sin^2 x}{\cos^2 x} = 3.$$

$$\tan^2 x = 3.$$

$$\tan x = \sqrt{3}.$$

$$x = 60^\circ.$$

$$22. 2 \sin^2 x + \cos^2 x = \frac{3}{2}.$$

$$2 \sin^2 x + 1 - \sin^2 x = \frac{3}{2}.$$

$$\sin^2 x = \frac{1}{2}.$$

$$\sin x = \frac{1}{2} \sqrt{2}.$$

$$x = 45^\circ.$$

$$23. 3 \tan^2 x - \sec^2 x = 1.$$

$$3 \tan^2 x - 1 - \tan^2 x = 1.$$

$$2 \tan^2 x = 2.$$

$$\tan x = 1.$$

$$x = 45^\circ$$

$$24. \tan x + \cot x = 2.$$

$$\tan x + \frac{1}{\tan x} = 2.$$

$$\tan^2 x - 2 \tan x + 1 = 0.$$

$$(\tan x - 1)^2 = 0.$$

$$\tan x = 1.$$

$$x = 45^\circ.$$

$$25. \tan x = 2 \sin x.$$

$$\frac{\tan x}{\sin x} = 2.$$

$$\frac{1}{\cos x} = 2.$$

$$\cos x = \frac{1}{2}.$$

$$x = 60^\circ.$$

$$26. \sec x = \sqrt{2} \tan x.$$

$$\frac{\sec x}{\tan x} = \sqrt{2}.$$

$$\frac{\sec x}{\tan x} = \csc x = \frac{1}{\sin x}.$$

$$\frac{1}{\sin x} = \sqrt{2}.$$

$$\sin x = \frac{1}{\sqrt{2}}.$$

$$= \frac{1}{2} \sqrt{2}.$$

$$x = 45^\circ.$$

$$30. \sin x + \sqrt{3} \cos x = 2.$$

$$\sqrt{1 - \cos^2 x} = 2 - \sqrt{3} \cos x.$$

$$1 - \cos^2 x = 4 - 4\sqrt{3} \cos x + 3 \cos^2 x.$$

$$4 \cos^2 x - 4\sqrt{3} \cos x + 3 = 0.$$

$$(2 \cos x - \sqrt{3})^2 = 0.$$

$$\cos x = \frac{\sqrt{3}}{2}.$$

$$x = 30^\circ.$$

$$31. \text{ Given } (\sin x + \cos x)^2 - 1 = (\sin x - \cos x)^2 + 1, \text{ find } x.$$

$$(\sin x + \cos x)^2 - 1 = (\sin x - \cos x)^2 + 1.$$

$$\sin^2 x + 2 \sin x \cos x + \cos^2 x - 1 = \sin^2 x - 2 \sin x \cos x + \cos^2 x + 1.$$

$$4 \sin x \cos x = 2.$$

$$2 \sin x \cos x = 1.$$

$$4 \sin^2 x \cos^2 x = 1.$$

$$4 \sin^2 x (1 - \sin^2 x) = 1.$$

$$4 \sin^2 x - 4 \sin^4 x = 1.$$

$$27. \sin^2 x - \cos x = \frac{1}{4}.$$

$$1 - \cos^2 x - \cos x = \frac{1}{4}.$$

$$4 \cos^2 x + 4 \cos x - 3 = 0.$$

$$(2 \cos x + 3)(2 \cos x - 1) = 0.$$

$$\cos x = \frac{1}{2} \text{ or } -\frac{3}{2}.$$

$$x = 60^\circ.$$

$$28. \tan^2 x - \sec x = 1.$$

$$1 - \cos^2 x - \cos x = \cos^2 x.$$

$$-2 \cos^2 x - \cos x + 1 = 0.$$

$$2 \cos^2 x + \cos x - 1 = 0.$$

$$(2 \cos x - 1)(\cos x + 1) = 0.$$

$$\cos x = \frac{1}{2}, -1.$$

$$x = 60^\circ \text{ or } 180^\circ.$$

$$29. \tan^2 x + \csc^2 x = 3.$$

$$\tan^2 x + 1 + \cot^2 x = 3.$$

$$\tan^4 x + \tan^2 x + 1 = 3 \tan^2 x.$$

$$\tan^4 x - 2 \tan^2 x + 1 = 0.$$

$$(\tan^2 x - 1)(\tan^2 x - 1) = 0.$$

$$\tan^2 x = 1.$$

$$\tan x = 1.$$

$$x = 45^\circ.$$

32. Given $2 \sin x = \cos x$, find $\sin x$ and $\cos x$.

$$4 \sin^2 x = -\sin^2 x + 1.$$

$$2\sqrt{1 - \cos^2 x} = \cos x.$$

$$5 \sin^2 x = 1.$$

$$4 - 4 \cos^2 x = \cos^2 x.$$

$$\sin^2 x = \frac{1}{5}.$$

$$-5 \cos^2 x = -4.$$

$$\sin x = \frac{1}{5} \sqrt{5}.$$

$$\cos x = \frac{2}{5} \sqrt{5}.$$

33. Given $4 \sin x = \tan x$, find $\sin x$ and $\tan x$.

$$\frac{4 \sin x}{\tan x} = 1.$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$4 \cos x = 1.$$

$$= \frac{1}{4} \sqrt{15} \div \frac{1}{4} = \sqrt{15}$$

$$\cos x = \frac{1}{4}.$$

$$\sin x = \sqrt{1 - \frac{1}{16}} = \frac{1}{4} \sqrt{15}.$$

34. Given $5 \sin x = \tan x$, find $\cos x$ and $\sec x$.

$$\frac{5 \sin x}{\tan x} = 1.$$

$$\sec x = \frac{1}{\cos x}.$$

$$5 \cos x = 1.$$

$$\sec x = 1 \div \frac{1}{5} = 5.$$

$$\cos x = \frac{1}{5}.$$

35. Given $4 \cot x = \tan x$, find the other functions.

$$\frac{4}{\tan x} = \tan x.$$

$$\frac{\sin x}{\cos x} = 2.$$

$$4 = \tan^2 x.$$

$$\sin x = 2 \cos x.$$

$$\tan x = 2.$$

$$\sin^2 x = 4 - 4 \sin^2 x.$$

$$\cot x = \frac{1}{2}.$$

$$\sin^2 x = \frac{4}{5}.$$

$$\sin x = \frac{2}{5} \sqrt{5}. \quad \csc x = \frac{5}{2} \sqrt{5}.$$

$$\cos x = \frac{1}{5} \sqrt{5}. \quad \sec x = \sqrt{5}.$$

36. Given $\sin x = 4 \cos x$, find $\sin x$ and $\cos x$.

$$1 - \cos^2 x = 16 \cos^2 x.$$

$$16 - 16 \sin^2 x = \sin^2 x.$$

$$17 \cos^2 x = 1.$$

$$17 \sin^2 x = 16.$$

$$\cos x = \frac{1}{\sqrt{17}}.$$

$$\sin x = \frac{4}{\sqrt{17}}.$$

37. If $\sin x : \cos x = 9 : 40$, find $\sin x$ and $\cos x$.

$$40 \sin x = 9 \cos x.$$

$$1600 - 1600 \cos^2 x = 81 \cos^2 x.$$

$$40 \sin x = \pm 9 \sqrt{1 - \sin^2 x}.$$

$$1681 \cos^2 x = 1600.$$

$$1600 \sin^2 x = 81 - 81 \sin^2 x.$$

$$\cos^2 x = \frac{1600}{1681}.$$

$$\sin^2 x = \frac{81}{1681}.$$

$$\cos x = \frac{40}{41}.$$

$$\sin x = \frac{9}{41}.$$

38. From the formula $\tan x = \pm \frac{\sin x}{\sqrt{1 - \sin^2 x}}$, find the condition under which $\tan x = \sin x$.

$$\sin x = \pm \frac{\sin x}{\sqrt{1 - \sin^2 x}}.$$

$$\sin^2 x = \frac{\sin^2 x}{1 - \sin^2 x}.$$

$$\sin^4 x = 0.$$

$$\sin x = 0.$$

$$x = 0^\circ.$$

Solve the following equations; that is, find the value of x when:

39. $\cos x = \sec x.$

$$\cos x = \frac{1}{\cos x}.$$

$$\cos^2 x = 1.$$

$$\cos x = \pm 1.$$

$$x = 0^\circ \text{ or } 180^\circ.$$

40. $\cos x = \tan x.$

$$\cos x = \frac{\sin x}{\cos x}.$$

$$\cos^2 x = \sin x.$$

$$1 - \sin^2 x = \sin x.$$

$$\sin^2 x + \sin x - 1 = 0.$$

$$\sin x = 0.6180; -1.6180.$$

$$x = 38^\circ 10'.$$

41. $\cos x = \sin x.$

$$\cos^2 x = 1 - \cos^2 x.$$

$$2 \cos^2 x = 1.$$

$$\cos^2 x = \frac{1}{2}.$$

$$\cos x = \pm \frac{1}{\sqrt{2}} \sqrt{2}.$$

$$x = 45^\circ, \text{ or } 225^\circ.$$

42. $\tan x = \cot x.$

$$\tan x = \frac{1}{\tan x}.$$

$$\tan^2 x = 1.$$

$$\tan x = \pm 1.$$

$$x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

43. $\sec x = \csc x.$

$$\frac{1}{\cos x} = \frac{1}{\sin x}.$$

$$\sin x = \cos x.$$

From Ex. 41, $x = 45^\circ \text{ or } 225^\circ.$

44. $2 \cos x + \sec x = 3.$

$$2 \cos x + \frac{1}{\cos x} = 3.$$

$$2 \cos^2 x + 1 = 3 \cos x.$$

$$2 \cos^2 x - 3 \cos x + 1 = 0.$$

$$(2 \cos x - 1)(\cos x - 1) = 0.$$

$$\cos x = 1 \text{ or } \frac{1}{2}.$$

$$x = 0^\circ \text{ or } 60^\circ$$

45. $\cos^2 x - \sin^2 x = \sin x.$

$$1 - \sin^2 x - \sin^2 x = \sin x.$$

$$2 \sin^2 x + \sin x - 1 = 0.$$

$$(2 \sin x - 1)(\sin x + 1) = 0.$$

$$\sin x = -1, \frac{1}{2}.$$

$$x = 270^\circ \text{ or } 30^\circ.$$

46. $2 \sin x + \cot x = 1 + 2 \cos x.$

$$2 \sin x + \frac{1}{\tan x} = 1 + 2 \frac{\sin x}{\tan x}.$$

$$2 \sin x \tan x + 1 = \tan x + 2 \sin x.$$

$$2 \sin x (\tan x - 1) = \tan x - 1.$$

$$2 \sin x = 1, \text{ or } \tan x - 1 = 0.$$

$$\sin x = \frac{1}{2}, \text{ or } \tan x = 1.$$

$$x = 30^\circ, 45^\circ, 150^\circ, \text{ or } 225^\circ.$$

$$47. \sin^2 x + \tan^2 x = 3 \cos^2 x.$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\tan^2 x}{\cos^2 x} = 3.$$

$$\frac{1 - \cos^2 x}{\cos^2 x} + \frac{1 - \cos^2 x}{\cos^4 x} = 3.$$

$$\cos^2 x - \cos^4 x + 1 - \cos^2 x = 3 \cos^4 x.$$

$$4 \cos^4 x = 1.$$

$$\cos^2 x = \frac{1}{2}.$$

$$\cos x = \pm \frac{1}{2} \sqrt{2}.$$

$$x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

$$48. \tan x + 2 \cot x = \frac{5}{2} \csc x.$$

$$\frac{\sin x}{\cos x} + \frac{2 \cos x}{\sin x} = \frac{5}{2 \sin x}.$$

$$2 \sin^2 x + 4 \cos^2 x = 5 \cos x.$$

$$2 - 2 \cos^2 x + 4 \cos^2 x = 5 \cos x.$$

$$2 \cos^2 x - 5 \cos x + 2 = 0.$$

$$(2 \cos x - 1)(\cos x - 2) = 0.$$

$$\cos x = \frac{1}{2}, 2.$$

$$x = 60^\circ.$$

Prove the following relations:

$$49. \sin A + \cos A = (1 + \tan A) \cos A.$$

$$\frac{\sin A}{\cos A} = \tan A.$$

$$\sin A = \cos A \tan A.$$

$$\cos A \tan A + \cos A = (1 + \tan A) \cos A.$$

$$\cos A (1 + \tan A) = (1 + \tan A) \cos A.$$

$$50. \frac{\cot x}{\cos x} = \sqrt{1 + \cot^2 x}.$$

$$\sqrt{1 + \cot^2 x} = \csc x.$$

$$\frac{\cot x}{\cos x} = \csc x.$$

$$\cot x = \cos x \csc x = \frac{\cos x}{\sin x}.$$

$$51. \cos x : \cot x = \sqrt{1 - \cos^2 x}.$$

$$\sqrt{1 - \cos^2 x} = \sin x.$$

$$\frac{\cos x}{\cot x} = \sin x.$$

$$\cos x \tan x = \sin x.$$

$$\cos x \frac{\sin x}{\cos x} = \sin x.$$

$$\sin x = \sin x.$$

$$52. \tan^2 x = \frac{1}{\cos^2 x} - 1.$$

$$\tan^2 x = \sec^2 x - 1.$$

$$\sec^2 x - 1 = \frac{1}{\cos^2 x} - 1.$$

$$\sec^2 x - 1 = \sec^2 x - 1$$

Find the values of the other functions of A when:

$$53. \sin A = \frac{2}{3}.$$

$$\cos A = \sqrt{1 - \frac{4}{9}} = \frac{1}{3} \sqrt{5}.$$

$$\tan A = \frac{2}{\frac{1}{3} \sqrt{5}} = \frac{2}{5} \sqrt{5}.$$

$$\cot A = \frac{1}{2} \sqrt{5}.$$

$$\sec A = \frac{3}{5} \sqrt{5}.$$

$$\csc A = \frac{3}{2}.$$

$$54. \cos A = \frac{3}{4}.$$

$$\sin A = \sqrt{1 - \frac{9}{16}} = \frac{1}{4}\sqrt{7}.$$

$$\tan A = \frac{1}{3}\sqrt{7}.$$

$$\csc A = \frac{4}{1}\sqrt{7}.$$

$$\sec A = \frac{4}{3}.$$

$$\cot A = \frac{3}{1}\sqrt{7}.$$

$$55. \tan A = 1.5.$$

$$\sec A = \sqrt{1 + \frac{9}{4}} = \frac{1}{2}\sqrt{13}.$$

$$\cos A = \frac{2}{\sqrt{13}} = \frac{2}{13}\sqrt{13}.$$

$$\cot A = \frac{2}{3}.$$

$$\frac{\cos A}{\sin A} = \frac{2}{3}.$$

$$\sin A = \frac{3}{13}\sqrt{13}.$$

$$\csc A = \frac{1}{3}\sqrt{13}.$$

$$56. \cot A = 0.75.$$

$$\tan A = \frac{4}{3}.$$

$$\sec A = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}.$$

$$\cos A = \frac{3}{5}.$$

$$\sin A = \frac{4}{5}.$$

$$\csc A = \frac{5}{4}.$$

$$57. \sec A = 1.5.$$

$$\cos A = \frac{2}{3}.$$

$$\sin A = \sqrt{1 - \frac{4}{9}} = \frac{1}{3}\sqrt{5}.$$

$$\tan A = \frac{1}{2}\sqrt{5}.$$

$$\csc A = \frac{3}{5}\sqrt{5}.$$

$$\cot A = \frac{2}{5}\sqrt{5}.$$

$$58. \sin A = \frac{12}{13}.$$

$$\csc A = \frac{13}{12}.$$

$$\cos A = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}.$$

$$\tan A = \frac{12}{5}.$$

$$\sec A = \frac{13}{5}.$$

$$\cot A = \frac{5}{12}.$$

$$59. \sin A = 0.8.$$

$$\csc A = \frac{5}{4}.$$

$$\cos A = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}.$$

$$\tan A = \frac{4}{3}.$$

$$\sec A = \frac{5}{3}.$$

$$\cot A = \frac{3}{4}.$$

$$60. \cos A = \frac{60}{61}.$$

$$\sec A = \frac{61}{60}.$$

$$\sin A = \sqrt{1 - \frac{3600}{3721}} = \frac{11}{37}.$$

$$\tan A = \frac{11}{60}.$$

$$\csc A = \frac{61}{11}.$$

$$\cot A = \frac{60}{11}.$$

$$61. \cos A = 0.28.$$

$$\sec A = \frac{25}{7}.$$

$$\sin A = \sqrt{1 - \frac{49}{625}} = \frac{24}{25}.$$

$$\tan A = \frac{24}{7}.$$

$$\csc A = \frac{25}{24}.$$

$$\cot A = \frac{7}{24}.$$

$$62. \tan A = \frac{4}{3}.$$

$$\cot A = \frac{3}{4}.$$

$$\sec A = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}.$$

$$\cos A = \frac{3}{5}.$$

$$\sin A = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}.$$

$$\csc A = \frac{5}{4}.$$

63. $\cot A = 1.$

$$\tan A = 1.$$

$$\sec A = \sqrt{1+1} = \sqrt{2}.$$

$$\cos A = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}.$$

$$\sin A = \sqrt{1 - \frac{2}{4}} = \frac{1}{2}\sqrt{2}.$$

$$\csc A = \sqrt{2}.$$

64. $\cot A = 0.5.$

$$\tan A = 2.$$

$$\sec A = \sqrt{1+4} = \sqrt{5}.$$

$$\cos A = \frac{1}{5}\sqrt{5}.$$

$$\sin A = \sqrt{1 - \frac{5}{25}} = \frac{2}{5}\sqrt{5}.$$

$$\csc A = \frac{1}{2}\sqrt{5}.$$

65. $\sec A = 2.$

$$\cos A = \frac{1}{2}.$$

$$\sin A = \sqrt{1 - \frac{1}{4}} = \frac{1}{2}\sqrt{3}.$$

$$\tan A = \sqrt{3}.$$

$$\csc A = \frac{2}{3}\sqrt{3}.$$

$$\cot A = \frac{1}{3}\sqrt{3}.$$

66. $\csc A = \sqrt{2}.$

$$\sin A = \frac{1}{2}\sqrt{2}.$$

$$\cos A = \sqrt{1 - \frac{2}{4}} = \frac{1}{2}\sqrt{2}.$$

$$\tan A = 1.$$

$$\sec A = \sqrt{2}.$$

$$\cot A = 1.$$

67. $\sin A = m.$

$$\csc A = \frac{1}{m}.$$

$$\cos A = \sqrt{1 - m^2}.$$

$$\sec A = \frac{1}{\sqrt{1 - m^2}}.$$

$$\tan A = \frac{m}{\sqrt{1 - m^2}}.$$

$$\cot A = \frac{\sqrt{1 - m^2}}{m}.$$

68. Given $\sin A = 2m : (1 + m^2)$, find the value of $\tan A$.

$$\sin A = \frac{2m}{1 + m^2}.$$

$$\tan A = \pm \frac{\sin A}{\sqrt{1 - \sin^2 A}} = \frac{2m}{\sqrt{(1 + m^2)^2 - 4m^2}} = \frac{2m}{1 - m^2}.$$

69. Given $\cos A = 2mn : (m^2 + n^2)$, find the value of $\sec A$.

$$\cos A = \frac{2mn}{m^2 + n^2}.$$

$$\sec A = \frac{m^2 + n^2}{2mn}.$$

70. Given $\sin 0^\circ = 0$, find the other functions of 0° .

$$\sin 0^\circ = 0.$$

$$\csc 0^\circ = \infty.$$

$$\cos 0^\circ = 1.$$

$$\sec 0^\circ = 1.$$

$$\tan 0^\circ = 0.$$

$$\cot 0^\circ = \infty$$

71. Given $\sin 90^\circ = 1$, find the other functions of 90° .

$$\sin 90^\circ = 1.$$

$$\csc 90^\circ = 1.$$

$$\cos 90^\circ = 0.$$

$$\sec 90^\circ = \infty.$$

$$\tan 90^\circ = \infty.$$

$$\cot 90^\circ = 0.$$

72. Given $\tan 90^\circ = \infty$, find the other functions of 90° .

$$\tan 90^\circ = \infty.$$

$$\cot 90^\circ = 0.$$

$$\sec 90^\circ = \infty.$$

$$\cos 90^\circ = 0.$$

$$\sin 90^\circ = 1.$$

$$\csc 90^\circ = 1.$$

73. Given $\cot 22^\circ 30' = \sqrt{2} + 1$, find the other functions of $22^\circ 30'$.

$$\cot 22^\circ 30' = \sqrt{2} + 1.$$

$$\tan 22^\circ 30' = \sqrt{2} - 1.$$

$$\csc 22^\circ 30' = \frac{\sqrt{1 + 2 + 2\sqrt{2} + 1}}{\sqrt{4 + 2\sqrt{2}}}.$$

$$\sin 22^\circ 30' = \frac{1}{\sqrt{4 + 2\sqrt{2}}}.$$

$$\sec 22^\circ 30' = \frac{\sqrt{1 + 2 + 2\sqrt{2} + 1}}{\sqrt{4 - 2\sqrt{2}}}.$$

$$= \frac{1}{\sqrt{4 - 2\sqrt{2}}}.$$

$$\cos 22^\circ 30' = \frac{1}{\sqrt{4 - 2\sqrt{2}}}.$$

74. Write $\tan^2 A + \cot^2 A$ so as to contain only $\cos A$.

$$\frac{1 - \cos^2 A}{\cos A} + \frac{\cos^2 A}{1 - \cos^2 A} \text{ since } \tan A = \pm \frac{\sqrt{1 - \cos^2 A}}{\cos A}.$$

and

$$\cot A = \pm \frac{\cos A}{\sqrt{1 - \cos^2 A}}.$$

In the triangle ABC, prove the following relations :

75. $\sin A = \sin (B + C).$

$$A = 180^\circ - (B + C).$$

$$\sin [180^\circ - (B + C)] = \sin (B + C).$$

76. $\cos A = -\cos (B + C).$

$$\cos [180^\circ - (B + C)] = -\cos (B + C).$$

77. $\tan A = -\tan (B + C).$

$$\tan [180^\circ - (B + C)] = -\tan (B + C).$$

78. $\cot A = -\cot (B + C).$

$$\cot [180^\circ - (B + C)] = -\cot (B + C).$$

79. $\sin A = -\sin (2A + B + C).$

$$A + B + C = 180^\circ; 2A + B + C = 180^\circ + A.$$

$$\sin A = -\sin (180^\circ + A) = -\sin (2A + B + C).$$

$$80. \sin B = -\sin(A + 2B + C).$$

$$A + B + C = 180^\circ; A + 2B + C = 180^\circ + B.$$

$$\sin B = -\sin(180^\circ + B) = -\sin(A + 2B + C).$$

$$81. \cos C = -\cos(A + B + 2C).$$

$$A + B + C = 180^\circ; A + B + 2C = 180^\circ + C.$$

$$\cos C = -\cos(180^\circ + C) = -\cos(A + B + 2C).$$

$$82. \cot B = \cot(A + 2B + C).$$

$$A + 2B + C = 180^\circ + B.$$

$$\cot B = \cot(180^\circ + B) = \cot(A + 2B + C).$$

$$83. \sin A = -\cos(\tfrac{3}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C).$$

$$\tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 90^\circ; 90^\circ + A = \tfrac{3}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C.$$

$$\sin A = -\cos(90^\circ + A) = -\cos(\tfrac{3}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C).$$

$$84. \cos A = -\cos(2A + B + C).$$

$$2A + B + C = 180^\circ + A.$$

$$\cos A = -\cos(180^\circ + A) = -\cos(2A + B + C).$$

$$85. \cos A = \sin(\tfrac{3}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C).$$

$$\tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 90^\circ; 90^\circ + A = \tfrac{3}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C.$$

$$\cos A = \sin(90^\circ + A) = \sin(\tfrac{3}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C).$$

$$86. \sin(\tfrac{1}{2}A + B) = \cos(\tfrac{1}{2}B - \tfrac{1}{2}C).$$

$$90^\circ = \tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C.$$

$$90^\circ + \tfrac{1}{2}B = \tfrac{1}{2}A + B + \tfrac{1}{2}C.$$

$$90^\circ + \tfrac{1}{2}B - \tfrac{1}{2}C = \tfrac{1}{2}A + B.$$

$$\sin(\tfrac{1}{2}A + B) = \sin(90^\circ + \tfrac{1}{2}B - \tfrac{1}{2}C) = \cos(\tfrac{1}{2}B - \tfrac{1}{2}C).$$

$$87. \sin(\tfrac{1}{2}C - \tfrac{1}{2}A) = -\cos(\tfrac{1}{2}B + C).$$

$$90^\circ + \tfrac{1}{2}C = \tfrac{1}{2}A + \tfrac{1}{2}B + C.$$

$$90^\circ + \tfrac{1}{2}C - \tfrac{1}{2}A = \tfrac{1}{2}B + C.$$

$$-\cos(\tfrac{1}{2}B + C) = -\cos(90^\circ + \tfrac{1}{2}C - \tfrac{1}{2}A) = \sin(\tfrac{1}{2}C - \tfrac{1}{2}A)$$

$$88. \cos B = -\cos(A + 2B + C).$$

$$A + 2B + C = 180^\circ + B.$$

$$\cos B = -\cos(180^\circ + B) = -\cos(A + 2B + C).$$

$$89. \tan A = \tan(2A + B + C).$$

$$2A + B + C = 180^\circ + A.$$

$$\tan A = \tan(180^\circ + A) = \tan 2A + B + C.$$

$$90. \cot A = \tan(\tfrac{3}{2}B + \tfrac{3}{2}C + \tfrac{1}{2}A).$$

$$\tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 90^\circ.$$

$$\tfrac{3}{2}B + \tfrac{3}{2}C + \tfrac{1}{2}A = 90^\circ + B + C = 270^\circ - A.$$

$$\cot A = \tan(270^\circ - A) = \tan(\tfrac{3}{2}B + \tfrac{3}{2}C + \tfrac{1}{2}A).$$

In the quadrilateral $ABCD$, prove the following relations :

91. $-\sin A = \sin(B + C + D).$

$$A = 360^\circ - (B + C + D).$$

$$-\sin A = -\sin[360^\circ - (B + C + D)] = \sin(B + C + D).$$

92. $\cos A = \cos(B + C + D).$

$$A = 360^\circ - (B + C + D).$$

$$\cos A = \cos[360^\circ - (B + C + D)] = \cos(B + C + D).$$

93. $-\tan A = \tan(B + C + D).$

$$A = 360^\circ - (B + C + D).$$

$$-\tan A = -\tan[360^\circ - (B + C + D)] = \tan(B + C + D).$$

94. $-\cot A = \cot(B + C + D).$

$$A = 360^\circ - (B + C + D).$$

$$-\cot A = -\cot[360^\circ - (B + C + D)] = \cot(B + C + D).$$

Exercise 41. Sines and Cosines (Page 98)

Given $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$, $\cos 30^\circ = \sin 60^\circ = \frac{1}{2}\sqrt{3}$, and $\sin 45^\circ = \cos 45^\circ = \frac{1}{2}\sqrt{2}$, find the values of the following:

1. $\sin 15^\circ.$

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos(-30^\circ) + \cos 45^\circ \sin(-30^\circ) \\ &= \frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{2}(-\frac{1}{2}) = \frac{\sqrt{6} - \sqrt{2}}{4} = 0.25875.\end{aligned}$$

2. $\cos 15^\circ.$

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos(-30^\circ) - \sin 45^\circ \sin(-30^\circ) \\ &= \frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{2}(-\frac{1}{2}) = \frac{\sqrt{6} + \sqrt{2}}{4} = 0.96575.\end{aligned}$$

3. $\sin 75^\circ.$

$$\begin{aligned}\sin 75^\circ &= \sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ \\ &= \frac{1}{2} \cdot \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{3} \cdot \frac{1}{2}\sqrt{2} = \frac{\sqrt{2} + \sqrt{6}}{4} = 0.96575.\end{aligned}$$

4. $\cos 75^\circ.$

$$\begin{aligned}\cos 75^\circ &= \cos(30^\circ + 45^\circ) = \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ \\ &= \frac{1}{2}\sqrt{3} \cdot \frac{1}{2}\sqrt{2} - \frac{1}{2} \cdot \frac{1}{2}\sqrt{2} = \frac{\sqrt{6} - \sqrt{2}}{4} = 0.25875.\end{aligned}$$

5. $\sin 90^\circ.$

$$\begin{aligned}\sin 90^\circ &= \sin(45^\circ + 45^\circ) = \sin 45^\circ \cos 45^\circ + \cos 45^\circ \sin 45^\circ \\ &= \frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{2} = \frac{1}{2} + \frac{1}{2} = 1.\end{aligned}$$

6. $\cos 90^\circ.$

$$\begin{aligned}\cos 90^\circ &= \cos(45^\circ + 45^\circ) = \cos 45^\circ \cos 45^\circ - \sin 45^\circ \sin 45^\circ \\ &= \frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{2} = \frac{1}{2} - \frac{1}{2} = 0.\end{aligned}$$

7. $\sin 105^\circ$.

$$\begin{aligned}\sin 105^\circ &= \sin(60^\circ + 45^\circ) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \\ &= \frac{1}{2}\sqrt{3} \cdot \frac{1}{2}\sqrt{2} + \frac{1}{2} \cdot \frac{1}{2}\sqrt{2} = \frac{\sqrt{6} + \sqrt{2}}{4} = 0.96575.\end{aligned}$$

8. $\cos 105^\circ$.

$$\begin{aligned}\cos 105^\circ &= \cos(60^\circ + 45^\circ) = \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ \\ &= \frac{1}{2} \cdot \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{3} \cdot \frac{1}{2}\sqrt{2} = \frac{\sqrt{2} - \sqrt{6}}{4} = -0.25875\end{aligned}$$

9. $\sin 120^\circ$.

$$\begin{aligned}\sin 120^\circ &= \sin(90^\circ + 30^\circ) = \sin 90^\circ \cos 30^\circ + \cos 90^\circ \sin 30^\circ \\ &= 1 \cdot \frac{1}{2}\sqrt{3} + 0 \cdot \frac{1}{2} = \frac{1}{2}\sqrt{3} = 0.866.\end{aligned}$$

10. $\cos 120^\circ$.

$$\begin{aligned}\cos 120^\circ &= \cos(90^\circ + 30^\circ) = \cos 90^\circ \cos 30^\circ - \sin 90^\circ \sin 30^\circ \\ &= 0 \cdot \frac{1}{2}\sqrt{3} - 1 \cdot \frac{1}{2} = -\frac{1}{2} = -0.5.\end{aligned}$$

11. $\sin 135^\circ$.

$$\begin{aligned}\sin 135^\circ &= \sin(90^\circ + 45^\circ) = \sin 90^\circ \cos 45^\circ + \cos 90^\circ \sin 45^\circ \\ &= 1 \cdot \frac{1}{2}\sqrt{2} + 0 \cdot \frac{1}{2}\sqrt{2} = \frac{1}{2}\sqrt{2} = 0.707.\end{aligned}$$

12. $\cos 135^\circ$.

$$\begin{aligned}\cos 135^\circ &= \cos(90^\circ + 45^\circ) = \cos 90^\circ \cos 45^\circ - \sin 90^\circ \sin 45^\circ \\ &= 0 \cdot \frac{1}{2}\sqrt{2} - 1 \cdot \frac{1}{2}\sqrt{2} = -\frac{1}{2}\sqrt{2} = -0.707.\end{aligned}$$

13. $\sin 150^\circ$.

$$\begin{aligned}\sin 150^\circ &= \sin(90^\circ + 60^\circ) = \sin 90^\circ \cos 60^\circ + \cos 90^\circ \sin 60^\circ \\ &= 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}\sqrt{3} = \frac{1}{2} = 0.5.\end{aligned}$$

14. $\cos 150^\circ$.

$$\begin{aligned}\cos 150^\circ &= \cos(90^\circ + 60^\circ) = \cos 90^\circ \cos 60^\circ - \sin 90^\circ \sin 60^\circ \\ &= 0 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2}\sqrt{3} = -\frac{1}{2}\sqrt{3} = -0.866.\end{aligned}$$

15. $\sin 165^\circ$.

$$\begin{aligned}\sin 165^\circ &= \sin[(90^\circ + 45^\circ) + 30^\circ] \\ &= \sin(90^\circ + 45^\circ) \cos 30^\circ + \cos(90^\circ + 45^\circ) \sin 30^\circ \\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\ &= \frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4} = 0.25875.\end{aligned}$$

16. $\cos 165^\circ$.

$$\begin{aligned}\cos 165^\circ &= \cos[(90^\circ + 45^\circ) + 30^\circ] \\ &= \cos(90^\circ + 45^\circ) \cos 30^\circ - \sin(90^\circ + 45^\circ) \sin 30^\circ \\ &= -\sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= -\frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{2} \cdot \frac{1}{2} = \frac{-\sqrt{6} - \sqrt{2}}{4} = -0.96575.\end{aligned}$$

Exercise 42. Tangents and Cotangents (Page 99)

Given $\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}}$, $\cot 30^\circ = \tan 60^\circ = \sqrt{3}$, $\tan 45^\circ = \cot 45^\circ = 1$, find the values of the following:

1. $\tan 15^\circ$.

$$\begin{aligned}\tan 15^\circ &= \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ + \tan(-30^\circ)}{1 - \tan 45^\circ \cdot \tan(-30^\circ)} = \frac{1 + (-\frac{1}{\sqrt{3}})}{1 - 1 \cdot (-\frac{1}{\sqrt{3}})} \\ &= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} = \frac{12 - 6\sqrt{3}}{6} = 2 - \sqrt{3} = 0.268.\end{aligned}$$

2. $\cot 15^\circ$.

$$\begin{aligned}\cot 15^\circ &= \cot(45^\circ - 30^\circ) = \frac{\cot 45^\circ \cdot \cot(-30^\circ) - 1}{\cot(-30^\circ) + \cot 45^\circ} = \frac{1 \cdot (-\sqrt{3}) - 1}{(-\sqrt{3}) + 1} \\ &= \frac{-1 - \sqrt{3}}{1 - \sqrt{3}} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3} = 3.732.\end{aligned}$$

3. $\tan 75^\circ$.

$$\begin{aligned}\tan 75^\circ &= \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \cdot \frac{1}{\sqrt{3}}} \\ &= \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3} = 3.732.\end{aligned}$$

4. $\cot 75^\circ$.

$$\begin{aligned}\cot 75^\circ &= \cot(45^\circ + 30^\circ) = \frac{\cot 45^\circ \cdot \cot 30^\circ - 1}{\cot 30^\circ + \cot 45^\circ} = \frac{1 \cdot \sqrt{3} - 1}{\sqrt{3} + 1} \\ &= \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3} = 0.268.\end{aligned}$$

5. $\tan 90^\circ$.

$$\tan 90^\circ = \tan(45^\circ + 45^\circ) = \frac{\tan 45^\circ + \tan 45^\circ}{1 - \tan 45^\circ \cdot \tan 45^\circ} = \frac{1 + 1}{1 - 1 \cdot 1} = \frac{2}{0} = \infty.$$

6. $\cot 90^\circ$.

$$\cot 90^\circ = \cot(45^\circ + 45^\circ) = \frac{\cot 45^\circ \cot 45^\circ - 1}{\cot 45^\circ + \cot 45^\circ} = \frac{1 \cdot 1 - 1}{1 + 1} = \frac{0}{2} = 0.$$

7. $\tan 105^\circ$.

$$\begin{aligned}\tan 105^\circ &= \tan(60^\circ + 45^\circ) = \frac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ \tan 45^\circ} = \frac{\sqrt{3} + 1}{1 - \sqrt{3} \cdot 1} = \frac{\sqrt{3} + 1}{1 - \sqrt{3}} \\ &= \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3} = -3.732.\end{aligned}$$

8. $\cot 105^\circ$.

$$\begin{aligned}\cot 105^\circ &= \cot(60^\circ + 45^\circ) = \frac{\cot 60^\circ \cot 45^\circ - 1}{\cot 45^\circ + \cot 60^\circ} = \frac{\frac{1}{\sqrt{3}} \cdot 1 - 1}{1 + \frac{1}{\sqrt{3}}} \\ &= \frac{\sqrt{3} - 3}{3 + \sqrt{3}} = \frac{12 - 6\sqrt{3}}{-6} = -2 + \sqrt{3} = -0.268.\end{aligned}$$

9. $\tan 120^\circ$.

$$\begin{aligned}\tan 120^\circ &= \tan(60^\circ + 60^\circ) = \frac{\tan 60^\circ + \tan 60^\circ}{1 - \tan 60^\circ \cdot \tan 60^\circ} = \frac{\sqrt{3} + \sqrt{3}}{1 - \sqrt{3} \cdot \sqrt{3}} = \frac{2\sqrt{3}}{1 - 3} \\ &= -\sqrt{3} = -1.732.\end{aligned}$$

10. $\cot 120^\circ$.

$$\begin{aligned}\cot 120^\circ &= \cot(60^\circ + 60^\circ) = \frac{\cot 60^\circ \cot 60^\circ - 1}{\cot 60^\circ + \cot 60^\circ} = \frac{\frac{1}{3}\sqrt{3} \cdot \frac{1}{3}\sqrt{3} - 1}{\frac{1}{3}\sqrt{3} + \frac{1}{3}\sqrt{3}} \\ &= \frac{\frac{1}{3} - 1}{\frac{2}{3}\sqrt{3}} = -\frac{1}{\sqrt{3}} = -0.5774.\end{aligned}$$

11. $\tan 135^\circ$.

$$\begin{aligned}\tan 135^\circ &= \tan[(45^\circ + 30^\circ) + 60^\circ] = \frac{\tan(45^\circ + 30^\circ) + \tan 60^\circ}{1 - \tan(45^\circ + 30^\circ) \tan 60^\circ} \\ \tan(45^\circ + 30^\circ) &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{3}\sqrt{3}}{1 - \frac{1}{3}\sqrt{3}} = 2 + \sqrt{3}. \\ \frac{\tan(45^\circ + 30^\circ) + \tan 60^\circ}{1 - \tan(45^\circ + 30^\circ) \tan 60^\circ} &= \frac{2 + \sqrt{3} + \sqrt{3}}{1 - (2 + \sqrt{3})\sqrt{3}} = -\frac{1 + \sqrt{3}}{1 + \sqrt{3}} = -1.\end{aligned}$$

12. $\cot 135^\circ$.

$$\begin{aligned}\cot 135^\circ &= \cot[(45^\circ + 30^\circ) + 60^\circ] = \frac{\cot(45^\circ + 30^\circ) \cot 60^\circ - 1}{\cot 60^\circ + \cot(45^\circ + 30^\circ)}. \quad (1) \\ \cot(45^\circ + 30^\circ) &= \frac{\cot 45^\circ \cot 30^\circ - 1}{\cot 30^\circ + \cot 45^\circ} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}.\end{aligned}$$

Substitute in (1),

$$\frac{(2 - \sqrt{3})\frac{1}{3}\sqrt{3} - 1}{\frac{1}{3}\sqrt{3} + (2 - \sqrt{3})} = \frac{\frac{2}{3}\sqrt{3} - 2}{-(\frac{2}{3}\sqrt{3} - 2)} = -1.$$

13. $\tan 150^\circ$.

$$\begin{aligned}\tan 150^\circ &= \tan[(60^\circ + 60^\circ) + 30^\circ] = \frac{\tan(60^\circ + 60^\circ) + \tan 30^\circ}{1 - \tan(60^\circ + 60^\circ) \tan 30^\circ} \\ \tan(60^\circ + 60^\circ) &= \frac{\tan 60^\circ + \tan 60^\circ}{1 - \tan 60^\circ \tan 60^\circ} = \frac{\sqrt{3} + \sqrt{3}}{1 - \sqrt{3} \cdot \sqrt{3}} = -\sqrt{3}. \\ \frac{\tan(60^\circ + 60^\circ) + \tan 30^\circ}{1 - \tan(60^\circ + 60^\circ) \tan 30^\circ} &= \frac{-\sqrt{3} + \frac{1}{3}\sqrt{3}}{1 - (-\sqrt{3} \cdot \frac{1}{3}\sqrt{3})} = -\frac{\sqrt{3}}{3} = -0.577\end{aligned}$$

14. $\cot 150^\circ$.

$$\begin{aligned}\cot 150^\circ &= \cot[(60^\circ + 60^\circ) + 30^\circ] = \frac{\cot(60^\circ + 60^\circ) \cot 30^\circ - 1}{\cot 30^\circ + \cot(60^\circ + 60^\circ)} \\ \cot(60^\circ + 60^\circ) &= \frac{\cot 60^\circ \cot 60^\circ - 1}{\cot 60^\circ + \cot 60^\circ} = \frac{\frac{1}{3}\sqrt{3} \cdot \frac{1}{3}\sqrt{3} - 1}{\frac{1}{3}\sqrt{3} + \frac{1}{3}\sqrt{3}} = -\frac{1}{\sqrt{3}}. \\ \frac{\cot(60^\circ + 60^\circ) \cot 30^\circ - 1}{\cot 30^\circ + \cot(60^\circ + 60^\circ)} &= \frac{-\frac{1}{\sqrt{3}} \cdot \sqrt{3} - 1}{\sqrt{3} - \frac{1}{\sqrt{3}}} = -\sqrt{3} = -1.732.\end{aligned}$$

15. $\tan 165^\circ$.

$$\begin{aligned}\tan 165^\circ &= \tan [(60^\circ + 60^\circ) + 45^\circ] = \frac{\tan (60^\circ + 60^\circ) + \tan 45^\circ}{1 - \tan (60^\circ + 60^\circ) \tan 45^\circ} \\ &= \frac{-\sqrt{3} + 1}{1 + \sqrt{3}} = \frac{4 - 2\sqrt{3}}{-2} = \sqrt{3} - 2 = -0.268.\end{aligned}$$

16. $\cot 165^\circ$.

$$\cot 165^\circ = \cot [(60^\circ + 60^\circ) + 45^\circ] = \frac{\cot (60^\circ + 60^\circ) \cot 45^\circ - 1}{\cot 45^\circ + \cot (60^\circ + 60^\circ)}. \quad (1)$$

$$\begin{aligned}\cot (60^\circ + 60^\circ) &= \frac{\cot 60^\circ \cot 60^\circ - 1}{\cot 60^\circ + \cot 60^\circ} = \frac{\frac{1}{3}\sqrt{3} \cdot \frac{1}{3}\sqrt{3} - 1}{\frac{1}{3}\sqrt{3} + \frac{1}{3}\sqrt{3}} \\ &= \frac{-\frac{2}{3}}{\frac{2}{3}\sqrt{3}} = -\frac{1}{\sqrt{3}}.\end{aligned}$$

Substitute in (1),

$$\cot 165^\circ = \frac{-\frac{1}{\sqrt{3}} - 1}{1 - \frac{1}{\sqrt{3}}} = -2 - \sqrt{3} = -3.732.$$

Exercise 43. The Addition Formulas (Page 102)Given $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$, $\sin y = \frac{5}{13}$, $\cos y = \frac{12}{13}$, find the value of:1. $\sin(x + y)$.

$$\sin(x + y) = \sin x \cos y + \cos x \sin y = \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \frac{5}{13} = \frac{56}{65}.$$

2. $\sin(x - y)$.

$$\sin(x - y) = \sin x \cos y - \cos x \sin y = \frac{3}{5} \cdot \frac{12}{13} - \frac{4}{5} \cdot \frac{5}{13} = \frac{16}{65}.$$

3. $\cos(x + y)$.

$$\cos(x + y) = \cos x \cos y - \sin x \sin y = \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{33}{65}.$$

4. $\cos(x - y)$.

$$\cos(x - y) = \cos x \cos y + \sin x \sin y = \frac{4}{5} \cdot \frac{12}{13} + \frac{3}{5} \cdot \frac{5}{13} = \frac{63}{65}.$$

5. $\tan(x + y)$.

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{56}{33} = 1\frac{23}{33}.$$

6. $\tan(x - y)$.

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{\frac{3}{4} - \frac{5}{12}}{1 + \frac{3}{4} \cdot \frac{5}{12}} = \frac{16}{63}.$$

By letting $x = 90^\circ$ in the formulas, find the following:7. $\sin(90^\circ - y)$.

$$\begin{aligned}\sin(90^\circ - y) &= \sin(x - y) = \sin x \cos y - \cos x \sin y \\ &= 1 \cdot \cos y - 0 \cdot \sin y = \cos y.\end{aligned}$$

$$3. \cos(90^\circ - y).$$

$$\begin{aligned}\cos(90^\circ - y) &= \cos(x - y) = \cos x \cos y + \sin x \sin y \\ &= 0 \cdot \cos y + 1 \cdot \sin y = \sin y.\end{aligned}$$

$$9. \tan(90^\circ - y).$$

$$\tan(90^\circ - y) = \frac{\sin(90^\circ - y)}{\cos(90^\circ - y)} = \frac{\cos y}{\sin y} = \cot y.$$

Similarly, by substituting in the formulas, find the following:

$$10. \sin(90^\circ + y).$$

$$\sin(90^\circ + y) = \sin x \cos y + \cos x \sin y = 1 \cdot \cos y + 0 \cdot \sin y = \cos y.$$

$$11. \sin(180^\circ - y).$$

$$\sin(180^\circ - y) = \sin x \cos y - \cos x \sin y = 0 \cdot \cos y - (-1) \sin y = \sin y.$$

$$12. \sin(180^\circ + y).$$

$$\sin(180^\circ + y) = \sin x \cos y + \cos x \sin y = 0 \cdot \cos y + (-1) \sin y = -\sin y$$

$$13. \sin(270^\circ - y).$$

$$\sin(270^\circ - y) = \sin x \cos y - \cos x \sin y = -1 \cdot \cos y - 0 \cdot \sin y = -\cos y.$$

$$14. \sin(270^\circ + y).$$

$$\sin(270^\circ + y) = \sin x \cos y + \cos x \sin y = -1 \cdot \cos y + 0 \cdot \sin y = -\cos y.$$

$$15. \sin(360^\circ - y).$$

$$\sin(360^\circ - y) = \sin x \cos y - \cos x \sin y = 0 \cdot \cos y - 1 \cdot \sin y = -\sin y.$$

$$16. \sin(360^\circ + y).$$

$$\sin(360^\circ + y) = \sin x \cos y + \cos x \sin y = 0 \cdot \cos y + 1 \cdot \sin y = \sin y.$$

$$17. \cos(x - 90^\circ).$$

$$\cos(x - 90^\circ) = \cos x \cos y + \sin x \sin y = \cos x \cdot 0 + \sin x \cdot 1 = \sin x.$$

$$18. \cos(x - 180^\circ).$$

$$\cos(x - 180^\circ) = \cos x \cos y + \sin x \sin y = \cos x (-1) + \sin x \cdot 0 = -\cos x$$

$$19. \cos(x - 270^\circ).$$

$$\cos(x - 270^\circ) = \cos x \cos y + \sin x \sin y = \cos x \cdot 0 + \sin x (-1) = -\sin x.$$

$$20. \tan(x - 90^\circ).$$

$$\begin{aligned}\tan(x - 90^\circ) &= \frac{\sin(x - 90^\circ)}{\cos(x - 90^\circ)} = \frac{\sin x \cos 90^\circ - \cos x \sin 90^\circ}{\cos x \cos 90^\circ + \sin x \sin 90^\circ} = \frac{-\cos x}{\sin x} \\ &= -\cot x.\end{aligned}$$

21. $\tan(x - 180^\circ)$.

$$\tan(x - 180^\circ) = \frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{\tan x - 0}{1 + \tan x \cdot 0} = \frac{\tan x}{1} = \tan x.$$

22. $\cot(x - 90^\circ)$.

$$\cot(x - 90^\circ) = \frac{\cot x \cot y + 1}{\cot y - \cot x} = \frac{\cot x \cdot 0 + 1}{0 - \cot x} = \frac{1}{-\cot x} = -\tan x.$$

23. $\cot(x - 180^\circ)$.

$$\begin{aligned}\cot(x - 180^\circ) &= \frac{\cos(x - 180^\circ)}{\sin(x - 180^\circ)} = \frac{\cos x \cos 180^\circ + \sin x \sin 180^\circ}{\sin x \cos 180^\circ - \cos x \sin 180^\circ} \\ &= \frac{-\cos x}{-\sin x} = \cot x.\end{aligned}$$

24. $\sin(-y)$.

$$\begin{aligned}\sin(-y) &= \sin(360^\circ - y) = \sin 360^\circ \cos y - \cos 360^\circ \sin y \\ &= 0 \cos y - 1 \cdot \sin y = -\sin y.\end{aligned}$$

25. $\sin(45^\circ - y)$.

$$\begin{aligned}\sin(45^\circ - y) &= \sin x \cos y - \cos x \sin y = \frac{1}{2} \sqrt{2} \cdot \cos y - \frac{1}{2} \sqrt{2} \sin y \\ &= \frac{1}{2} \sqrt{2} (\cos y - \sin y).\end{aligned}$$

26. $\cos(45^\circ - y)$.

$$\begin{aligned}\cos(45^\circ - y) &= \cos x \cos y + \sin x \sin y = \frac{1}{2} \sqrt{2} \cos y + \frac{1}{2} \sqrt{2} \sin y \\ &= \frac{1}{2} \sqrt{2} (\cos y + \sin y).\end{aligned}$$

27. $\tan(45^\circ - y)$.

$$\tan(45^\circ - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{1 - \tan y}{1 + 1 \cdot \tan y} = \frac{1 - \tan y}{1 + \tan y}.$$

28. $\cot(30^\circ + y)$.

$$\cot(30^\circ + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x} = \frac{\sqrt{3} \cdot \cot y - 1}{\cot y + \sqrt{3}} = \frac{\sqrt{3} \cot y - 1}{\cot y + \sqrt{3}}.$$

29. $\cot(60^\circ - y)$.

$$\cot(60^\circ - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x} = \frac{\frac{1}{3} \sqrt{3} \cot y + 1}{\cot y - \frac{1}{3} \sqrt{3}}.$$

30. $\cot(90^\circ - y)$.

$$\cot(90^\circ - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x} = \frac{0 \cdot \cot y + 1}{\cot y - 0} = \frac{1}{\cot y} = \tan y.$$

31. If $\tan x = 0.5$ and $\tan y = 0.25$, find $\tan(x + y)$ and $\tan(x - y)$.

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{0.5 + 0.25}{1 - 0.5 \cdot 0.25} = \frac{0.75}{0.875} = 0.8571.$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{0.5 - 0.25}{1 + 0.5 \cdot 0.25} = \frac{0.25}{1.125} = 0.2222.$$

32. If $\tan x = 1$ and $\tan y = \frac{1}{3} \sqrt{3}$, find $\tan(x + y)$ and $\tan(x - y)$.

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{1 + \frac{1}{3}\sqrt{3}}{1 - 1 \cdot \frac{1}{3}\sqrt{3}} = \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3} = 3.732$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{1 - \frac{1}{3}\sqrt{3}}{1 + 1 \cdot \frac{1}{3}\sqrt{3}} = \frac{12 - 6\sqrt{3}}{6} = 2 - \sqrt{3} = 0.268.$$

33. If $\tan x = \frac{5}{6}$ and $\tan y = \frac{1}{11}$, find $\tan(x + y)$ and $\tan(x - y)$, and find the number of degrees in $x + y$.

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{5}{6} + \frac{1}{11}}{1 - \frac{5}{6} \cdot \frac{1}{11}} = \frac{\frac{6}{6} \frac{1}{6}}{\frac{6}{6} \frac{6}{6}} = 1; \quad x + y = 45^\circ.$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{\frac{5}{6} - \frac{1}{11}}{1 + \frac{5}{6} \cdot \frac{1}{11}} = \frac{\frac{4}{6} \frac{9}{6}}{\frac{7}{6} \frac{1}{6}} = \frac{4}{7}.$$

34. If $\tan x = 2$ and $\tan y = \frac{1}{2}$, what is the nature of the angle $x + y$? Consider the same question when $\tan x = 3$ and $\tan y = \frac{1}{3}$, and when $\tan x = a$ and $\tan y = \frac{1}{a}$.

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{2 + \frac{1}{2}}{1 - 2 \cdot \frac{1}{2}} = \frac{\frac{5}{2}}{0} = \infty;$$

$$\tan 90^\circ \text{ and } 270^\circ = \infty.$$

$$x + y = 90^\circ, 270^\circ.$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{3 + \frac{1}{3}}{1 - 3 \cdot \frac{1}{3}} = \frac{\frac{10}{3}}{0} = \infty.$$

$$\tan 90^\circ \text{ and } 270^\circ = \infty.$$

$$x + y = 90^\circ, 270^\circ.$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{a + \frac{1}{a}}{1 - a \cdot \frac{1}{a}} = \frac{\frac{a^2 + 1}{a}}{0} = \infty.$$

$$\tan 90^\circ \text{ and } 270^\circ = \infty.$$

$$x + y = 90^\circ, 270^\circ.$$

35. Prove that the sum of $\tan(x - 45^\circ)$ and $\cot(x + 45^\circ)$ is zero.

$$\tan(x - 45^\circ) = \frac{\tan x - \tan 45^\circ}{1 + \tan x \tan 45^\circ} = \frac{\tan x - 1}{1 + \tan x};$$

$$\cot(x + 45^\circ) = \frac{\cot x \cot y - 1}{\cot y + \cot x} = \frac{\cot x - 1}{1 + \cot x};$$

$$\begin{aligned} \frac{\tan x - 1}{1 + \tan x} + \frac{\cot x - 1}{1 + \cot x} &= \frac{\tan x - 1}{1 + \tan x} + \frac{\frac{1}{\tan x} - 1}{1 + \frac{1}{\tan x}} \\ &= \frac{\tan x - 1 + 1 - \tan x}{1 + \tan x} = 0. \end{aligned}$$

36. Prove that the sum of $\cot(x - 45^\circ)$ and $\tan(x + 45^\circ)$ is zero.

$$\begin{aligned}\cot(x - 45^\circ) &= \frac{\cot x \cot y + 1}{\cot y - \cot x} = \frac{\cot x \cdot 1 + 1}{1 - \cot x} = \frac{\cot x + 1}{1 - \cot x}; \\ \tan(x + 45^\circ) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\tan x + 1}{1 - \tan x \cdot 1} = \frac{\tan x + 1}{1 - \tan x}; \\ \frac{\cot x + 1}{1 - \cot x} + \frac{\tan x + 1}{1 - \tan x} &= \frac{\frac{1}{\tan x} + 1}{1 - \frac{1}{\tan x}} + \frac{\tan x + 1}{1 - \tan x} \\ &= \frac{1 + \tan x - \tan x - 1}{\tan x - 1} = 0.\end{aligned}$$

37. If $\sin x = 0.2\sqrt{5}$ and $\sin y = 0.1\sqrt{10}$, prove that $x + y = 45^\circ$. May $x + y$ have other values? If so, state two of these values.

$$\begin{aligned}\sin x &= \frac{1}{5}\sqrt{5}. \\ \cos x &= \sqrt{1 - \left(\frac{1}{5}\sqrt{5}\right)^2} = \sqrt{\frac{4}{5}} = \frac{2}{5}\sqrt{5}. \\ \sin y &= \frac{1}{10}\sqrt{10}. \\ \cos y &= \sqrt{1 - \left(\frac{1}{10}\sqrt{10}\right)^2} = \sqrt{\frac{9}{10}} = \frac{3}{10}\sqrt{10}. \\ \sin(x + y) &= \sin x \cos y + \cos x \sin y = \frac{1}{5}\sqrt{5} \cdot \frac{3}{10}\sqrt{10} + \frac{2}{5}\sqrt{5} \cdot \frac{1}{10}\sqrt{10} \\ &= \frac{3}{50}\sqrt{50} + \frac{2}{50}\sqrt{50} = \frac{1}{10}\sqrt{50} = \frac{1}{2}\sqrt{2}. \\ \sin 45^\circ &= \frac{1}{2}\sqrt{2}. \quad x + y \text{ may have other values, as } 225^\circ, 405^\circ. \\ \sin(x + y) &= \frac{1}{2}\sqrt{2}. \\ x + y &= 45^\circ.\end{aligned}$$

38. Prove that if an angle x is decreased by 45° the cotangent of the resulting angle is equal to $-\frac{\cot x + 1}{\cot x - 1}$.

$$\cot(x - 45^\circ) = \frac{\cot x \cot 45^\circ + 1}{\cot 45^\circ - \cot x} = \frac{\cot x + 1}{1 - \cot x} = -\frac{\cot x + 1}{\cot x - 1}.$$

39. Prove that if an angle x is increased by 45° the cotangent of the resulting angle is equal to $\frac{\cot x - 1}{\cot x + 1}$.

$$\cot(x + 45^\circ) = \frac{\cot x \cot 45^\circ - 1}{\cot y + \cot x} = \frac{\cot x - 1}{1 + \cot x} = \frac{\cot x - 1}{\cot x + 1}.$$

40. If $\tan x = \frac{a}{1+a}$ and $\tan y = \frac{1}{1+2a}$, prove that $\tan(x + y) = 1$.

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{a}{1+a} + \frac{1}{1+2a}}{1 - \frac{a}{1+a} \cdot \frac{1}{1+2a}} = 1.$$

41. If a right angle is divided into any three angles x, y, z , prove that

$$\tan x = \frac{1 - \tan y \tan z}{\tan y + \tan z}.$$

$$90^\circ = x + y + z.$$

$$x = 90^\circ - (y + z).$$

$$\tan(y + z) = \frac{\tan y + \tan z}{1 - \tan y \tan z}.$$

$$\tan x = \tan[90^\circ - (y + z)] = \frac{1}{\tan y + \tan z} = \frac{1 - \tan y \tan z}{\tan y + \tan z}.$$

Exercise 44. Functions of Twice an Angle (Page 103)

As suggested above, deduce the formulas for the following:

1. $\sin 2x$.

$$\sin 2x = \sin(x + x) = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x.$$

2. $\cos 2x$.

$$\cos 2x = \cos(x + x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$$

3. $\tan 2x$.

$$\tan 2x = \tan(x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x} = \frac{2 \tan x}{1 - \tan^2 x}.$$

4. $\cot 2x$.

$$\cot 2x = \cot(x + x) = \frac{\cot x \cot x - 1}{\cot x + \cot x} = \frac{\cot^2 x - 1}{2 \cot x}.$$

Find $\sin 2x$, given the following values of $\sin x$ and $\cos x$:

5. $\sin x = \frac{1}{2}\sqrt{2}, \cos x = \frac{1}{2}\sqrt{2}.$

$$\sin 2x = 2 \sin x \cos x = 2 \cdot \frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{2} = 1.$$

6. $\sin x = \frac{1}{2}, \cos x = \frac{1}{2}\sqrt{3}.$

$$\sin 2x = 2 \sin x \cos x = 2 \cdot \frac{1}{2} \cdot \frac{1}{2}\sqrt{3} = \frac{1}{2}\sqrt{3}.$$

Find $\cos 2x$, given the following values of $\sin x$ and $\cos x$:

7. $\sin x = \frac{1}{2}\sqrt{3}, \cos x = \frac{1}{2}.$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\sqrt{3}\right)^2 \\ &= -\frac{1}{2}. \end{aligned}$$

8. $\sin x = \frac{3}{5}, \cos x = \frac{4}{5}.$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 \\ &= \frac{7}{25}. \end{aligned}$$

Find $\tan 2x$, given the following values of $\tan x$:

9. $\tan x = 0.3673$.

$$\begin{aligned}\tan 2x &= \frac{2 \tan x}{1 - \tan^2 x} \\ &= \frac{2 \cdot 0.3673}{1 - (0.3673)^2} \\ &= \frac{0.7346}{0.8651} \\ &= 0.8492.\end{aligned}$$

10. $\tan x = 0.2701$.

$$\begin{aligned}\tan 2x &= \frac{2 \tan x}{1 - \tan^2 x} \\ &= \frac{2 \cdot 0.2701}{1 - (0.2701)^2} \\ &= \frac{0.5402}{0.9270} \\ &= 0.5827.\end{aligned}$$

Find $\cot 2x$, given the following values of $\cot x$ and $\tan x$:

11. $\cot x = 0.3673$.

$$\begin{aligned}\cot 2x &= \frac{\cot^2 x - 1}{2 \cot x} \\ &= \frac{(0.3673)^2 - 1}{2 \cdot 0.3673} \\ &= \frac{-0.8651}{0.7346} \\ &= -1.1776.\end{aligned}$$

12. $\tan x = 0.2701$.

$$\begin{aligned}\tan 2x &= \frac{2 \tan x}{1 - \tan^2 x} \\ \cot 2x &= \frac{1}{\tan 2x} \\ &= \frac{1 - \tan^2 x}{2 \tan x} \\ &= \frac{1 - (0.2701)^2}{2 \times 0.2701} \\ &= 1.7161.\end{aligned}$$

Find $\sin 2x$, given the following values of $\sin x$:

13. $\sin x = \frac{5}{13}$.

$$\begin{aligned}\sin 2x &= 2 \sin x \cos x, \\ \cos x &= \sqrt{1 - \left(\frac{5}{13}\right)^2} = \frac{12}{13}, \\ \sin 2x &= 2 \cdot \frac{5}{13} \cdot \frac{12}{13} = \frac{120}{169} \\ &= 0.71006.\end{aligned}$$

14. $\sin x = \frac{1}{3}$.

$$\begin{aligned}\sin 2x &= 2 \sin x \cos x, \\ \cos x &= \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3} = \frac{2.828}{3}, \\ \sin 2x &= 2 \cdot \frac{1}{3} \cdot \frac{2.828}{3} = \frac{1.131}{3} \\ &= 0.71006.\end{aligned}$$

15. As suggested in § 101, find $\sin 3x$ in terms of $\sin x$.

$$\begin{aligned}\sin 3x &= \sin (x + 2x) \\ &= \sin x \cos 2x + \cos x \sin 2x \\ &= \sin x (\cos^2 x - \sin^2 x) + 2 \sin x \cos^2 x \\ &= \sin x (1 - \sin^2 x) - \sin^3 x + 2 \sin x (1 - \sin^2 x) \\ &= \sin x - \sin^3 x - \sin^3 x + 2 \sin x - 2 \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x.\end{aligned}$$

16. As suggested in § 101, find $\cos 3x$ in terms of $\cos x$.

$$\begin{aligned}\cos 3x &= \cos (x + 2x) \\ &= \cos x \cos 2x - \sin x \sin 2x \\ &= \cos x (\cos^2 x - \sin^2 x) - \sin x (2 \sin x \cos x) \\ &= \cos x (2 \cos^2 x - 1) - 2 \sin^2 x \cos x \\ &= 2 \cos^3 x - \cos x - 2 \cos x (1 - \cos^2 x) \\ &= 4 \cos^3 x - 3 \cos x.\end{aligned}$$

Exercise 45. Functions of Half an Angle (Page 104)

Given $\sin 30^\circ = \frac{1}{2}$, find the values of the following :

1. $\sin 15^\circ$.

$$\sin 15^\circ = \sin \frac{1}{2} \cdot 30^\circ = \sqrt{\frac{1 - \cos 30^\circ}{2}} = \sqrt{\frac{1 - \frac{1}{2}\sqrt{3}}{2}} = \frac{1}{2}\sqrt{2 - \sqrt{3}} = 0.2588.$$

2. $\cos 15^\circ$.

$$\cos 15^\circ = \cos \frac{1}{2} \cdot 30^\circ = \sqrt{\frac{1 + \cos 30^\circ}{2}} = \sqrt{\frac{1 + \frac{1}{2}\sqrt{3}}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{3}} = 0.9659.$$

3. $\tan 15^\circ$.

$$\tan 15^\circ = \tan \frac{1}{2} \cdot 30^\circ = \sqrt{\frac{1 - \cos 30^\circ}{1 + \cos 30^\circ}} = \sqrt{\frac{1 - \frac{1}{2}\sqrt{3}}{1 + \frac{1}{2}\sqrt{3}}} = \sqrt{7 - 4\sqrt{3}} = 0.2679.$$

4. $\cot 15^\circ$.

$$\cot 15^\circ = \cot \frac{1}{2} \cdot 30^\circ = \sqrt{\frac{1 + \cos 30^\circ}{1 - \cos 30^\circ}} = \sqrt{\frac{1 + \frac{1}{2}\sqrt{3}}{1 - \frac{1}{2}\sqrt{3}}} = \sqrt{7 + 4\sqrt{3}} = 3.7321.$$

5. $\cot 7\frac{1}{2}^\circ$.

$$\cot 7\frac{1}{2}^\circ = \cot \frac{1}{2} \cdot 15^\circ = \sqrt{\frac{1 + \cos 15^\circ}{1 - \cos 15^\circ}} = \sqrt{\frac{1 + 0.9659}{1 - 0.9659}} = 7.5928.$$

Given $\tan 45^\circ = 1$, find the values of the following :

6. $\sin 22.5^\circ$.

$$\sin 22.5^\circ = \sin \frac{1}{2} \cdot 45^\circ = \sqrt{\frac{1 - \cos 45^\circ}{2}} = \sqrt{\frac{1 - \frac{1}{2}\sqrt{2}}{2}} = \frac{1}{2}\sqrt{2 - \sqrt{2}} = 0.3827.$$

7. $\cos 22.5^\circ$.

$$\cos 22.5^\circ = \cos \frac{1}{2} \cdot 45^\circ = \sqrt{\frac{1 + \cos 45^\circ}{2}} = \sqrt{\frac{1 + \frac{1}{2}\sqrt{2}}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{2}} = 0.9239.$$

8. $\tan 22.5^\circ$.

$$\tan 22.5^\circ = \tan \frac{1}{2} \cdot 45^\circ = \sqrt{\frac{1 - \cos 45^\circ}{1 + \cos 45^\circ}} = \sqrt{\frac{1 - \frac{1}{2}\sqrt{2}}{1 + \frac{1}{2}\sqrt{2}}} = \sqrt{3 - 2\sqrt{2}} = 0.4142.$$

9. $\cot 22.5^\circ$.

$$\cot 22.5^\circ = \cot \frac{1}{2} \cdot 45^\circ = \sqrt{\frac{1 + \cos 45^\circ}{1 - \cos 45^\circ}} = \sqrt{\frac{1 + \frac{1}{2}\sqrt{2}}{1 - \frac{1}{2}\sqrt{2}}} = \sqrt{3 + 2\sqrt{2}} = 2.4142.$$

10. $\cot 11\frac{1}{4}^\circ$.

$$\cot 11\frac{1}{4}^\circ = \cot \frac{1}{2} \cdot 22\frac{1}{2}^\circ = \sqrt{\frac{1 + \cos 22\frac{1}{2}^\circ}{1 - \cos 22\frac{1}{2}^\circ}} = \sqrt{\frac{1 + 0.9239}{1 - 0.9239}} = 5.0280.$$

11. Given $\sin x = 0.2$, find $\sin \frac{1}{2}x$ and $\cos \frac{1}{2}x$.

$$\cos^2 x = 1 - \sin^2 x = 1 - 0.04.$$

$$\cos x = \sqrt{0.96}.$$

$$\sin \frac{1}{2}x = \sqrt{\frac{1 - \cos x}{2}} = \sqrt{\frac{1 - \sqrt{0.96}}{2}} = \sqrt{\frac{1 - 0.4\sqrt{6}}{2}} = 0.10051.$$

$$\cos \frac{1}{2}x = \sqrt{\frac{1 + \cos x}{2}} = \sqrt{\frac{1 + 0.4\sqrt{6}}{2}} = 0.99493.$$

12. Given $\cos x = 0.7$, find $\sin \frac{1}{2}x$, $\cos \frac{1}{2}x$, $\tan \frac{1}{2}x$, and $\cot \frac{1}{2}x$.

$$\sin \frac{1}{2}x = \sqrt{\frac{1 - \cos x}{2}} = \sqrt{\frac{1 - 0.7}{2}} = 0.3873.$$

$$\cos \frac{1}{2}x = \sqrt{\frac{1 + \cos x}{2}} = \sqrt{\frac{1 + 0.7}{2}} = 0.92196.$$

$$\tan \frac{1}{2}x = \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \sqrt{\frac{1 - 0.7}{1 + 0.7}} = 0.42009.$$

$$\cot \frac{1}{2}x = \sqrt{\frac{1 + \cos x}{1 - \cos x}} = \sqrt{\frac{1 + 0.7}{1 - 0.7}} = 2.3805.$$

Exercise 46. Formulas (Page 105)

Prove the following formulas:

$$1. \sin 2x = \frac{2 \tan x}{1 + \tan^2 x}.$$

$$\sin 2x = 2 \sin x \cos x = \frac{2 \sin x}{\cos x} \cos^2 x = 2 \tan x \cos^2 x$$

$$= \frac{2 \tan x}{\sec^2 x} = \frac{2 \tan x}{1 + \tan^2 x}.$$

$$2. \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}.$$

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - \cos^2 x \tan^2 x = \cos^2 x (1 - \tan^2 x)$$

$$= \frac{1 - \tan^2 x}{\sec^2 x} = \frac{1 - \tan^2 x}{1 + \tan^2 x}.$$

$$3. \tan \frac{1}{2}x = \frac{\sin x}{1 + \cos x}.$$

$$\tan \frac{1}{2}x = \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \sqrt{\frac{(1 - \cos x)(1 + \cos x)}{(1 + \cos x)(1 + \cos x)}} = \sqrt{\frac{1 - \cos^2 x}{(1 + \cos x)^2}}$$

$$= \sqrt{\frac{\sin^2 x}{(1 + \cos x)^2}} = \frac{\sin x}{1 + \cos x}.$$

$$4. \cot \frac{1}{2} x = \frac{\sin x}{1 - \cos x}.$$

$$\begin{aligned} \cot \frac{1}{2} x &= \sqrt{\frac{1 + \cos x}{1 - \cos x}} = \sqrt{\frac{(1 + \cos x)(1 - \cos x)}{(1 - \cos x)^2}} = \sqrt{\frac{1 - \cos^2 x}{(1 - \cos x)^2}} \\ &= \sqrt{\frac{\sin^2 x}{(1 - \cos x)^2}} = \frac{\sin x}{1 - \cos x}. \end{aligned}$$

If A, B, C are the angles of a triangle, prove that:

$$5. \sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C.$$

$$\begin{aligned} \sin A + \sin B + \sin C \\ &= \sin A + \sin B + \sin [180^\circ - (A + B)] \\ &= \sin A + \sin B + \sin (A + B); \end{aligned}$$

by [42] and [30],

$$\begin{aligned} &= 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) + 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A + B) \\ &= 2 \sin \frac{1}{2} (A + B) [\cos \frac{1}{2} (A - B) + \cos \frac{1}{2} (A + B)]; \end{aligned}$$

by [44],

$$\begin{aligned} &= 2 \sin \frac{1}{2} (A + B) 2 \cos \frac{1}{2} A \cos \frac{1}{2} B \\ &= 4 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} A \cos \frac{1}{2} B. \end{aligned}$$

$$\text{But} \quad \cos \frac{1}{2} C = \cos [90^\circ - \frac{1}{2} (A + B)] = \sin \frac{1}{2} (A + B).$$

$$\therefore \sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C.$$

$$6. \cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C.$$

$$\cos C = \cos [180^\circ - (A + B)] = -\cos (A + B).$$

$$\therefore \cos A + \cos B + \cos C$$

$$= \cos A + \cos B - \cos (A + B);$$

by [44],

$$= 2 \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) - \cos (A + B);$$

by [32],

$$\begin{aligned} &= 2 \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) - 2 \cos^2 \frac{1}{2} (A + B) + 1 \\ &= [2 \cos \frac{1}{2} (A + B)] [\cos \frac{1}{2} (A - B) - \cos \frac{1}{2} (A + B)] + 1; \end{aligned}$$

by [45],

$$\begin{aligned} &= [2 \cos \frac{1}{2} (A + B)] \times [2 \sin \frac{1}{2} A \sin \frac{1}{2} B] + 1 \\ &= (2 \sin \frac{1}{2} C) (2 \sin \frac{1}{2} A \sin \frac{1}{2} B) + 1 \\ &= 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C. \end{aligned}$$

$$7. \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

$$\text{Since} \quad A + B + C = 180^\circ, \quad C = 180^\circ - (A + B).$$

$$\therefore \tan C = \tan [180^\circ - (A + B)] = -\tan (A + B);$$

by [26],

$$\begin{aligned} \tan A + \tan B &= \tan (A + B) (1 - \tan A \tan B) \\ &= \tan (A + B) - \tan (A + B) \tan A \tan B. \end{aligned}$$

$$\begin{aligned} \therefore \tan A + \tan B + \tan C &= \tan (A + B) - \tan (A + B) \\ &\quad - \tan (A + B) \tan A \tan B \\ &= -\tan (A + B) \tan A \tan B \\ &= \tan A \tan B \tan C. \end{aligned}$$

8. Given $\tan \frac{1}{2}x = 1$, find $\cos x$. 9. Given $\cot \frac{1}{2}x = \sqrt{3}$, find $\sin x$.

$$\tan \frac{1}{2}x = \sqrt{\frac{1 - \cos x}{1 + \cos x}}.$$

$$1 = \sqrt{\frac{1 - \cos x}{1 + \cos x}}.$$

$$1 = \frac{1 - \cos x}{1 + \cos x}.$$

$$1 + \cos x = 1 - \cos x.$$

$$2 \cos x = 0.$$

$$\cos x = 0.$$

$$\cot \frac{1}{2}x = \sqrt{\frac{1 + \cos x}{1 - \cos x}}.$$

$$\sqrt{3} = \sqrt{\frac{1 + \cos x}{1 - \cos x}}.$$

$$3 = \frac{1 + \cos x}{1 - \cos x}.$$

$$3 - 3 \cos x = 1 + \cos x.$$

$$-4 \cos x = -2.$$

$$\cos x = \frac{1}{2}.$$

$$\sin^2 x = 1 - \cos^2 x$$

$$= 1 - \frac{1}{4} = \frac{3}{4}.$$

$$\sin x = \frac{1}{2} \sqrt{3}.$$

10. Prove that $\tan 18^\circ = \frac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ}$.

Let $x = 18^\circ,$

and $y = 15^\circ.$

Then $2 \sin x \cos y = \sin(x + y) + \sin(x - y). \quad (1)$

$2 \cos x \cos y = \cos(x + y) + \cos(x - y). \quad (2)$

Divide (1) by (2), $\tan x = \frac{\sin(x + y) + \sin(x - y)}{\cos(x + y) + \cos(x - y)}.$

Substitute values of x and y ,

$$\tan 18^\circ = \frac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ}.$$

11. Prove that $\sin \frac{1}{2}x \pm \cos \frac{1}{2}x = \sqrt{1 \pm \sin x}.$

$$\begin{aligned} \sin \frac{1}{2}x \pm \cos \frac{1}{2}x &= \sqrt{(\sin \frac{1}{2}x \pm \cos \frac{1}{2}x)^2} \\ &= \sqrt{\sin^2 \frac{1}{2}x + \cos^2 \frac{1}{2}x \pm 2 \sin \frac{1}{2}x \cos \frac{1}{2}x} \\ &= \sqrt{1 \pm 2 \sin \frac{1}{2}x \cos \frac{1}{2}x} \\ &= \sqrt{1 \pm \sin x}. \end{aligned}$$

12. Prove that $\frac{\tan x \pm \tan y}{\cot x \pm \cot y} = \pm \tan x \tan y.$

$$\begin{aligned} \frac{\tan x \pm \tan y}{\cot x \pm \cot y} &= \frac{\frac{\sin x}{\cos x} \pm \frac{\sin y}{\cos y}}{\frac{\cos x}{\sin x} \pm \frac{\cos y}{\sin y}} \\ &= \frac{\sin x \cos y \pm \sin y \cos x}{\cos x \cos y} \times \frac{\sin x \sin y}{\cos x \sin y \pm \sin x \cos y} \\ &= \pm \frac{\sin x \sin y}{\cos x \cos y} = \pm \tan x \tan y. \end{aligned}$$

13. Prove that $\tan(45^\circ - x) = \frac{1 - \tan x}{1 + \tan x}$.

$$\tan(45^\circ - x) = \frac{\tan 45^\circ - \tan x}{1 + \tan 45^\circ \tan x} = \frac{1 - \tan x}{1 + \tan x}.$$

14. In the triangle ABC prove that

$$\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C = \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C.$$

Since $\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C = 90^\circ$, $\frac{1}{2}C = 90^\circ - \frac{1}{2}(A + B)$.

$$\therefore \cot \frac{1}{2}C = \tan \frac{1}{2}(A + B),$$

$$\cot \frac{1}{2}B = \tan \frac{1}{2}(A + C),$$

and $\cot \frac{1}{2}A = \tan \frac{1}{2}(B + C)$.

$$\therefore \cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C$$

$$= \tan \frac{1}{2}(A + B) + \tan \frac{1}{2}(A + C) + \tan \frac{1}{2}(B + C);$$

by Prob. 7, $= \tan \frac{1}{2}(A + B) \times \tan \frac{1}{2}(A + C) \times \tan \frac{1}{2}(B + C)$

$$= \cot \frac{1}{2}A \times \cot \frac{1}{2}B \times \cot \frac{1}{2}C.$$

Change to a form involving products instead of sums, and hence more convenient for computation by logarithms:

15. $\cot x + \tan x$.

$$\cot x + \tan x$$

$$= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$$

$$= \frac{2(\cos^2 x + \sin^2 x)}{2 \sin x \cos x};$$

by [16] and [30],

$$= \frac{2}{\sin 2x}.$$

16. $\cot x - \tan x$.

$$\cot x - \tan x$$

$$= \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}$$

$$= \frac{\cos^2 x - \sin^2 x}{\sin x \cos x};$$

by [32],

$$= \frac{\cos 2x}{\sin x \cos x} \\ = \frac{2 \cos 2x}{2 \sin x \cos x},$$

by [30],

$$= \frac{2 \cos 2x}{\sin 2x} \\ = 2 \cot 2x.$$

17. $\cot x + \tan y$.

$$\cot x + \tan y$$

$$= \frac{\cos x}{\sin x} + \frac{\sin y}{\cos y}$$

$$= \frac{\cos x \cos y + \sin x \sin y}{\sin x \cos y};$$

by [25], $= \frac{\cos(x - y)}{\sin x \cos y}.$

18. $\cot x - \tan y$.

$$\cot x - \tan y$$

$$= \frac{\cos x}{\sin x} - \frac{\sin y}{\cos y}$$

$$= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y};$$

by [24], $= \frac{\cos(x + y)}{\sin x \cos y}.$

19. $\frac{1 - \cos 2x}{1 + \cos 2x}$.

$$\frac{1 - \cos 2x}{1 + \cos 2x} = \frac{1 - \cos 2x}{1 + \cos 2x} \cdot \frac{2}{2}$$

by [34] and [35], $= \frac{\sin^2 x}{\cos^2 x} \\ = \tan^2 x.$

20. $1 + \tan x \tan y.$

$$\begin{aligned} & 1 + \tan x \tan y \\ &= 1 + \frac{\sin x}{\cos x} \times \frac{\sin y}{\cos y} \\ &= \frac{\cos x \cos y + \sin x \sin y}{\cos x \cos y}; \end{aligned}$$

by [25], $= \frac{\cos(x-y)}{\cos x \cos y}.$

21. $1 - \tan x \tan y.$

$$\begin{aligned} & 1 - \tan x \tan y \\ &= 1 - \frac{\sin x \sin y}{\cos x \cos y} \\ &= \frac{\cos x \cos y - \sin x \sin y}{\cos x \cos y} \\ &= \frac{\cos(x+y)}{\cos x \cos y}. \end{aligned}$$

22. $\cot x \cot y + 1.$

$$\begin{aligned} & \cot x \cot y + 1 \\ &= \frac{\cos x}{\sin x} \times \frac{\cos y}{\sin y} + 1 \\ &= \frac{\cos x \cos y + \sin x \sin y}{\sin x \sin y}; \end{aligned}$$

by [25], $= \frac{\cos(x-y)}{\sin x \sin y}.$

25. Prove that $\tan x + \tan y = \frac{\sin(x+y)}{\cos x \cos y}.$

$$\tan x + \tan y = \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y} = \frac{\sin(x+y)}{\cos x \cos y}.$$

26. Prove that $\cot y - \cot x = \frac{\sin(x-y)}{\sin x \sin y}.$

$$\cot y - \cot x = \frac{\cos y}{\sin y} - \frac{\cos x}{\sin x} = \frac{\sin x \cos y - \cos x \sin y}{\sin x \sin y} = \frac{\sin(x-y)}{\sin x \sin y}.$$

27. Given $\tan(x+y) = 3$, and $\tan x = 2$, find $\tan y.$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

$$3 = \frac{2 + \tan y}{1 - 2 \tan y}.$$

$$3 - 6 \tan y = 2 + \tan y.$$

$$-7 \tan y = -1.$$

$$\tan y = \frac{1}{7}.$$

23. $\cot x \cot y - 1.$

$$\begin{aligned} & \cot x \cot y - 1 \\ &= \frac{\cos x \cos y}{\sin x \sin y} - 1 \\ &= \frac{\cos x \cos y - \sin x \sin y}{\sin x \sin y}; \end{aligned}$$

by [24], $= \frac{\cos(x+y)}{\sin x \sin y}.$

24. $\frac{\tan x + \tan y}{\cot x + \cot y}.$

$$\begin{aligned} & \frac{\tan x + \tan y}{\cot x + \cot y} \\ &= \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{\frac{\cos x}{\sin x} + \frac{\cos y}{\sin y}} \\ &= \frac{\frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}}{\frac{\sin x \cos y + \cos x \sin y}{\sin x \sin y}} \\ &= \frac{\sin x \sin y}{\cos x \cos y} \\ &= \tan x \tan y. \end{aligned}$$

28. Prove that $(\sin x + \cos x)^2 = 1 + \sin 2x$.

$$\sin^2 x + 2 \sin x \cos x + \cos^2 x = 1 + 2 \sin x \cos x = 1 + \sin 2x.$$

29. Prove that $(\sin x - \cos x)^2 = 1 - \sin 2x$.

$$\sin^2 x - 2 \sin x \cos x + \cos^2 x = 1 - 2 \sin x \cos x = 1 - \sin 2x.$$

30. Prove that $\tan x + \cot x = 2 \csc 2x$.

$$\begin{aligned} \tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\ &= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \\ &= \frac{1}{\sin x \cos x} \\ &= \frac{2}{\sin 2x} = 2 \csc 2x. \end{aligned}$$

31. Prove that $\cot x - \tan x = 2 \cos 2x \csc 2x$.

$$\begin{aligned} \cot x - \tan x &= \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} \\ &= \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} \\ &= \frac{\cos 2x}{\sin x \cos x} \\ &= \frac{2 \cos 2x}{\sin 2x} = 2 \cos 2x \csc 2x. \end{aligned}$$

32. Prove that $2 \sin^2(45^\circ - x) = 1 - \sin 2x$.

$$\begin{aligned} \sin(45^\circ - x) &= \sin 45^\circ \cos x - \cos 45^\circ \sin x \\ &= \frac{1}{2} \sqrt{2} (\cos x - \sin x). \\ \sin^2(45^\circ - x) &= \frac{1}{2} (\cos^2 x - 2 \sin x \cos x + \sin^2 x) \\ &= \frac{1}{2} (1 - 2 \sin x \cos x) \\ &= \frac{1}{2} (1 - \sin 2x). \\ 2 \sin^2(45^\circ - x) &= 1 - \sin 2x. \end{aligned}$$

33. Prove that $\cos 45^\circ + \cos 75^\circ = \cos 15^\circ$.

$$\begin{aligned} \cos 45^\circ + \cos 75^\circ &= 2 \cos \frac{1}{2} (45^\circ + 75^\circ) \cos \frac{1}{2} (75^\circ - 45^\circ) \\ &= 2 \cos 60^\circ \cos 15^\circ \\ &= 2 \cdot \frac{1}{2} \cos 15^\circ \\ &= \cos 15^\circ. \end{aligned}$$

34. Prove that $1 + \tan x \tan 2x = \tan 2x \cot x - 1$.

$$\text{By [31],} \quad 1 + \tan x \tan 2x = 1 + \frac{2 \tan^2 x}{1 - \tan^2 x} = \frac{1 + \tan^2 x}{1 - \tan^2 x}. \quad (1)$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

$$\tan^2 x = 1 - \frac{2 \tan x}{\tan 2x}.$$

$$\text{Substitute in (1),} \quad \frac{1 + \tan^2 x}{1 - \tan^2 x} = \frac{2 - \frac{2 \tan x}{\tan 2x}}{\frac{2 \tan x}{\tan 2x}}.$$

$$\therefore 1 + \tan x \tan 2x = \frac{\tan 2x}{\tan x} - 1 = \tan 2x \cot x - 1.$$

Prove the following formulas :

$$35. (\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 2 + 2 \cos(x - y).$$

$$\begin{aligned} & (\cos x + \cos y)^2 + (\sin x + \sin y)^2 \\ &= \cos^2 x + 2 \cos x \cos y + \cos^2 y + \sin^2 x + 2 \sin x \sin y + \sin^2 y \\ &= 1 + 1 + 2 \cos x \cos y + 2 \sin x \sin y \\ &= 2 + 2 \cos(x - y) \end{aligned}$$

$$36. (\sin x + \cos y)^2 + (\sin y + \cos x)^2 = 2 + 2 \sin(x + y).$$

$$\begin{aligned} & (\sin x + \cos y)^2 + (\sin y + \cos x)^2 \\ &= \sin^2 x + 2 \sin x \cos y + \cos^2 y + \sin^2 y + 2 \sin y \cos x + \cos^2 x \\ &= 1 + 1 + 2 \sin x \cos y + 2 \cos x \sin y \\ &= 2 + 2 \sin(x + y). \end{aligned}$$

$$37. \sin(x + y) + \cos(x - y) = (\sin x + \cos x)(\sin y + \cos y).$$

$$\begin{aligned} & \sin(x + y) + \cos(x - y) \\ &= \sin x \cos y + \cos x \sin y + \cos x \cos y + \sin x \sin y \\ &= \sin x (\cos y + \sin y) + \cos x (\sin y + \cos y) \\ &= (\sin x + \cos x)(\sin y + \cos y). \end{aligned}$$

$$38. \sin(x + y) \cos y - \cos(x + y) \sin y = \sin x.$$

$$\begin{aligned} & \sin(x + y) \cos y - \cos(x + y) \sin y \\ &= \cos y (\sin x \cos y + \cos x \sin y) - \sin y (\cos x \cos y - \sin x \sin y) \\ &= \sin x \cos^2 y + \cos x \cos y \sin y - \cos x \cos y \sin y + \sin x \sin^2 y \\ &= \sin x (\sin^2 y + \cos^2 y) \\ &= \sin x (1) = \sin x. \end{aligned}$$

Exercise 47. Law of Sines (Page 109)

1. Consider the formula $\frac{a}{b} = \frac{\sin A}{\sin B}$ when $B = 90^\circ$; when $A = 90^\circ$; when $A = B$; when $a = b$.

$$\begin{array}{lll} \frac{a}{b} = \frac{\sin A}{\sin 90^\circ} = \frac{\sin A}{1} & \frac{a}{b} = \frac{\sin 90^\circ}{\sin B} = \frac{1}{\sin B} & \frac{a}{b} = \frac{\sin A}{\sin A} \cdot \frac{a}{a} = \frac{\sin A}{\sin B} \\ a = b \sin A & b = a \sin B & a = b \sin A = \sin B \end{array}$$

2. Prove by the Law of Sines that the bisector of an angle of a triangle divides the opposite side into parts proportional to the adjacent sides.

Let CD bisect angle C ;

then $\frac{AD}{CD} = \frac{\sin \frac{1}{2}C}{\sin A},$

and $\frac{DB}{CD} = \frac{\sin \frac{1}{2}C}{\sin B}.$

By division,

$$\frac{AD}{DB} = \frac{\sin B}{\sin A}.$$

But $\frac{\sin B}{\sin A} = \frac{b}{a}.$

$$\therefore \frac{AD}{DB} = \frac{b}{a}.$$

3. Prove Ex. 2 for the bisector of an exterior angle of a triangle.

Let CD bisect the external angle at C ;

then $\frac{BD}{CD} = \frac{\sin \frac{1}{2}C}{\sin B},$

and $\frac{AB + BD}{CD} = \frac{\sin \frac{1}{2}C}{\sin A}.$

By division,

$$\frac{BD}{AB + BD} = \frac{\sin A}{\sin B}.$$

But $\frac{\sin A}{\sin B} = \frac{a}{b}.$

$$\therefore \frac{BD}{AD} = \frac{a}{b}.$$

4. The triangle ABC has $A = 78^\circ$, $B = 72^\circ$, and $c = 4$ in. Find the diameter of the circumscribed circle.

$$C = 180^\circ - (78^\circ + 72^\circ) = 30^\circ.$$

By § 106, $2R = \frac{c}{\sin C} = \frac{4}{\frac{1}{2}} = 8.$

The diameter of the circumscribed circle is 8 in.

5. The triangle ABC has $A = 76^\circ 37'$, $B = 81^\circ 46'$, and $c = 368.4$ ft. Find the diameter of the circumscribed circle.

$$C = 180^\circ - (76^\circ 37' + 81^\circ 46') = 21^\circ 37'.$$

$$2R = \frac{c}{\sin C} = \frac{368.4}{0.3684} = 1000.$$

The diameter of the circumscribed circle is 1000 ft.

6. What is the diameter of the circle circumscribed about an equilateral triangle of side 7.4 in.? What is the diameter of the circle inscribed in the same triangle?

$$\begin{aligned} A &= B = C = 60^\circ. \\ 2R &= \frac{a}{\sin A} = \frac{7.4}{0.8660} = 8.5450. \\ 2R &= 2\sqrt{(4.2725)^2 - (3.7)^2} = 2(2.1364) = 4.2728. \end{aligned}$$

The diameter of the circumscribed circle is 8.5450 in., and of the inscribed circle 4.2728 in.

7. What is the diameter of the circle circumscribed about an isosceles triangle of base 4.8 in. and vertical angle 10° ?

$$2R = \frac{c}{\sin C} = \frac{4.8}{0.1736} = 27.6498.$$

The diameter of the circle is 27.6498 in.

8. What is the diameter of the circle circumscribed about an isosceles triangle whose vertical angle is 18° and the sum of the two equal sides 18 in.?

$$\begin{aligned} \frac{1}{2} \text{ of } 18 \text{ in.} &= 9 \text{ in.} \\ \frac{1}{2}(180^\circ - 18^\circ) &= 81^\circ. \\ 2R &= \frac{a}{\sin A} = \frac{9}{0.9877} = 9.1121. \end{aligned}$$

The diameter of the circle is 9.1121 in.

Exercise 48. Law of Sines (Page 110)

Solve the triangle ABC, given the following parts:

1. $a = 500$, $A = 10^\circ 12'$, $B = 46^\circ 36'$.

$a = 500$	$\log a = 2.69897$	$\log a = 2.69897$
$A = 10^\circ 12'$	$\text{colog } \sin A = 0.75182$	$\text{colog } \sin A = 0.75182$
$B = 46^\circ 36'$	$\log \sin B = 9.86128$	$\log \sin C = 9.92260$
$A + B = 56^\circ 48'$	$\log b = 13.31207$	$\log c = 13.37339$
$\therefore C = 123^\circ 12'$	$\therefore b = 2051.5.$	$\therefore c = 2362.6.$

2. $a = 795$, $A = 79^\circ 59'$, $B = 44^\circ 41'$.

$a = 795$	$\log a = 2.90037$	$\log a = 2.90037$
$A = 79^\circ 59'$	$\text{colog } \sin A = 0.00667$	$\text{colog } \sin A = 0.00667$
$B = 44^\circ 41'$	$\log \sin B = 9.84707$	$\log \sin C = 9.91512$
$A + B = 124^\circ 40'$	$\log b = 12.75411$	$\log c = 12.82216$
$\therefore C = 55^\circ 20'$	$\therefore b = 567.69,$	$\therefore c = 663.99.$

3. $a = 804$, $A = 99^\circ 55'$, $B = 45^\circ 1'$.

$a = 804$	$\log a = 2.90526$	$\log a = 2.90526$
$A = 99^\circ 55'$	$\text{colog sin } A = 0.00654$	$\text{colog sin } A = 0.00654$
$B = 45^\circ 1'$	$\log \sin B = 9.84961$	$\log \sin C = 9.75931$
$A + B = 144^\circ 56'$	$\log b = 12.76141$	$\log c = 12.67111$
$\therefore C = 35^\circ 4'$	$\therefore b = 577.31.$	$\therefore c = 468.93.$

4. $a = 820$, $A = 12^\circ 49'$, $B = 141^\circ 59'$.

$a = 820$	$\log a = 2.91381$	$\log a = 2.91381$
$A = 12^\circ 49'$	$\text{colog sin } A = 0.65398$	$\text{colog sin } A = 0.65398$
$B = 141^\circ 59'$	$\log \sin B = 9.78950$	$\log \sin C = 9.62918$
$A + B = 154^\circ 48'$	$\log b = 13.35729$	$\log c = 13.19697$
$\therefore C = 25^\circ 12'$	$\therefore b = 2276.6.$	$\therefore c = 1573.9.$

5. $c = 1005$, $A = 78^\circ 19'$, $B = 54^\circ 27'$.

$c = 1005$	$\log c = 3.00217$	$\log c = 3.00217$
$A = 78^\circ 19'$	$\text{colog sin } C = 0.13423$	$\text{colog sin } C = 0.13423$
$B = 54^\circ 27'$	$\log \sin A = 9.99091$	$\log \sin B = 9.91042$
$A + B = 132^\circ 46'$	$\log a = 13.12731$	$\log b = 13.04682$
$\therefore C = 47^\circ 14'$	$\therefore a = 1340.6.$	$\therefore b = 1113.8.$

6. $b = 13.57$, $B = 13^\circ 57'$, $C = 57^\circ 13'$.

$b = 13.57$	$\log b = 1.13258$	$\log b = 1.13258$
$B = 13^\circ 57'$	$\text{colog sin } B = 0.61785$	$\text{colog sin } B = 0.61785$
$C = 57^\circ 13'$	$\log \sin A = 9.97610$	$\log \sin C = 9.92465$
$B + C = 71^\circ 10'$	$\log a = 11.72653$	$\log c = 11.67508$
$\therefore A = 108^\circ 50'$	$\therefore a = 53.276.$	$\therefore c = 47.324.$

7. $a = 6412$, $A = 70^\circ 55'$, $C = 52^\circ 9'$.

$a = 6412$	$\log a = 3.80699$	$\log a = 3.80699$
$A = 70^\circ 55'$	$\text{colog sin } A = 0.02455$	$\text{colog sin } A = 0.02455$
$C = 52^\circ 9'$	$\log \sin B = 9.92326$	$\log \sin C = 9.89742$
$A + C = 123^\circ 4'$	$\log b = 13.75480$	$\log c = 13.72896$
$\therefore B = 56^\circ 56'$	$\therefore b = 5685.9.$	$\therefore c = 5357.5.$

8. $b = 999$, $A = 37^\circ 58'$, $C = 65^\circ 2'$.

$b = 999$	$\log b = 2.99957$	$\log b = 2.99957$
$A = 37^\circ 58'$	$\text{colog sin } B = 0.01128$	$\text{colog sin } B = 0.01128$
$C = 65^\circ 2'$	$\log \sin A = 9.78902$	$\log \sin C = 9.95739$
$A + C = 103^\circ$	$\log a = 12.79987$	$\log c = 12.96824$
$\therefore B = 77^\circ.$	$\therefore a = 630.77$	$\therefore c = 929.48.$

Solve Exs. 9-14 without using logarithms:

9. Given $b = 7.071$, $A = 30^\circ$, and $C = 105^\circ$, find a and c .

$$\begin{aligned} B &= 180^\circ - (A + C) = 45^\circ, \\ \frac{a}{b} &= \frac{\sin A}{\sin B}, \\ \frac{a}{7.071} &= \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{2}} = \sqrt{2}, \\ a &= \frac{7.071}{\sqrt{2}} = 4.9991 = 5. \end{aligned}$$

Let p and q denote the segments of c made by the $\perp$ dropped from C .

$$\begin{aligned} \frac{p}{b} &= \cos A = \frac{1}{2}\sqrt{3} = 0.86603, \\ p &= b \times 0.86603 \\ &= 7.071 \times 0.86603 \\ &= 6.1237, \\ \frac{q}{a} &= \cos B = \frac{1}{2}\sqrt{2} = 0.7071, \\ q &= a \times 0.7071 \\ &= 5 \times 0.7071 = 3.5355, \\ c &= p + q \\ &= 6.1237 + 3.5355 \\ &= 9.659. \end{aligned}$$

10. Given $c = 9.562$, $A = 45^\circ$, and $B = 60^\circ$, find a and b .

$$\begin{aligned} C &= 75^\circ, \\ a &= \frac{c \sin A}{\sin C}, \\ \sin C &= \sin(45^\circ + 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ \\ &\quad + \cos 45^\circ \sin 30^\circ \\ &= \frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{2} \times \frac{1}{2} \\ &= \frac{1}{4}(\sqrt{6} + \sqrt{2}), \\ \therefore a &= \frac{9.562 \times \frac{1}{2}\sqrt{2}}{\frac{1}{4}(\sqrt{6} + \sqrt{2})} \\ &= \frac{19.124 \times \sqrt{2}}{\sqrt{6} + \sqrt{2}} \\ &= \frac{(19.124 \times \sqrt{2})(\sqrt{6} - \sqrt{2})}{6 - 2} \\ &= 9.562(\sqrt{3} - 1) \\ &= 6.99986 = 7, \\ b &= \frac{a \sin B}{\sin A} = \frac{7 \times \frac{1}{2}\sqrt{3}}{\frac{1}{2}\sqrt{2}} \\ &= \frac{7\sqrt{3}}{\sqrt{2}} = \frac{7\sqrt{6}}{2} \\ &= 3.5\sqrt{6} = 8.573. \end{aligned}$$

11. The base of a triangle is 600 ft. and the angles at the base are 30° and 120° . Find the other sides and the altitude.

$$\begin{aligned} C &= 180^\circ - (A + B) = 30^\circ, \\ \therefore a &= c = 600 \text{ ft.} \\ \frac{b}{c} &= \frac{\sin B}{\sin C}, \\ \frac{b}{600} &= \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}, \\ b &= 600 \cdot \sqrt{3} = 1039.2 \text{ ft.} \\ h &= a \sin B = 600 \times \frac{1}{2}\sqrt{3} \\ &= 519.6 \text{ ft.} \end{aligned}$$

12. Two angles of a triangle are 20° and 40° . Find the ratio of the opposite sides.

$$\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{0.3420}{0.6428} = 855:1607.$$

13. The angles of a triangle are as 5 : 10 : 21, and the side opposite the smallest angle is 3. Find the other sides.

Since the angles A, B, C are as 5 : 10 : 21,

$$A = \frac{5}{36} \text{ of } 180^\circ = 25^\circ.$$

$$B = \frac{10}{36} \text{ of } 180^\circ = 50^\circ.$$

$$C = \frac{21}{36} \text{ of } 180^\circ = 105^\circ.$$

$$b = \frac{a \sin B}{\sin A} = \frac{3 \times 0.7660}{0.4226} = 5.438.$$

$$c = \frac{a \sin C}{\sin A} = \frac{3 \times 0.9659}{0.4226} = 6.857.$$

14. Given one side of a triangle 27 in., and the adjacent angles each equal to 30° , find the radius of the circumscribed circle.

$$2R = \frac{a}{\sin A}.$$

$$\sin A = \sin 120^\circ = \sin(180^\circ - 120^\circ) = \sin 60^\circ.$$

$$\sin 60^\circ = \frac{1}{2} \sqrt{3}.$$

$$\therefore 2R = \frac{27}{\frac{1}{2} \sqrt{3}} = \frac{54}{\sqrt{3}} = 18 \sqrt{3}.$$

$$\therefore R = 9 \sqrt{3} = 15.588 \text{ in.}$$

15. The angles B and C of a triangle ABC are $50^\circ 30'$ and $122^\circ 9'$ respectively, and BC is 9 mi. Find AB and AC .

$$C = 122^\circ 9'$$

$$B = 50^\circ 30'$$

$$B + C = 172^\circ 39'$$

$$\therefore A = 7^\circ 21'.$$

$$\log BC = 0.95424$$

$$\text{colog } \sin A = 0.89303$$

$$\log \sin B = 9.88741$$

$$\log b = 1.73468$$

$$b = 54.285.$$

$$\therefore AC = 54.285 \text{ mi.}$$

$$\log BC = 0.95424$$

$$\text{colog } \sin A = 0.89303$$

$$\log \sin C = 9.92771$$

$$\log c = 1.77498$$

$$c = 59.564.$$

$$\therefore AB = 59.564 \text{ mi.}$$

16. In a parallelogram, given a diagonal d and the angles x and y which this diagonal makes with the sides, find the sides. Compute the results when $d = 11.2$, $x = 19^\circ 1'$, and $y = 42^\circ 54'$.

$$\frac{a}{d} = \frac{\sin x}{\sin z}.$$

$$\therefore a = \frac{d \sin x}{\sin z}.$$

$$\frac{c}{d} = \frac{\sin y}{\sin z}.$$

$$\therefore c = \frac{d \sin y}{\sin z}.$$

$$d = 11.2.$$

$$x = 19^\circ 1'$$

$$y = 42^\circ 54'$$

$$x + y = 61^\circ 55'$$

$$\therefore z = 118^\circ 5'.$$

$$\log d = 1.04922$$

$$\text{colog } \sin z = 0.05440$$

$$\log \sin x = 9.51301$$

$$\log a = 0.61663$$

$$a = 4.1365.$$

$$\log d = 1.04922$$

$$\text{colog } \sin z = 0.05440$$

$$\log \sin y = 9.83297$$

$$\log c = 0.93659$$

$$c = 8.6416.$$

17. A lighthouse was observed from a ship to bear N. 34° E.; after the ship sailed due south 3 mi. the lighthouse bore N. 23° E. Find the distance from the lighthouse to the ship in each position.

$$c = 3.$$

$$A = 23^\circ$$

$$B = (180^\circ - 34^\circ) = 146^\circ$$

$$A + B = 169^\circ$$

$$\therefore C = 11^\circ.$$

$$\log c = 0.47712$$

$$\text{colog sin } C = 0.71940$$

$$\log \sin A = 9.59188$$

$$\log a = 0.78840$$

$$a = 6.1433.$$

$$\log c = 0.47712$$

$$\text{colog sin } C = 0.71940$$

$$\log \sin B = 9.74756$$

$$\log b = 0.94408$$

$$b = 8.7918.$$

Therefore the required distances are 6.1433 mi. and 8.7918 mi.

18. A headland was observed from a ship to bear directly east; after the ship had sailed 5 mi. N. 31° E. the headland bore S. 42° E. Find the distance from the headland to the ship in each position.

$$A = 59^\circ.$$

$$B = 48^\circ.$$

$$C = 73^\circ.$$

$$b = 5.$$

$$C = \frac{b \sin C}{\sin B}.$$

$$\log b = 0.69897$$

$$\log \sin C = 9.98060$$

$$\text{colog sin } B = 0.12893$$

$$0.80850$$

$$c = 6.4343.$$

$$a = \frac{b \sin A}{\sin B}$$

$$\log b = 0.69897$$

$$\log \sin A = 9.93307$$

$$\text{colog sin } B = 0.12893$$

$$0.76097$$

$$a = 5.7673.$$

19. In a trapezoid, given the parallel sides a and b , and the angles x and y at the ends of one of the parallel sides, find the non-parallel sides. Compute the results when $a = 15$, $b = 7$, $x = 70^\circ$, $y = 40^\circ$.

Given parallel sides,

$$AB = 7 \quad \text{and} \quad DC = 15;$$

also $\angle ADC = 40^\circ$ and $\angle BCD = 70^\circ$, required AD and BC .

Draw $AE \parallel BC$;

then $AB = EC$ ($\parallel$ s comp. bet. $\parallel$ s),

and $DE = DC - AB$

$$= 15 - 7 = 8.$$

Also $\angle AED = \angle BCD$

$$= 70^\circ \text{ (ext. int. } \angle \text{)}.$$

Now $\angle DAE = 180^\circ - (40^\circ + 70^\circ)$

$$= 70^\circ.$$

But since

$$\angle AED = \angle DAE = 70^\circ,$$

the $\triangle AED$ is isosceles and

$$DA = DE = 8.$$

$$AE = BC,$$

and we are to find BC .

$$\frac{AE}{DE} = \frac{\sin \angle ADE}{\sin \angle DAE}.$$

$$\log DE = 0.90309$$

$$\log \sin \angle ADE = 9.80807$$

$$\text{colog sin } \angle DAE = 0.02701$$

$$\log AE = 0.73817$$

$$AE = BC = 5.4723.$$

20. Two observers 5 mi. apart on a plain, and facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 55° and 58° respectively. Find the distance from the balloon to each observer, and also the height of the balloon above the plain.

$$B = 58^\circ$$

$$A = 55^\circ$$

$$A + B = 113^\circ$$

$$\therefore C = 67^\circ.$$

$$\log c = 0.69897$$

$$\text{colog } \sin C = 0.03597$$

$$\log \sin A = 9.91336$$

$$\log a = 0.64830$$

$$a = 4.4494.$$

$$\therefore BC = 4.4494 \text{ mi.}$$

$$\log c = 0.69897$$

$$\text{colog } \sin C = 0.03597$$

$$\log \sin B = 9.92842$$

$$\log b = 0.66336$$

$$b = 4.6064.$$

$$\therefore AC = 4.6064 \text{ mi.}$$

To find h .

$$\frac{h}{a} = \sin B.$$

$$\therefore h = a \sin B.$$

$$\log a = 0.64830$$

$$\log \sin B = 9.92842$$

$$\log h = 0.57672$$

$$h = 3.7733.$$

$$\therefore \text{height} = 3.7733 \text{ mi.}$$

21. A balloon is directly above a straight road $7\frac{1}{4}$ mi. long, joining two towns. The balloonist observes that the first town makes an angle of 42° and the second town an angle of 38° with the perpendicular. Find the distance from the balloon to each town, and also the height of the balloon above the plain.

$$C = 80^\circ.$$

$$A = 90^\circ - 42^\circ = 48^\circ.$$

$$B = 90^\circ - 38^\circ = 52^\circ.$$

$$c = 7.25 \text{ mi.}$$

$$\frac{a}{c} = \frac{\sin A}{\sin C}.$$

$$a = \frac{7.25 \sin 48^\circ}{\sin 80^\circ}.$$

$$\log 7.25 = 0.86034$$

$$\log \sin 48^\circ = 9.87107$$

$$\text{colog } \sin 80^\circ = 0.00665$$

$$\log a = 0.73806$$

$$a = 5.4709 \text{ mi.}$$

$$b = \frac{7.25 \sin 52^\circ}{\sin 80^\circ}$$

$$\log c = 0.86034$$

$$\log \sin 52^\circ = 9.89653$$

$$\text{colog } \sin 80^\circ = 0.00665$$

$$\log b = 0.76352$$

$$b = 5.8013 \text{ mi.}$$

$$\frac{h}{a} = \sin B.$$

$$h = a \sin 52^\circ.$$

$$\log a = 0.73806$$

$$\log \sin 52^\circ = 9.89653$$

$$\log h = 0.63459$$

$$h = 4.3111 \text{ mi.}$$

Exercise 49. The Ambiguous Case (Page 112)

In the triangle ABC given a , b , and A , prove that :

1. If $a > b$, then $A > B$, B is acute, and there is one, and only one, triangle which will satisfy the given conditions.

$$\frac{\sin A}{\sin B} = \frac{a}{b};$$

if $a > b$, $\sin A > \sin B$ and $A > B$, and B must be acute, whatever the value of A ; a triangle can have only one obtuse angle. Hence there is only one triangle that will satisfy the given conditions.

2. If $a = b$, both A and B are acute, and there is one and only one triangle which will satisfy the given conditions, and this triangle is isosceles.

$$a = b, \sin A = \sin B.$$

$$\therefore A = B.$$

Both A and B must be acute, and the required triangle is isosceles.

3. If $a < b$, then A must be acute to have the triangle possible, and there are in general two triangles which satisfy the given conditions.

$$a < b, \sin A < \sin B.$$

$A < B$, and A must be acute in order that the triangle may be possible.

There are two triangles: (1) when B is acute; (2) when B is obtuse.

4. If $a = b \sin A$, the required triangle is a right triangle.

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \sin B = \frac{b \sin A}{a} = \frac{a}{a} = 1, \quad B = 90^\circ.$$

$\therefore ABC$ is a right triangle.

5. If $a < b \sin A$, the triangle is impossible.

If $a < b \sin A$, then $\sin B$ would be greater than 1, and the triangle is impossible.

6. If $A = B$, there is one, and only one, triangle.

$$\frac{\sin A}{\sin B} = 1 = \frac{a}{b}, \quad a = b, \text{ and there is only one isosceles triangle.}$$

Exercise 50. The Oblique Triangle (Page 115)

Find the number of solutions, given the following :

1. $a = 80, b = 100, A = 30^\circ.$

$\therefore a < b$, but $a > b \sin A = 100 \times \frac{1}{2}$,
and $A < 90^\circ$.

$\therefore$ two solutions.

2. $a = 50, b = 100, A = 30^\circ.$

$\therefore a = b \sin A$
 $= 100 \times \frac{1}{2}.$

$\therefore$ one solution.

3. $a = 40, b = 100, A = 30^\circ.$

$\therefore a < b \sin A = 100 \times \frac{1}{2}$, and $A < 90^\circ$.
 $\therefore$ no solution.

4. $a = 100$, $b = 100$, $A = 30^\circ$.
 $a = b$, and A is an acute angle.
 $\therefore$ one solution.

5. $a = 13.4$, $b = 11.46$, $A = 77^\circ 20'$.
 $\therefore a > b$.
 $\therefore$ one solution.

6. $a = 70$, $b = 75$, $A = 60^\circ$.
 $\therefore a < b$,
 but $a > b \sin A = 75 \times \frac{1}{2} \sqrt{3}$,
 and $A < 90^\circ$.
 $\therefore$ two solutions.

Solve the triangles, given the following:

9. $a = 840$, $b = 485$, $A = 21^\circ 31'$.
 $\log a = 7.07572 - 10$
 $\log b = 2.68574$
 $\log \sin A = 9.56440$
 $\log \sin B = 9.32586$
 $B = 12^\circ 13' 34''$.
 $\therefore C = 146^\circ 15' 26''$.
 $\log a = 2.92428$
 $\log \sin C = 9.74463$
 $\text{colog} \sin A = 0.43560$
 $\log c = 3.10454$
 $c = 1272.1$.

10. $a = 9.399$, $b = 9.197$,
 $A = 120^\circ 35'$.
 $\log a = 9.02692 - 10$
 $\log b = 0.96365$
 $\log \sin A = 9.93495$
 $\log \sin B = 9.92552$
 $B = 57^\circ 23' 40''$.
 $\therefore C = 2^\circ 1' 20''$.
 $\log a = 0.97308$
 $\log \sin C = 8.54761$
 $\text{colog} \sin A = 0.06505$
 $\log c = 9.58574 - 10$
 $c = 0.38525$.

7. $a = 134.16$, $b = 84.54$, $B = 52^\circ 9'$.
 $b < a$,
 $B < 90^\circ$,
 $\sin B = 0.7896$.
 $84.54 < 134.16 \times 0.7896$.
 $\therefore b < a \sin B$.
 $\therefore$ no solution.

8. $a = 200$, $b = 100$, $A = 30^\circ$.
 $a > b$.
 $\therefore$ one solution.

11. $a = 91.06$, $b = 77.04$,
 $A = 51^\circ 9'$.
 $\log a = 8.04067 - 10$
 $\log b = 1.88672$
 $\log \sin A = 9.89142$
 $\log \sin B = 9.81881$
 $B = 41^\circ 12' 56''$.
 $\therefore C = 87^\circ 38' 4''$.
 $\log a = 1.95933$
 $\log \sin C = 9.99963$
 $\text{colog} \sin A = 0.10858$
 $\log c = 2.06754$
 $c = 116.83$.

12. $a = 55.55$, $b = 66.66$,
 $B = 77^\circ 44'$.
 $\log a = 1.74468$
 $\log \sin B = 9.98997$
 $\log \sin b = 8.17613 - 10$
 $\log \sin A = 9.91078$
 $A = 54^\circ 31'$.
 $\therefore C = 47^\circ 45'$.
 $\log a = 1.74468$
 $\log \sin C = 9.86936$
 $\text{colog} \sin A = 0.08922$
 $\log c = 1.70326$
 $c = 50.496$.

$$13. a = 309, b = 360, \\ A = 21^\circ 14'.$$

There are two solutions,
for $a < b$,
but $a > b \sin A$,
and $A < 90^\circ$.

First solution.

$$\log b = 2.55630 \\ \log \sin A = 9.55891 \\ \text{colog } a = \underline{7.51004 - 10} \\ \log \sin B = 9.62525$$

$$B = 24^\circ 57' 26''.$$

$$\therefore C = 133^\circ 48' 34''.$$

$$\log a = 2.48996 \\ \log \sin C = 9.85832 \\ \text{colog } \sin A = \underline{0.44109} \\ \log C = \underline{2.78937} \\ c = 615.7.$$

Second solution.

$$B' = 180^\circ - B = 155^\circ 2' 34''.$$

$$C' = B - A = 3^\circ 43' 26''.$$

$$\log a = 2.48996 \\ \log \sin C' = 8.81257 \\ \text{colog } \sin A = \underline{0.44109} \\ \log c' = \underline{1.74362} \\ c' = 55.414.$$

$$14. a = 34, b = 22, B = 30^\circ 20'.$$

Here $b < a$, but $> a \sin B$, and $B < 90^\circ$.

$\therefore$ two solutions.

$$\log a = 1.53148 \\ \log \sin B = 9.70332 \\ \text{colog } b = \underline{8.65758 - 10} \\ \log \sin A = 9.89238$$

$$A = 51^\circ 18' 27''.$$

$$\therefore A' = 128^\circ 41' 33''.$$

$$\therefore C = 98^\circ 21' 33''.$$

$$\therefore C' = 20^\circ 58' 27''.$$

$$\log a = 1.53148$$

$$\log \sin C = 9.99536$$

$$\text{colog } \sin A = \underline{0.10762}$$

$$\log c = \underline{1.63446}$$

$$c = 43.098.$$

$$\log a = 1.53148$$

$$\log \sin C' = 9.55382$$

$$\text{colog } \sin A = \underline{0.10762}$$

$$\log c' = \underline{1.19292}$$

$$c' = 15.593.$$

$$15. b = 19, c = 18, C = 15^\circ 49'$$

There are two solutions,

for $c < b$,
but $c > b \sin C$,
and $C < 90^\circ$.

$$\log b = 1.27875 \\ \log \sin C = 9.43546 \\ \text{colog } c = \underline{8.74473 - 10} \\ \log \sin B = \underline{9.45894}$$

$$B = 16^\circ 43' 13''.$$

$$\therefore B' = 163^\circ 16' 47''.$$

$$\therefore A = 147^\circ 27' 47''.$$

$$\therefore A' = 0^\circ 54' 13''.$$

$$\log b = 1.27875 \\ \text{colog } \sin B = 0.54106 \\ \log \sin A = \underline{9.73065} \\ \log a = \underline{1.55046}$$

$$a = 35.519.$$

$$\log b = 1.27875 \\ \text{colog } \sin B' = 0.54106$$

$$\log \sin A' = \underline{8.19784}$$

$$\log a' = \underline{0.01765}$$

$$a' = 1.0415.$$

$$16. a = 8.716, b = 9.787, \\ A = 38^\circ 14' 12''.$$

There are two solutions,

for $a < b$,
but $a > b \sin A$,
and $A < 90^\circ$

$$\begin{aligned}
 \log b &= 0.99065 \\
 \log \sin A &= 9.79163 \\
 \text{colog } a &= \underline{9.05968 - 10} \\
 \log \sin B &= 9.84196
 \end{aligned}$$

$$\begin{aligned}
 B &= 44^\circ 1' 28''. \\
 \therefore C &= 97^\circ 44' 20''.
 \end{aligned}$$

$$\begin{aligned}
 \log a &= 0.94032 \\
 \log \sin C &= 9.99602 \\
 \text{colog } \sin A &= \underline{0.20837} \\
 \log c &= 1.14471 \\
 c &= 13.954.
 \end{aligned}$$

$$\begin{aligned}
 B' &= 180^\circ - B = 135^\circ 58' 32''. \\
 C' &= B - A = 5^\circ 47' 16''.
 \end{aligned}$$

$$\begin{aligned}
 \log a &= 0.94032 \\
 \log \sin C' &= 9.00365 - 10 \\
 \text{colog } \sin A &= \underline{0.20837} \\
 \log c' &= 0.15234 \\
 c' &= 1.4202.
 \end{aligned}$$

$$\begin{aligned}
 17. \ a &= 4.4, \ b = 5.21, \\
 A &= 57^\circ 37' 17''.
 \end{aligned}$$

$$\begin{aligned}
 \log b &= 0.71684 \\
 \log \sin A &= 9.92661 \\
 \text{colog } a &= \underline{9.35655 - 10} \\
 \log \sin B &= 0.00000
 \end{aligned}$$

$$\begin{aligned}
 B &= 90^\circ. \\
 C &= 32^\circ 22' 43''.
 \end{aligned}$$

$$\begin{aligned}
 \log b &= 0.71684 \\
 \log \cos A &= \underline{9.72877} \\
 \log c &= 0.44561 \\
 c &= 2.7901.
 \end{aligned}$$

18. Given $a = 75$, $b = 29$, $B = 16^\circ 15'$; find the difference between the areas of the two triangles which meet these conditions.

The triangle which is the difference of the two triangles has for its altitude $a \sin B$, and two of its sides are of length 29.

$$\begin{aligned}
 \log a &= 1.87506 \\
 \log \sin B &= \underline{9.44689} \\
 \log (a \sin B) &= 1.32195 \\
 a \sin B &= 20.987 = 21.
 \end{aligned}$$

$$\begin{aligned}
 29^2 - 21^2 &= (29 - 21)(29 + 21) \\
 &= 8 \times 50 = 400.
 \end{aligned}$$

$$\therefore \sqrt{29^2 - 21^2} = 20.$$

Hence the base of the triangle is $2 \times 20 = 40$, and its altitude 21. Its area is therefore $\frac{1}{2} \times 40 \times 21 = 420$.

19. In a parallelogram, given the side a , a diagonal d , and the angle A made by the two diagonals, find the other diagonal. As a special case consider the parallelogram in which $a = 35$, $d = 63$, and $A = 21^\circ 36'$.

$$\begin{aligned}
 a &= 35. \\
 \frac{1}{2}d &= 31.5. \\
 A &= 21^\circ 36'.
 \end{aligned}$$

$$\begin{aligned}
 \text{colog } a &= 8.45593 - 10 \\
 \log \frac{1}{2}d &= 1.49831 \\
 \log \sin A &= \underline{9.56599} \\
 \log \sin B &= 9.52023
 \end{aligned}$$

$$\begin{aligned}
 B &= 19^\circ 20' 53''. \\
 C &= 139^\circ 3' 7''.
 \end{aligned}$$

$$\begin{aligned}
 \log a &= 1.54407 \\
 \log \sin C &= 9.81648 \\
 \text{colog } \sin A &= \underline{0.43401} \\
 \log \frac{1}{2}d' &= 1.79456
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2}d' &= 62.31. \\
 d' &= 124.62.
 \end{aligned}$$

20. In a parallelogram $ABCD$, given $AD = 3$ in., $BD = 2.5$ in., and $A = 47^\circ 20'$, find AB .

$$\begin{aligned}\sin ABD &= \frac{AD \sin A}{BD} \\ &= \frac{3 \times \sin 47^\circ 20'}{2.5}.\end{aligned}$$

$$\begin{aligned}\log 3 &= 0.47712 \\ \log \sin 47^\circ 20' &= 9.86647 \\ \text{colog } 2.5 &= 9.60206\end{aligned}$$

$$\log \sin ABD = 9.94565$$

$$\therefore ABD = 61^\circ 55' 43''.$$

$$\therefore ADC = 70^\circ 44' 17''.$$

$$\begin{aligned}AB &= \frac{BD \sin D}{\sin A} \\ &= \frac{2.5 \times \sin 70^\circ 44' 17''}{\sin 47^\circ 20'}.\end{aligned}$$

$$\begin{aligned}\log 2.5 &= 0.39794 \\ \log \sin 70^\circ 44' 17'' &= 9.97498 \\ \text{colog } \sin 47^\circ 20' &= 0.13353 \\ \log AB &= 0.50645 \\ \therefore AB &= 3.2096.\end{aligned}$$

21. In a quadrilateral $ABCD$, given $AC = 4$ in., $\angle BAC = 35^\circ$, $\angle B = 75^\circ 20'$, $\angle D = 38^\circ 30'$, and $\angle BAD = 70^\circ 40'$, find the length of each of the four sides.

$$BC = \frac{AC \sin BAC}{\sin B} = \frac{4 \sin 35^\circ}{\sin 75^\circ 20'}.$$

$$\begin{aligned}\log 4 &= 0.60206 \\ \log \sin 35^\circ &= 9.75859 \\ \text{colog } \sin 75^\circ 20' &= 0.01439 \\ \log BC &= 0.37504\end{aligned}$$

$$BC = 2.3716 \text{ in.}$$

$$CD = \frac{AC \sin DAC}{\sin D} = \frac{4 \sin 35^\circ 40'}{38^\circ 30'}.$$

$$\begin{aligned}\log 4 &= 0.60206 \\ \log \sin 35^\circ 40' &= 9.76572 \\ \text{colog } \sin 38^\circ 30' &= 0.20585 \\ \log CD &= 0.57363\end{aligned}$$

$$CD = 3.7465 \text{ in.}$$

$$AD = \frac{AC \sin DCA}{\sin D} = \frac{4 \sin 105^\circ 50'}{\sin 38^\circ 30'}$$

$$\begin{aligned}\log 4 &= 0.60206 \\ \log \sin 105^\circ 50' &= 9.98320 \\ \text{colog } \sin 38^\circ 30' &= 0.20585 \\ \log AD &= 0.79111\end{aligned}$$

$$AD = 6.1817 \text{ in.}$$

$$AB = \frac{AC \sin ACB}{\sin B} = \frac{4 \sin 69^\circ 40'}{\sin 75^\circ 20'}.$$

$$\begin{aligned}\log 4 &= 0.60206 \\ \log \sin 69^\circ 40' &= 9.97206 \\ \text{colog } \sin 75^\circ 20' &= 0.01439 \\ \log AB &= 0.58851\end{aligned}$$

$$AB = 3.8771 \text{ in.}$$

22. In a pentagon $ABCDE$, given $\angle A = 110^\circ 50'$, $\angle B = 106^\circ 30'$, $\angle E = 104^\circ 10'$, $\angle BAC = 30^\circ$, $\angle DAE = 34^\circ 56'$, $\angle ADC = 52^\circ 30'$, and $AC = 6$ in., find the sides and the remaining angles of the pentagon.

$$\begin{aligned}\angle D &= \angle ADE + \angle ADC \\ &= 40^\circ 54' + 52^\circ 30' \\ &= 93^\circ 24'.\end{aligned}$$

$$BC = \frac{AC \sin A}{\sin B} = \frac{6 \sin 30^\circ}{\sin 106^\circ 30'}.$$

$$\begin{aligned}\log AC &= 0.77815 \\ \log \sin 30^\circ &= 9.69897 \\ \text{colog } \sin 106^\circ 30' &= 0.01826 \\ \log BC &= 0.49538\end{aligned}$$

$$BC = 3.1288 \text{ in.}$$

$$CD = \frac{AC \sin DAC}{\sin ADC} = \frac{6 \sin 45^\circ 54'}{\sin 52^\circ 30'}.$$

$$\begin{aligned}\log 6 &= 0.77815 \\ \log \sin 45^\circ 54' &= 9.85620 \\ \text{colog } \sin 52^\circ 30' &= 0.10053 \\ \log CD &= 0.73488\end{aligned}$$

$$CD = 5.421 \text{ in.}$$

$$AE = \frac{AD \sin ADE}{\sin E}$$

$$= \frac{7.4817 \sin 40^\circ 54'}{\sin 104^\circ 10'}$$

$$\log 7.4817 = 0.87400$$

$$\log \sin 40^\circ 54' = 9.81607$$

$$\text{colog} \sin 104^\circ 10' = 0.01341$$

$$\log AE = 0.70348$$

$$AE = 5.0522 \text{ in.}$$

$$\angle C = \angle ACD + \angle ACB$$

$$= 81^\circ 36' + 43^\circ 30'$$

$$= 125^\circ 6'.$$

$$AB = \frac{AC \sin ACB}{\sin B} = \frac{6 \sin 43^\circ 30'}{\sin 106^\circ 30'}$$

$$\log AC = 0.77815$$

$$\log \sin 43^\circ 30' = 9.83781$$

$$\text{colog} \sin 106^\circ 30' = 0.01826$$

$$\log AB = 0.63422$$

$$AB = 4.3075 \text{ in.}$$

$$AD = \frac{AC \sin ACD}{\sin ADC} = \frac{6 \sin 81^\circ 36'}{\sin 52^\circ 30'}$$

$$\log 6 = 0.77815$$

$$\log \sin 81^\circ 36' = 9.99532$$

$$\text{colog} \sin 52^\circ 30' = 0.10053$$

$$\log AD = 0.87400$$

$$AD = 7.4817 \text{ in.}$$

$$DE = \frac{AD \sin DAE}{\sin E}$$

$$= \frac{7.4817 \sin 34^\circ 56'}{\sin 104^\circ 10'}$$

$$\log 7.4817 = 0.87400$$

$$\log \sin 34^\circ 56' = 9.75787$$

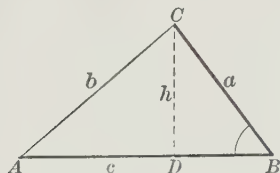
$$\text{colog} \sin 104^\circ 10' = 0.01341$$

$$\log DE = 0.64528$$

$$DE = 4.4186 \text{ in.}$$

Exercise 51. Law of Cosines (Page 117)

1. Using the figures on page 116, prove that, whether the angle B is acute or obtuse, $c = a \cos B + b \cos A$.



When angle B is acute,

$$\cos B = \frac{DB}{a}$$

$$\cos A = \frac{AD}{b}$$

$$\therefore DB = a \cos B,$$

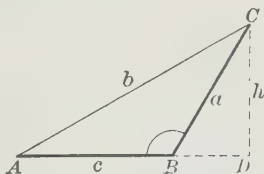
$$AD = b \cos A.$$

and

$$\therefore DB + AD = a \cos B + b \cos A.$$

$$\text{But } DB + AD = c.$$

$$\therefore c = a \cos B + b \cos A.$$



When angle B is obtuse,

$$\cos A = \frac{AD}{b}$$

$$\frac{BD}{a} = \cos(180^\circ - B)$$

$$= -\cos B.$$

$$\therefore AD = b \cos A,$$

and

$$BD = -a \cos B.$$

$$\therefore AD - BD = b \cos A + a \cos B$$

$$\text{But } AD - BD = c.$$

$$\therefore c = a \cos B + b \cos A$$

2. What are the two symmetrical formulas obtained by changing the letters in Ex. 1? What does the formula in Ex. 1 become when $B = 90^\circ$?

The symmetrical formulas are

$$b = a \cos C + c \cos A,$$

and

$$a = b \cos C + c \cos B.$$

When $B = 90^\circ$,

$$\cos A = \frac{c}{b}.$$

$$\therefore c = b \cos A.$$

3. Show that the sum of the squares on the sides of a triangle is equal to $2(ab \cos C + bc \cos A + ca \cos B)$.

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$b^2 = c^2 + a^2 - 2ca \cos B.$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

$$\begin{aligned} a^2 + b^2 + c^2 &= b^2 + c^2 - 2bc \cos A + c^2 + a^2 - 2ca \cos B + a^2 + b^2 - 2ab \cos C \\ &= 2(ab \cos C + bc \cos A + ca \cos B). \end{aligned}$$

4. Consider the Law of Cosines in the case of the triangle $a = 5$, $b = 12$, $c = 6$.

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$25 = 144 + 36 - 144 \cos A.$$

$$144 \cos A = 155.$$

$$\cos A = \frac{155}{144} = 1\frac{11}{144}.$$

That is, $\cos A > 1$, which is impossible, since the cosine of an angle is never greater than 1. Hence the given triangle is impossible.

5. Given $a = 5$, $b = 5$, and $C = 60^\circ$, find c .

$$c^2 = a^2 + b^2 - 2ab \cos C = 25 + 25 - 50 \cdot \frac{1}{2} = 50 - 25 = 25.$$

$$c = 5.$$

6. Given $a = 10$, $b = 10$, and $C = 45^\circ$, find c .

$$c^2 = a^2 + b^2 - 2ab \cos C = 100 + 100 - 200 \cdot \frac{1}{2} \sqrt{2} = 200 - 100 \sqrt{2} = 58.6.$$

$$c = 7.655.$$

7. Given $a = 8$, $b = 5$, and $C = 60^\circ$, find c .

$$c^2 = a^2 + b^2 - 2ab \cos C = 64 + 25 - 80 \cdot \frac{1}{2} = 89 - 40 = 49.$$

$$c = 7.$$

8. From the formula $a^2 = b^2 + c^2 - 2bc \cos A$ deduce a formula for $\cos A$. From this result find the value of A when $b^2 + c^2 = a^2$.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\text{When } b^2 + c^2 = a^2,$$

$$2bc \cos A = b^2 + c^2 - a^2.$$

$$\cos A = \frac{a^2 - a^2}{2bc} = 0.$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$A = 90^\circ.$$

9. Prove that if $\frac{\cos A}{b} = \frac{\cos B}{a}$ the triangle is either isosceles or right.

$$\frac{\cos A}{b} = \frac{\cos B}{a}, \quad a \cos A = b \cos B.$$

$$\frac{a(b^2 + c^2 - a^2)}{2bc} = \frac{b(c^2 + a^2 - b^2)}{2ca}.$$

$$a^2b^2 + a^2c^2 - a^4 = b^2c^2 + a^2b^2 - b^4.$$

$$(a^2c^2 - b^2c^2) - (a^4 - b^4) = 0.$$

$$(a^2 - b^2)[c^2 - (a^2 + b^2)] = 0,$$

$$a^2 - b^2 = 0, \quad c^2 - (a^2 + b^2) = 0.$$

Factoring,
or

Whence, $a = b$, or $c^2 = a^2 + b^2$, and hence the triangle is either isosceles or right.

10. Prove that $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$.

$$2bc \cos A + 2ac \cos B + 2ab \cos C = a^2 + b^2 + c^2$$

$$2bc \cos A = b^2 + c^2 - a^2$$

$$2ca \cos B = c^2 + a^2 - b^2$$

$$2ab \cos C = a^2 + b^2 - c^2$$

$$2bc \cos A + 2ac \cos B + 2ab \cos C = a^2 + b^2 + c^2$$

11. Prove that $\frac{b^2}{a} \cos A + \frac{c^2}{b} \cos B + \frac{a^2}{c} \cos C = \frac{a^4 + b^4 + c^4}{2abc}$.

$$2b^3c \cos A + 2ac^3 \cos B + 2a^3b \cos C = a^4 + b^4 + c^4$$

$$2b^3c \cos A = b^2(b^2 + c^2 - a^2)$$

$$2c^3a \cos B = c^2(c^2 + a^2 - b^2)$$

$$2a^3b \cos C = a^2(a^2 + b^2 - c^2)$$

$$2b^3c \cos A + 2ac^3 \cos B + 2a^3b \cos C = a^4 + b^4 + c^4$$

12. From the Law of Cosines prove that the square on the side opposite an acute angle of a triangle is equal to the sum of the squares on the other two sides minus twice the product of either side and the projection of the other side upon it.

Let AD be the projection of b on c .

To prove $a^2 = b^2 + c^2 - 2cAD$.

$$a^2 = b^2 + c^2 - 2bc \cos A. \quad (1)$$

$$\cos A = \frac{AD}{b}, \quad b \cos A = AD.$$

Substitute the value of $b \cos A$ in (1), $a^2 = b^2 + c^2 - 2cAD$.

13. As in Ex. 12, consider the geometric proposition relating to the square on the side opposite an obtuse angle.

To prove

$$\begin{aligned} b^2 &= a^2 + c^2 + 2cBD. \\ b^2 &= c^2 + a^2 - 2ca \cos B. \end{aligned} \quad (1)$$

$$\cos B = -\frac{BD}{a}.$$

Substitute the value of $\cos B$ in (1),

$$b^2 = c^2 + a^2 + 2cBD.$$

14. In the parallelogram $ABCD$, given $AB = 4$ in., $AD = 5$ in., and $A = 38^\circ 40'$, find the two diagonals.

$$\overline{BD}^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cos 38^\circ 40' = 16 + 25 - 31.232 = 9.768.$$

$$BD = 3.1254 \text{ in.}$$

$$\overline{AC}^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cos 141^\circ 20' = 16 + 25 + 31.232 = 72.232.$$

$$AC = 8.499 \text{ in.}$$

15. In the parallelogram $ABCD$, given $AB = 7$ in., $AC = 10$ in., and $\angle BAC = 36^\circ 7'$, find the side BC and the diagonal BD .

$$\overline{BC}^2 = 7^2 + 10^2 - 2 \cdot 7 \cdot 10 \cos 36^\circ 7'$$

$$= 49 + 100 - 113.092$$

$$= 35.908.$$

$$BC = 5.9924 \text{ in.}$$

$$7^2 = 10^2 + (5.9924)^2 - 2 \cdot 10 \cdot 5.9924 \cos ACB.$$

$$49 = 100 + 35.908 - 119.848 \cos ACB.$$

$$119.848 \cos ACB = 86.908.$$

$$\cos ACB = 0.7252.$$

$$ACB = 43^\circ 31'.$$

$$C = 43^\circ 31' + 36^\circ 7'$$

$$= 79^\circ 38'.$$

$$\overline{BD}^2 = 7^2 + (5.9924)^2 - 2 \cdot 7 \cdot 5.9924 \cos 79^\circ 38'$$

$$= 49 + 35.908 - 15.0915$$

$$= 69.8165.$$

$$BD = 8.3556 \text{ in.}$$

16. In the quadrilateral $ABCD$, given $AC = 3.6$ in., $AD = 4$ in., $BC = 2.4$ in., $\angle ACB = 29^\circ 40'$, and $\angle CAD = 71^\circ 20'$, find the other two sides and all four angles of the quadrilateral.

$$\overline{AB}^2 = 2.4^2 + 3.6^2 - 2 \cdot 2.4 \cdot 3.6 \cos 29^\circ 40'$$

$$= 5.76 + 12.96 - 15.0146$$

$$= 3.7054.$$

$$AB = 1.9249 \text{ in.}$$

$$\overline{CD}^2 = 4^2 + 3.6^2 - 2 \cdot 4 \cdot 3.6 \cos 71^\circ 20'$$

$$= 16 + 12.96 - 9.2189$$

$$= 19.7411.$$

$$CD = 4.4431 \text{ in.}$$

$$3.6^2 = (1.9249)^2 + 2.4^2 - 2 \cdot 2.4 \cdot 1.9249 \cos B.$$

$$12.96 = 3.7054 + 5.76 - 9.2395 \cos B.$$

$$9.2395 \cos B = -3.4946.$$

$$\cos B = -0.3782.$$

$$B = 112^\circ 13' 40''.$$

$$4^2 = (4.4431)^2 + 3.6^2 - 2 \cdot 3.6 \cdot 4.4431 \cos ACD.$$

$$16 = 19.7411 + 12.96 - 31.9903 \cos ACD.$$

$$31.9903 \cos ACD = 16.7011.$$

$$\cos ACD = 0.5221.$$

$$ACD = 58^\circ 31' 40''.$$

$$C = 58^\circ 31' 40'' + 29^\circ 40'$$

$$= 88^\circ 11' 40''.$$

$$2.4^2 = (1.9249)^2 + 3.6^2 - 2 \cdot 3.6 \cdot 1.9249 \cos BAC.$$

$$5.76 = 3.7054 + 12.96 - 13.8593 \cos BAC.$$

$$13.8593 \cos BAC = 10.9054.$$

$$\cos BAC = 0.7869.$$

$$BAC = 38^\circ 6'.$$

$$A = 38^\circ 6' + 71^\circ 20' = 109^\circ 26'.$$

$$D = 360^\circ - (109^\circ 26' + 88^\circ 11' 40'' + 112^\circ 13' 40'')$$

$$= 50^\circ 8' 40''.$$

17. In the pentagon $ABCDE$, given $AB = 3.4$ in., $AC = 4.1$ in., $AD = 3.9$ in., $AE = 2.2$ in., $\angle BAC = 38^\circ 7'$, $\angle CAD = 41^\circ 22'$, and $\angle DAE = 32^\circ 5'$, find the perimeter of the pentagon.

$$\overline{BC}^2 = 4.1^2 + 3.4^2 - 2 \cdot 4.1 \cdot 3.4 \cos 38^\circ 7'$$

$$= 16.81 + 11.56 - 27.88 \cdot 0.7868$$

$$= 28.37 - 21.9360$$

$$= 6.4340.$$

$$BC = 2.5365 \text{ in.}$$

$$\overline{CD}^2 = 4.1^2 + 3.9^2 - 2 \cdot 4.1 \cdot 3.9 \cos 41^\circ 22'$$

$$= 16.81 + 15.21 - 31.98 \cdot 0.7505$$

$$= 32.02 - 24.0010$$

$$= 8.0190.$$

$$CD = 2.8318 \text{ in.}$$

$$\overline{DE}^2 = 3.9^2 + 2.2^2 - 2 \cdot 3.9 \cdot 2.2 \cos 32^\circ 5'$$

$$= 15.21 + 4.84 - 17.16 \cdot 0.8473.$$

$$= 20.05 - 14.5397$$

$$= 5.5103.$$

$$DE = 2.3474 \text{ in.}$$

$$\text{Perimeter} = (AB + BC + CD + DE + EA) \text{ in.}$$

$$= (3.4 + 2.5365 + 2.8318 + 2.3474 + 2.2) \text{ in.}$$

$$= 13.3157 \text{ in.}$$

Exercise 52. Law of Tangents (Page 119)

Find the form to which $\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$ reduces when:

1. $C = 90^\circ$.

$$A + B = 90^\circ.$$

$$B = 90^\circ - A.$$

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}[A - (90^\circ - A)]}{\tan 45^\circ} = \frac{\tan(A - 45^\circ)}{1} = \tan(A - 45^\circ)$$

2. $a = b$.

$$a - a = 0.$$

$$a + a = 2a.$$

$$\frac{0}{2a} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$$

$$0 = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$$

$$\tan \frac{1}{2}(A-B) = 0.$$

3. $A = B = C$.

$$A = 180^\circ \div 3 = 60^\circ = B = C.$$

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(0^\circ)}{\tan 60^\circ}.$$

$$\frac{a-b}{a+b} = \frac{\tan 0^\circ}{\tan 60^\circ}$$

$$= 0.$$

$$a - b = 0.$$

$$a = b.$$

4. $A - B = 90^\circ$, and $B = C$.

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$$

$$A + B + C = 180^\circ;$$

or $A + 2B = 180^\circ$

$$A - B = 90^\circ$$

$$3B = 90^\circ$$

$$B = 30^\circ.$$

$$C = 30^\circ.$$

$$A = 120^\circ.$$

$$\frac{a-b}{a+b} = \frac{\tan 45^\circ}{\tan 75^\circ}$$

$$= \frac{\tan 45^\circ}{\cot 15^\circ}$$

$$= \frac{1}{2 + \sqrt{3}}.$$

$$\therefore a + b = (a - b)(2 + \sqrt{3}).$$

Prove the following formulas:

5. $\frac{b-c}{b+c} = \tan \frac{1}{2}(B-C) \cot \frac{1}{2}(B+C).$

$$\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} = \frac{\tan \frac{1}{2}(B-C)}{\frac{1}{\cot \frac{1}{2}(B+C)}} = \tan \frac{1}{2}(B-C) \cot \frac{1}{2}(B+C)$$

6. $\tan \frac{1}{2}(B-C) = \frac{b-c}{b+c} \cot \frac{1}{2}A.$

$$\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}.$$

$$\tan \frac{1}{2}(B-C) = \frac{b-c}{b+c} \tan \frac{1}{2}(B+C) = \frac{b-c}{b+c} \tan \frac{1}{2}(180^\circ - A) = \frac{b-c}{b+c} \cot \frac{1}{2}A.$$

$$7. \frac{a+b}{a-b} = \frac{\cot \frac{1}{2}(A-B)}{\cot \frac{1}{2}(A+B)}.$$

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \frac{\cot \frac{1}{2}(A+B)}{\cot \frac{1}{2}(A-B)} = \frac{1}{\cot \frac{1}{2}(A-B)} = \cot \frac{1}{2}(A-B).$$

$$8. \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

$$\therefore \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

$$9. \frac{\sin B + \sin C}{\sin B - \sin C} = \frac{2 \sin \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C)}{2 \cos \frac{1}{2}(B+C) \sin \frac{1}{2}(B-C)}$$

$$\sin B + \sin C = 2 \sin \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C). \quad (1)$$

$$\sin B - \sin C = 2 \cos \frac{1}{2}(B+C) \sin \frac{1}{2}(B-C). \quad (2)$$

Divide (1) by (2),

$$\frac{\sin B + \sin C}{\sin B - \sin C} = \frac{2 \sin \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C)}{2 \cos \frac{1}{2}(B+C) \sin \frac{1}{2}(B-C)}.$$

$$10. \frac{\sin A + \sin B}{\sin A - \sin B} = \tan \frac{1}{2}(A+B) \cot \frac{1}{2}(A-B).$$

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B). \quad (1)$$

$$\sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B). \quad (2)$$

Divide (1) by (2),

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \tan \frac{1}{2}(A+B) \cot \frac{1}{2}(A-B).$$

11. To what does the formula in Ex. 8 reduce when $A = B$?

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

$$\frac{\sin A + \sin A}{\sin A - \sin A} = \frac{\tan \frac{1}{2}(2A)}{\tan \frac{1}{2}(0^\circ)}.$$

$$\frac{2 \sin A}{0} = \frac{\tan A}{0}.$$

$$\infty = \infty$$

12. To what does the formula in Ex. 9 reduce when $B = C = 60^\circ$?

$$\begin{aligned}\frac{\sin B + \sin C}{\sin B - \sin C} &= \frac{2 \sin \frac{1}{2}(B + C) \cos \frac{1}{2}(B - C)}{2 \cos \frac{1}{2}(B + C) \sin \frac{1}{2}(B - C)} \\ \frac{\sin 60^\circ + \sin 60^\circ}{\sin 60^\circ - \sin 60^\circ} &= \frac{2 \sin \frac{1}{2}(120^\circ) \cos \frac{1}{2}(0^\circ)}{2 \cos \frac{1}{2}(120^\circ) \sin \frac{1}{2}(0^\circ)} \\ \frac{2 \sin 60^\circ}{0} &= \frac{2 \sin 60^\circ \cos 0^\circ}{2 \cos 60^\circ \sin 0^\circ} \\ \frac{\sqrt{3}}{0} &= \frac{\sqrt{3}}{0}.\end{aligned}$$

13. To what does the formula in Ex. 10 reduce when the triangle is equilateral?

$$\begin{aligned}\frac{\sin A + \sin B}{\sin A - \sin B} &= \tan \frac{1}{2}(A + B) \cot \frac{1}{2}(A - B). \\ \frac{\sin 60^\circ + \sin 60^\circ}{\sin 60^\circ - \sin 60^\circ} &= \tan 60^\circ \cot 0^\circ. \\ \frac{\sqrt{3}}{0} &= \sqrt{3} \cdot \infty.\end{aligned}$$

14. To what does the Law of Tangents, page 119, reduce in the case of an isosceles triangle in which $a = b$? What does this prove with respect to the angles opposite the equal sides?

$$\begin{aligned}\frac{a - b}{a + b} &= \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} \\ \frac{a - a}{a + a} &= \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} \\ \frac{0}{2a} &= \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} \\ 0 &= \tan \frac{1}{2}(A - B). \\ \therefore A = B \text{ for } \tan 0^\circ = 0.\end{aligned}$$

15. By the help of the Law of Tangents prove that an equilateral triangle is also equiangular.

$$\begin{aligned}a &= b = c. \\ \frac{a - a}{2a} &= \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} \\ \frac{0}{2a} &= \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} \\ \tan \frac{1}{2}(A - B) &= 0, \quad A = B. \\ \frac{b - b}{2b} &= \frac{\tan \frac{1}{2}(B - C)}{\tan \frac{1}{2}(B + C)} \\ \tan \frac{1}{2}(B - C) &= 0, \quad B = C. \\ \therefore A = B = C.\end{aligned}$$

16. By the help of the Law of Tangents prove that an equiangular triangle is also equilateral.

$$\begin{aligned}
 A &= B = C. \\
 \frac{a-b}{a+b} &= \frac{\tan \frac{1}{2}(A-A)}{\tan \frac{1}{2}(A+A)} = \frac{\tan 0^\circ}{\tan A} = 0. \\
 a-b &= 0. \\
 a &= b. \\
 \frac{b-c}{b+c} &= \frac{\tan \frac{1}{2}(B-B)}{\tan \frac{1}{2}(B+B)} = \frac{\tan 0^\circ}{\tan B} = 0. \\
 b-c &= 0. \\
 b &= c. \\
 \therefore a &= b = c.
 \end{aligned}$$

17. Given any three sides and any three angles of a quadrilateral, show how the fourth side and the fourth angle can be found. Show also that it is not necessary to have so many parts given, and find the smallest number of parts that will solve the quadrilateral.

Let AB, BC, CD be the given sides,
and A, B, D the given angles.

With these three angles given, C can be found by subtracting $A + B + D$ from 360° .

$AB - BC$ and $AB + BC$ can be found. $\angle BAC + \angle ACB = 180^\circ - B$. From the Law of Tangents, $\angle BAC - \angle ACB$ can be found, and the angles BAC and ACB .

$$\begin{aligned}
 \angle DAC &= \angle DAB - \angle BAC, \text{ known.} \\
 \angle ACD &= 180^\circ - (D + \angle DAC), \text{ known.} \\
 \frac{AD}{DC} &= \frac{\sin ACD}{\sin DAC}, \therefore AD \text{ can be found.}
 \end{aligned}$$

The side CD is not necessary, for the diagonal AC can be found in triangle ABC by the law of sines; and with the angles of ADC known, the sides AD and CD can be found. Therefore the smallest number of parts that will solve the quadrilateral is five.

18. What sides, what diagonals, and what angles of a pentagon is it necessary to know in order, by the aid of the Law of Tangents alone, to solve the pentagon?

To solve the pentagon we need sides AB, BC, AE , angles B, CAD, DAE , and diagonal AD .

Exercise 53. Solving Triangles (Page 121)*Solve these triangles, given the following parts:*

1. $a = 77.99$, $b = 83.39$, $C = 72^\circ 15'$. 3. $a = 17$, $b = 12$, $C = 59^\circ 17'$.

$$b + a = 161.38.$$

$$b - a = 5.4.$$

$$B + A = 107^\circ 45'.$$

$$\frac{1}{2}(B + A) = 53^\circ 52' 30''.$$

$$\log(b - a) = 0.73239$$

$$\text{colog}(b + a) = 7.79215 - 10$$

$$\log \tan \frac{1}{2}(B + A) = \underline{0.13675}$$

$$\log \tan \frac{1}{2}(B - A) = \underline{8.66129}$$

$$\frac{1}{2}(B - A) = 2^\circ 37' 30''.$$

$$A = 51^\circ 15'.$$

$$B = 56^\circ 30'.$$

$$\log b = 1.92111$$

$$\log \sin C = 9.97882$$

$$\text{colog} \sin B = \underline{0.07889}$$

$$\log c = 1.97882$$

$$c = 95.24.$$

2. $b = 872.5$, $c = 632.7$, $A = 80^\circ$.

$$b - c = 239.8.$$

$$b + c = 1505.2.$$

$$B + C = 100^\circ.$$

$$\frac{1}{2}(B + C) = 50^\circ.$$

$$\log(b - c) = 2.37985$$

$$\text{colog}(b + c) = 6.82240 - 10$$

$$\log \tan \frac{1}{2}(B + C) = \underline{0.07619}$$

$$\log \tan \frac{1}{2}(B - C) = \underline{9.27844}$$

$$\frac{1}{2}(B - C) = 10^\circ 45' 2''.$$

$$B = 60^\circ 45' 2''.$$

$$C = 39^\circ 14' 58''.$$

$$\log b = 2.94077$$

$$\log \sin A = 9.99335$$

$$\text{colog} \sin B = 0.05924$$

$$\log a = 2.99336$$

$$a = 984.83.$$

$$a + b = 29.$$

$$a - b = 5.$$

$$A + B = 120^\circ 43'.$$

$$\frac{1}{2}(A + B) = 60^\circ 21' 30''.$$

$$\log(a - b) = 0.69897$$

$$\text{colog}(a + b) = 8.53760 - 10$$

$$\log \tan \frac{1}{2}(A + B) = \underline{10.24486}$$

$$\log \tan \frac{1}{2}(A - B) = \underline{9.48143}$$

$$\frac{1}{2}(A - B) = 16^\circ 51' 23''.$$

$$A = 77^\circ 12' 53''.$$

$$B = 43^\circ 30' 7''.$$

$$\log b = 1.07918$$

$$\log \sin C = 9.93435$$

$$\text{colog} \sin B = \underline{0.16217}$$

$$\log c = 1.17570$$

$$c = 14.987.$$

4. $b = \sqrt{5}$, $c = \sqrt{3}$, $A = 35^\circ 53'$.

$$\sqrt{5} = 2.2361.$$

$$\sqrt{3} = 1.7321.$$

$$b + c = 3.9681.$$

$$b - c = 0.5040.$$

$$B + C = 144^\circ 7'.$$

$$\frac{1}{2}(B + C) = 72^\circ 3' 30''.$$

$$\log(b - c) = 9.70243 - 10$$

$$\text{colog}(b + c) = 9.40142 - 10$$

$$\log \tan \frac{1}{2}(B + C) = \underline{10.48973}$$

$$\log \tan \frac{1}{2}(B - C) = \underline{9.59358}$$

$$\frac{1}{2}(B - C) = 21^\circ 25' 6''.$$

$$B = 93^\circ 28' 36''.$$

$$C = 50^\circ 38' 24''.$$

$$\log c = 0.23856$$

$$\log \sin A = 9.76800$$

$$\text{colog} \sin C = 0.11172$$

$$\log a = 0.11828$$

$$a = 1.3131.$$

5. $a = 0.917, b = 0.312, C = 33^\circ 7' 9''$.

$$a + b = 1.229.$$

$$a - b = 0.605.$$

$$A + B = 146^\circ 52' 51''.$$

$$\frac{1}{2}(A + B) = 73^\circ 26' 25''.$$

$$\log(a - b) = 9.78176 - 10$$

$$\operatorname{colog}(a + b) = 9.91045 - 10$$

$$\log \tan \frac{1}{2}(A + B) = \frac{10.52674}{}$$

$$\log \tan \frac{1}{2}(A - B) = \frac{10.21895}{}$$

$$\frac{1}{2}(A - B) = 58^\circ 52' 2''.$$

$$A = 132^\circ 18' 27''.$$

$$B = 14^\circ 34' 24''.$$

$$\log b = 9.49415 - 10$$

$$\log \sin C = 9.73750$$

$$\operatorname{colog} \sin B = \frac{0.59926}{}$$

$$\log c = \frac{9.83091}{} - 10$$

$$c = 0.6775.$$

7. $b = 3000.9, c = 1587.2,$

$$A = 86^\circ 4' 4''.$$

$$b + c = 4588.1.$$

$$b - c = 1413.7.$$

$$B + C = 93^\circ 55' 56''.$$

$$\frac{1}{2}(B + C) = 46^\circ 57' 58''.$$

$$\log(b - c) = 3.15036$$

$$\operatorname{colog}(b + c) = 6.33837 - 10$$

$$\log \tan \frac{1}{2}(B + C) = \frac{10.02983}{}$$

$$\log \tan \frac{1}{2}(B - C) = \frac{9.51856}{}$$

$$\frac{1}{2}(B - C) = 18^\circ 15' 53''.$$

$$C = 28^\circ 42' 5''.$$

$$B = 65^\circ 13' 51''.$$

$$\log b = 3.47726$$

$$\log \sin A = 9.99898$$

$$\operatorname{colog} \sin B = \frac{0.04191}{}$$

$$\log a = \frac{3.51815}{}$$

$$a = 3297.2.$$

6. $a = 13.715, c = 11.214,$

$$B = 15^\circ 22' 36''.$$

$$a - c = 2.501.$$

$$a + c = 24.929.$$

$$A + C = 164^\circ 37' 24''.$$

$$\frac{1}{2}(A + C) = 82^\circ 18' 42''.$$

$$\log(a - c) = 0.39811$$

$$\operatorname{colog}(a + c) = 8.60330 - 10$$

$$\log \tan \frac{1}{2}(A + C) = \frac{10.86968}{}$$

$$\log \tan \frac{1}{2}(A - C) = \frac{9.87109}{}$$

$$\frac{1}{2}(A - C) = 36^\circ 37' 7''.$$

$$A = 118^\circ 55' 49''.$$

$$C = 45^\circ 41' 35''.$$

$$\log a = 1.13720$$

$$\log \sin B = 9.42352$$

$$\operatorname{colog} \sin A = \frac{0.05789}{}$$

$$\log b = \frac{0.61861}{}$$

$$b = 4.1554.$$

8. $a = 4527, b = 3465,$

$$C = 66^\circ 6' 27''.$$

$$a + b = 7992.$$

$$a - b = 1062.$$

$$A + B = 113^\circ 53' 33''.$$

$$\frac{1}{2}(A + B) = 56^\circ 56' 47''.$$

$$\log(a - b) = 3.02612$$

$$\operatorname{colog}(a + b) = 6.09734 - 10$$

$$\log \tan \frac{1}{2}(A + B) = \frac{10.18659}{}$$

$$\log \tan \frac{1}{2}(A - B) = \frac{9.31005}{}$$

$$\frac{1}{2}(A - B) = 11^\circ 32' 28''.$$

$$A = 68^\circ 29' 15''.$$

$$B = 45^\circ 24' 18''.$$

$$\log a = 3.65581$$

$$\log \sin C = 9.96109$$

$$\operatorname{colog} \sin A = \frac{0.03136}{}$$

$$\log c = \frac{3.64826}{}$$

$$c = 4449.$$

9. $a=55.14$, $b=33.09$, $C=30^\circ 24'$. 11. $a=210$, $b=105$, $C=36^\circ 52' 12''$.

$$a + b = 88.23.$$

$$a - b = 22.05.$$

$$A + B = 149^\circ 36'.$$

$$\frac{1}{2}(A + B) = 74^\circ 48'.$$

$$\log(a - b) = 1.34341$$

$$\text{colog}(a + b) = 8.05438 - 10$$

$$\log \tan \frac{1}{2}(A + B) = \frac{10.56592}{}$$

$$\log \tan \frac{1}{2}(A - B) = \frac{9.96371}{}$$

$$\frac{1}{2}(A - B) = 42^\circ 36' 32''.$$

$$A = 117^\circ 24' 32''.$$

$$B = 32^\circ 11' 28''.$$

$$\log b = 1.51970$$

$$\log \sin C = 9.70418$$

$$\text{colog} \sin B = \frac{0.27348}{}$$

$$\log c = \frac{1.49736}{}$$

$$c = 31.431.$$

$$a + b = 315.$$

$$a - b = 105.$$

$$A + B = 143^\circ 7' 48''.$$

$$\frac{1}{2}(A + B) = 71^\circ 33' 54'.$$

$$\log(a - b) = 2.02119$$

$$\text{colog}(a + b) = 7.50169 - 10$$

$$\log \tan \frac{1}{2}(A + B) = \frac{10.47712}{}$$

$$\log \tan \frac{1}{2}(A - B) = \frac{10.00000}{}$$

$$\frac{1}{2}(A - B) = 45^\circ.$$

$$A = 116^\circ 33' 54''.$$

$$B = 26^\circ 33' 54''.$$

$$\log b = 2.02119$$

$$\log \sin C = 9.77815$$

$$\text{colog} \sin B = \frac{0.34948}{}$$

$$\log c = \frac{2.14882}{}$$

$$c = 140.87.$$

10. $a = 47.99$, $b = 33.14$,
 $C = 175^\circ 19' 10''.$

$$a + b = 81.13.$$

$$a - b = 14.85.$$

$$A + B = 4^\circ 40' 50''.$$

$$\frac{1}{2}(A + B) = 2^\circ 20' 25''.$$

$$\log(a - b) = 1.17173$$

$$\text{colog}(a + b) = 8.09082 - 10$$

$$\log \tan \frac{1}{2}(A + B) = \frac{8.61138}{}$$

$$\log \tan \frac{1}{2}(A - B) = \frac{7.87393}{}$$

$$\frac{1}{2}(A - B) = 0^\circ 25' 43''.$$

$$A = 2^\circ 46' 8''.$$

$$B = 1^\circ 54' 42''.$$

$$\log b = 1.52035$$

$$\log \sin C = 8.91169$$

$$\text{colog} \sin B = \frac{1.47680}{}$$

$$\log c = \frac{1.90884}{}$$

$$c = 81.066.$$

12. $a = 100$, $b = 900$, $C = 65^\circ.$

$$b + a = 1000.$$

$$b - a = 800.$$

$$B + A = 115^\circ.$$

$$\frac{1}{2}(B + A) = 57^\circ 30'.$$

$$\log(b - a) = 2.90309$$

$$\text{colog}(b + a) = 7.00000$$

$$\log \tan \frac{1}{2}(B + A) = \frac{10.19581}{}$$

$$\log \tan \frac{1}{2}(B - A) = \frac{10.09890}{}$$

$$\frac{1}{2}(B - A) = 51^\circ 28' 5''$$

$$B = 108^\circ 58' 5''$$

$$A = 6^\circ 1' 55''.$$

$$\log b = 2.95424$$

$$\log \sin C = 9.95728$$

$$\text{colog} \sin B = \frac{0.02424}{}$$

$$\log c = \frac{2.93576}{}$$

$$c = 862.5.$$

$$13. a = 409, b = 169, C = 117.7^\circ.$$

$$a + b = 578.$$

$$a - b = 240.$$

$$A + B = 62^\circ 18'.$$

$$\frac{1}{2}(A + B) = 31^\circ 9'.$$

$$\log(a - b) = 2.38021$$

$$\text{colog}(a + b) = 7.23807$$

$$\log \tan \frac{1}{2}(A + B) = 9.78135$$

$$\log \tan \frac{1}{2}(A - B) = 9.39963$$

$$\frac{1}{2}(A - B) = 14^\circ 5' 20''.$$

$$A = 45^\circ 14' 20''.$$

$$B = 17^\circ 3' 40''.$$

$$\log b = 2.22789$$

$$\log \sin C = 9.94714$$

$$\text{colog} \sin B = \frac{0.53256}{}$$

$$\log c = 2.70759$$

$$c = 510.02.$$

$$15. a = 3718, b = 1507, C = 95.86^\circ.$$

$$a + b = 5225.$$

$$a - b = 2211.$$

$$A + B = 84^\circ 8' 24''.$$

$$\frac{1}{2}(A + B) = 42^\circ 4' 12''.$$

$$\log(a - b) = 3.34459$$

$$\text{colog}(a + b) = 6.28191$$

$$\log \tan \frac{1}{2}(A + B) = \frac{9.95550}{}$$

$$\log \tan \frac{1}{2}(A - B) = \frac{9.58200}{}$$

$$\frac{1}{2}(A - B) = 20^\circ 54' 14''.$$

$$A = 62^\circ 58' 26''.$$

$$B = 21^\circ 9' 58''.$$

$$\log b = 3.17811$$

$$\log \sin C = 9.99772$$

$$\text{colog} \sin B = \frac{0.44240}{}$$

$$\log c = \frac{3.61823}{}$$

$$c = 4151.7.$$

$$14. a = 6.25, b = 5.05, C = 105.77^\circ.$$

$$a + b = 11.30.$$

$$a - b = 1.20.$$

$$A + B = 74^\circ 13' 48''.$$

$$\frac{1}{2}(A + B) = 37^\circ 6' 54''.$$

$$\log(a - b) = 0.07918$$

$$\text{colog}(a + b) = 8.94692$$

$$\log \tan \frac{1}{2}(A + B) = 9.87892$$

$$\log \tan \frac{1}{2}(A - B) = 8.90502$$

$$\frac{1}{2}(A - B) = 4^\circ 35' 39''.$$

$$A = 41^\circ 42' 33''.$$

$$B = 32^\circ 31' 15''.$$

$$\log b = 0.70329$$

$$\log \sin C = 9.98333$$

$$\text{colog} \sin B = \frac{0.26954}{}$$

$$\log c = \frac{0.95616}{}$$

$$c = 9.0398.$$

$$16. a = 46.07, b = 22.29, C = 66.36^\circ.$$

$$a + b = 68.36.$$

$$a - b = 23.78.$$

$$A + B = 113^\circ 38' 24''.$$

$$\frac{1}{2}(A + B) = 56^\circ 49' 12''.$$

$$\log(a - b) = 1.37621$$

$$\text{colog}(a + b) = 8.16520$$

$$\log \tan \frac{1}{2}(A + B) = \frac{10.18450}{}$$

$$\log \tan \frac{1}{2}(A - B) = \frac{9.72591}{}$$

$$\frac{1}{2}(A - B) = 28^\circ 46''.$$

$$A = 84^\circ 49' 58''.$$

$$B = 28^\circ 48' 26''.$$

$$\log b = 1.34811$$

$$\log \sin C = 9.96192$$

$$\text{colog} \sin B = \frac{0.31707}{}$$

$$\log c = 1.62710$$

$$c = 42.374.$$

$$17. \ b = 445, \ c = 624, \ A = 10.88^\circ.$$

$$b + c = 1069.$$

$$c - b = 179.$$

$$C + B = 169^\circ 7' 12''.$$

$$\frac{1}{2}(C + B) = 84^\circ 33' 36''.$$

$$\log(c - b) = 2.25285$$

$$\text{colog}(c + b) = 6.97102$$

$$\log \tan \frac{1}{2}(C + B) = \frac{11.02121}{10}$$

$$\log \tan \frac{1}{2}(C - B) = \frac{10.24508}{10}$$

$$\frac{1}{2}(C - B) = 60^\circ 22' 16''.$$

$$B = 24^\circ 11' 20''.$$

$$C = 144^\circ 55' 52''.$$

$$\log b = 2.64836$$

$$\log \sin A = 9.27589$$

$$\text{colog} \sin B = \frac{0.38749}{10}$$

$$\log a = \frac{2.31174}{10}$$

$$a = 205.$$

$$18. \ b = 15.7, \ c = 43.6, \ A = 57.22^\circ.$$

$$c + b = 59.3.$$

$$c - b = 27.9.$$

$$C + B = 122^\circ 46' 48''$$

$$\frac{1}{2}(C + B) = 61^\circ 23' 24''.$$

$$\log(c - b) = 1.44560$$

$$\text{colog}(c + b) = 8.22695$$

$$\log \tan \frac{1}{2}(C + B) = \frac{10.26325}{10}$$

$$\log \tan \frac{1}{2}(C - B) = \frac{9.93580}{10}$$

$$\frac{1}{2}(C - B) = 40^\circ 46' 50''.$$

$$B = 20^\circ 36' 34''.$$

$$C = 102^\circ 10' 14''.$$

$$\log b = 1.19590$$

$$\log \sin A = 9.92467$$

$$\text{colog} \sin B = \frac{0.45346}{10}$$

$$\log a = \frac{1.57403}{10}$$

$$a = 37.5.$$

19. If two sides of a triangle are each equal to 6, and the included angle is 60° , find the third side by two different methods.

Since

$$a = b,$$

$$A = B,$$

$$A + B = 120^\circ.$$

$$\therefore A = B = C = 60^\circ.$$

$$\therefore a = b = c = 6.$$

$$A = 60^\circ, \ b = c = 6.$$

$$a = \frac{b \sin A}{\sin B} = \frac{6 \sin 60^\circ}{\sin 60^\circ} = 6.$$

20. If two sides of a triangle are each equal to 6, and the included angle is 120° , find the third side by three different methods.

$$A + B = 60^\circ.$$

$$\therefore A = B = 30^\circ,$$

$$a = 6 = b.$$

$$\log a = 0.77815$$

$$\log \sin C = 9.93753$$

$$\text{colog} \sin A = \frac{0.30103}{10}$$

$$\log c = \frac{1.01671}{10}$$

$$c = 10.392.$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$= 36 + 36 - 72(-\frac{1}{2})$$

$$= 72 + 36$$

$$= 108.$$

$$c = 10.392.$$

$$\frac{c - b}{c + b} = \frac{\tan \frac{1}{2}(C - B)}{\tan \frac{1}{2}(C + B)}$$

$$= \frac{\tan 45^\circ}{\tan 75^\circ}.$$

$$\tan(45^\circ + 30^\circ) = 2 + \sqrt{3}.$$

$$\frac{c - 6}{c + 6} = \frac{1}{2 + \sqrt{3}}.$$

$$2c - 12 + \sqrt{3}c - 6\sqrt{3} = c + 6.$$

$$2c - 12 + \sqrt{3}c - 6\sqrt{3} = c + 6.$$

$$c = \frac{6(3 + \sqrt{3})}{1 + \sqrt{3}}$$

$$= 6\sqrt{3}$$

$$= 10.392.$$

21. Apply the first method given on page 120 to the case in which a is equal to b ; that is, the case in which the triangle is isosceles.

If $a = b$, the formula

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \times \tan \frac{1}{2}(A + B)$$

becomes

$$\tan \frac{1}{2}(A - B) = 0.$$

$$\therefore A - B = 0.$$

$$A = B$$

$$= \frac{1}{2}(180^\circ - C)$$

$$= 90^\circ - \frac{1}{2}C.$$

$$c = \frac{a \sin C}{\sin A}.$$

22. If two sides of a triangle are 10 and 11, and the included angle is 50° , find the third side.

$$a + b = 21.$$

$$a - b = 1.$$

$$A + B = 130^\circ.$$

$$\frac{1}{2}(A + B) = 65^\circ.$$

$$\log(a - b) = 0.00000$$

$$\text{colog}(a + b) = 8.67778 - 10$$

$$\log \tan \frac{1}{2}(A + B) = \frac{10.33133}{}$$

$$\log \tan \frac{1}{2}(A - B) = 9.00911$$

$$\frac{1}{2}(A - B) = 5^\circ 49' 51''.$$

$$A = 70^\circ 49' 51''.$$

$$B = 59^\circ 10' 9''.$$

$$\log b = 1.00000$$

$$\log \sin C = 9.88425$$

$$\text{colog} \sin B = 0.06617$$

$$\log c = \frac{0.95042}{}$$

$$c = 8.9212.$$

23. If two sides of a triangle are 43.301 and 25, and the included angle is 30° , find the third side.

$$a + b = 68.301.$$

$$a - b = 18.301.$$

$$A + B = 150^\circ.$$

$$\frac{1}{2}(A + B) = 75^\circ.$$

$$\log(a - b) = 1.26247$$

$$\text{colog}(a + b) = 8.16557 - 10$$

$$\log \tan \frac{1}{2}(A + B) = \frac{10.57195}{}$$

$$\log \tan \frac{1}{2}(A - B) = 9.99999$$

$$\frac{1}{2}(A - B) = 45^\circ.$$

$$A = 120^\circ.$$

$$B = 30^\circ.$$

$\therefore$ in isosceles triangle ABC

$$c = b = 25.$$

24. In order to find the distance between two objects, A and B , separated by a swamp, a station C was chosen, and the distances $CA = 3825$ yd., $CB = 3475.6$ yd., together with the angle $ACB = 62^\circ 31'$, were measured. Find the distance from A to B .

$$b + a = 7300.6.$$

$$b - a = 349.4.$$

$$B + A = 117^\circ 29'.$$

$$\frac{1}{2}(B + A) = 58^\circ 44' 30''.$$

$$\log(b - a) = 2.54332$$

$$\text{colog}(b + a) = 6.13664 - 10$$

$$\log \tan \frac{1}{2}(B + A) = \frac{10.21680}{}$$

$$\log \tan \frac{1}{2}(B - A) = 8.89676$$

$$\frac{1}{2}(B - A) = 4^\circ 30' 29''.$$

$$B = 63^\circ 14' 59''.$$

$$A = 54^\circ 14' 1''.$$

$$\log b = 3.58263$$

$$\log \sin C = 9.94799$$

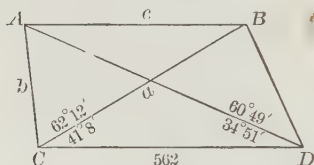
$$\text{colog} \sin B = \frac{0.04916}{}$$

$$\log c = 3.57978$$

$$c = 3800.$$

$$\therefore AB = 3800 \text{ yd.}$$

25. Two inaccessible objects, A and B , are each viewed from two stations, C and D , on the same side of AB and 562 yd. apart. The angle ACB is $62^\circ 12'$, BCD $41^\circ 8'$, ADB $60^\circ 49'$, and ADC $34^\circ 51'$. Required the distance AB .



In triangle ACD

$$A = 180^\circ - (C + D) \\ = 41^\circ 49'.$$

$$\frac{b}{562} = \frac{\sin 34^\circ 51'}{\sin 41^\circ 49'}.$$

$$\therefore b = \frac{562 \sin 34^\circ 51'}{\sin 41^\circ 49'}.$$

$$\log 562 = 2.74974$$

$$\log \sin 34^\circ 51' = 9.75696$$

$$\text{colog} \sin 41^\circ 49' = 0.17604$$

$$\log b = 2.68274$$

$$b = 481.66.$$

In the triangle CBD

$$B = 180^\circ - (C + D) \\ = 43^\circ 12'.$$

$$\frac{a}{562} = \frac{\sin 95^\circ 40'}{\sin 43^\circ 12'}.$$

$$\therefore a = \frac{562 \cos 5^\circ 40'}{\sin 43^\circ 12'}.$$

$$\log 562 = 2.74974$$

$$\log \cos 5^\circ 40' = 9.99787$$

$$\text{colog} \sin 43^\circ 12' = 0.16460$$

$$\log a = 2.91221$$

$$a = 816.98.$$

In triangle ACB

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \times \tan \frac{1}{2}(A + B)$$

$$\frac{1}{2}(A + B) = \frac{1}{2}(180^\circ - C) \\ = 58^\circ 54'.$$

$$a - b = 816.98 - 481.66 \\ = 335.32.$$

$$a + b = 816.98 + 481.66 \\ = 1298.64.$$

$$\log(a - b) = 2.52548$$

$$\text{colog}(a + b) = 6.88651 - 10$$

$$\log \tan \frac{1}{2}(A + B) = 10.21951$$

$$\log \tan \frac{1}{2}(A - B) = 9.63150$$

$$\frac{1}{2}(A - B) = 23^\circ 10' 26''.$$

$$A = 82^\circ 4' 26''.$$

$$\log a = 2.91221$$

$$\log \sin C = 9.94674$$

$$\text{colog} \sin A = 0.00417$$

$$\log c = 2.86312$$

$$c = 729.67.$$

$$\therefore AB = 729.67 \text{ yd.}$$

26. In order to find the distance between two objects, A and B , separated by a pond, a station C was chosen, and it was found that $CA = 426$ yd., $CB = 322.4$ yd., and $ACB = 68^\circ 42'$. Required the distance from A to B .

$$b + a = 748.4.$$

$$b - a = 103.6.$$

$$B + A = 111^\circ 18'.$$

$$\frac{1}{2}(B + A) = 55^\circ 39'.$$

$$\log(b - a) = 2.01536$$

$$\text{colog}(b + a) = 7.12587 - 10$$

$$\log \tan \frac{1}{2}(B + A) = 10.16530$$

$$\log \tan \frac{1}{2}(B - A) = 9.30653$$

$$\frac{1}{2}(B - A) = 11^\circ 27' 1''.$$

$$\therefore B = 67^\circ 6' 1''.$$

$$\begin{aligned}\log b &= 2.62941 \\ \log \sin C &= 9.96927 \\ \text{colog } \sin B &= 0.03565 \\ \log c &= 2.63433\end{aligned}$$

$$\begin{aligned}c &= 430.85. \\ \therefore AB &= 430.85 \text{ yd.}\end{aligned}$$

27. Two trains start at the same time from the same station and move along straight tracks that form an angle of 30° , one train at the rate of 30 mi. an hour, the other at the rate of 40 mi. an hour. How far apart are the trains at the end of half an hour?

$$\begin{aligned}a + b &= 35. \\ a - b &= 5. \\ A + B &= 150^\circ. \\ \frac{1}{2}(A + B) &= 75^\circ.\end{aligned}$$

$$\begin{aligned}\log(a - b) &= 0.69897 \\ \text{colog}(a + b) &= 8.45593 - 10 \\ \log \tan \frac{1}{2}(A + B) &= \frac{10.57195}{9.72685} \\ \log \tan \frac{1}{2}(A - B) &= \frac{10.57195}{9.72685}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}(A - B) &= 28^\circ 3' 52''. \\ B &= 46^\circ 56' 8''. \\ A &= 103^\circ 3' 52''.\end{aligned}$$

$$\begin{aligned}\log b &= 1.17609 \\ \log \sin C &= 9.69897 \\ \text{colog } \sin B &= 0.13633 \\ \log c &= 1.01139 \\ c &= 10.266.\end{aligned}$$

Therefore the trains are 10.266 mi. apart.

28. In a parallelogram, given the two diagonals 5 and 6 and the angle which they form $49^\circ 18'$, find the sides.

In the parallelogram $ABDE$
let $EB = 6$, and $AD = 5$,
and $\angle BCA = 49^\circ 18'$.

In triangle ACB
let $BC = a = 3$,
 $AC = b = 2.5$.

Find $AB = c$,
 $a - b = 0.5$,
 $a + b = 5.5$,
 $A + B = 130^\circ 42'$.

$$\begin{aligned}\frac{1}{2}(A + B) &= 65^\circ 21'. \\ \log(a - b) &= 9.69897 - 10 \\ \text{colog}(a + b) &= 9.25964 - 10 \\ \log \tan \frac{1}{2}(A + B) &= \frac{10.33829}{9.29690} \\ \log \tan \frac{1}{2}(A - B) &= \frac{10.33829}{9.29690} \\ \frac{1}{2}(A - B) &= 11^\circ 12' 20''. \\ A &= 76^\circ 33' 20''. \\ B &= 54^\circ 8' 40''.\end{aligned}$$

$$\begin{aligned}\log a &= 0.47712 \\ \log \sin C &= 9.87975 \\ \text{colog } \sin A &= 0.01207 \\ \log c &= 0.36894 \\ c &= AB = 2.3385.\end{aligned}$$

In triangle AEC
 $EC = a = 3$,
 $AC = b = 2.5$,
 $\angle ACE = 130^\circ 42'$,
 $A + E = 49^\circ 18'$.

$$\begin{aligned}\frac{1}{2}(A + E) &= 24^\circ 39'. \\ \log(a - b) &= 9.69897 - 10 \\ \text{colog}(a + b) &= 9.25964 - 10 \\ \log \tan \frac{1}{2}(A + E) &= \frac{9.66171}{8.62032} \\ \log \tan \frac{1}{2}(A - E) &= \frac{9.66171}{8.62032} \\ \frac{1}{2}(A - E) &= 2^\circ 23' 20''. \\ A &= 27^\circ 2' 20''.\end{aligned}$$

$$\begin{aligned}\log a &= 0.47712 \\ \log \sin C &= 9.87975 \\ \text{colog } \sin A &= 0.34238 \\ \log c &= 0.69925 \\ c &= EA = 5.0032.\end{aligned}$$

Exercise 54. Formulas of the Triangle (Page 125)

1. Given $\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, express the value of $\log \tan \frac{1}{2} A$

$$\log \tan \frac{1}{2} A = \frac{1}{2} [\log(s-b) + \log(s-c) + \operatorname{colog} s + \operatorname{colog}(s-a)]$$

2. Given $\sin \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}$, express the value of $\log \sin \frac{1}{2} A$.

$$\log \sin \frac{1}{2} A = \frac{1}{2} [\log(s-b) + \log(s-c) + \operatorname{colog} b + \operatorname{colog} c].$$

3. Given $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$, express the value of $\log r$.

$$\log r = \frac{1}{2} [\log(s-a) + \log(s-b) + \log(s-c) + \operatorname{colog} s]$$

4. Given $\tan \frac{1}{2} A = \frac{r}{s-a}$, express the value of $\log \tan \frac{1}{2} A$.

$$\log \tan \frac{1}{2} A = \log r + \operatorname{colog}(s-a).$$

5. Given $\tan \frac{1}{2} A = \frac{r}{s-a}$, express the value of $\log r$.

$$\tan \frac{1}{2} A = \frac{r}{s-a}.$$

$$r = (s-a) \tan \frac{1}{2} A.$$

$$\log r = \log(s-a) + \log \tan \frac{1}{2} A.$$

6. Of the three values for $\tan \frac{1}{2} A$,

$$\sqrt{\frac{1 - \cos A}{1 + \cos A}}, \quad (\S 192)$$

$$\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad (\S 115)$$

and $\frac{1}{s-a} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \quad (\S 116)$

which is the easiest to treat by logarithms? Express the logarithms of the results and show why your answer is correct.

The first formula is the easiest to treat by logarithms.

I. $\log \tan \frac{1}{2} A = \frac{1}{2} [\log(1 - \cos A) + \operatorname{colog}(1 + \cos A)].$

II. $\log \tan \frac{1}{2} A = \frac{1}{2} [\log(s-b) + \log(s-c) + \operatorname{colog} s + \operatorname{colog}(s-a)].$

III. $\log \tan \frac{1}{2} A = \log 1 + \operatorname{colog}(s-a) + \frac{1}{2} [\log(s-a) + \log(s-b) + \log(s-c) + \operatorname{colog} s].$

7 Given $a = 4$, $b = 5$, and $c = 6$, find the value of $\tan \frac{1}{2}A$, and then find the value of A .

$$a + b + c = 2s.$$

$$4 + 5 + 6 = 15.$$

$$s = 7\frac{1}{2}.$$

$$\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\log(s-b) = 0.39794$$

$$\log(s-c) = 0.17609$$

$$\text{colog } s = 9.12494$$

$$\text{colog}(s-a) = 9.45593$$

$$2 \overline{)19.15490}$$

$$\log \tan \frac{1}{2}A = 9.57745.$$

$$\tan \frac{1}{2}A = 20^\circ 42' 17''.$$

$$A = 41^\circ 24' 34''.$$

9. Discuss the formula

$$\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{1}{s-a} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

for the case of an equilateral triangle, say when $a = 4$.

$$s = \frac{1}{2}(4 + 4 + 4) = 6.$$

$$\tan \frac{1}{2}A = \frac{1}{6-4} \sqrt{\frac{(6-4)^3}{6}} = \frac{1}{2} \sqrt{\frac{4}{3}} = \frac{1}{3} \sqrt{3}.$$

$$\frac{1}{2}A = 30^\circ, \quad A = 60^\circ.$$

8. Deduce the equation

$$\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

from the equation

$$\tan \frac{1}{2}A = \sqrt{\frac{1-\cos A}{1+\cos A}}.$$

$$1-\cos A = \frac{2(s-b)(s-c)}{bc}. \quad (1)$$

$$1+\cos A = \frac{2s(s-a)}{bc}. \quad (2)$$

Divide (1) by (2) and take the square root.

$$\sqrt{\frac{1-\cos A}{1+\cos A}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\therefore \tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Exercise 55. Finding the Angles (Page 127)

Find the three angles of a triangle, given the three sides as follows:

1. 51, 65, 20.

$$2s = 136, s = 68.$$

$$s-a = 17, s-b = 3, s-c = 48.$$

$$\text{colog } s = 8.16749$$

$$\text{colog}(s-a) = 8.76955$$

$$\log(s-b) = 0.47712$$

$$\log(s-c) = 1.68124$$

$$2 \overline{)19.09540}$$

$$\log \tan \frac{1}{2}A = 9.54770$$

$$\frac{1}{2}A = 19^\circ 26' 24''.$$

$$A = 38^\circ 52' 48''.$$

$$\text{colog } s = 8.16749$$

$$\log(s-a) = 1.23045$$

$$\text{colog}(s-b) = 9.52288$$

$$\log(s-c) = 1.68124$$

$$2 \overline{)20.60206}$$

$$\log \tan \frac{1}{2}B = 10.30103$$

$$\frac{1}{2}B = 63^\circ 26' 6''.$$

$$B = 126^\circ 52' 12''.$$

$$A + B = 165^\circ 45', \text{ and } C = 14^\circ 15'.$$

2. 78, 101, 29.

$$2s = 208, s = 104.$$

$$s - a = 26, s - b = 3, s - c = 75.$$

$$\begin{aligned}\operatorname{colog} s &= 7.98297 \\ \operatorname{colog}(s - a) &= 8.58503 \\ \log(s - b) &= 0.47712 \\ \log(s - c) &= 1.87506\end{aligned}$$

$$2 \overline{)18.92018}$$

$$\log \tan \frac{1}{2} A = 9.46009$$

$$\frac{1}{2} A = 16^\circ 5' 27.5''.$$

$$A = 32^\circ 10' 55''.$$

$$\begin{aligned}\operatorname{colog} s &= 7.98297 \\ \log(s - a) &= 1.41497 \\ \operatorname{colog}(s - b) &= 9.52288 \\ \log(s - c) &= 1.87506\end{aligned}$$

$$2 \overline{)20.79588}$$

$$\log \tan \frac{1}{2} B = 10.39794$$

$$\frac{1}{2} B = 68^\circ 11' 55''.$$

$$B = 136^\circ 23' 50''.$$

$$A + B = 168^\circ 34' 45'', \text{ and } C = 11^\circ 25' 15''.$$

3. 111, 145, 40.

$$2s = 296, s = 148.$$

$$s - a = 37, s - b = 3, s - c = 108.$$

$$\begin{aligned}\operatorname{colog} s &= 7.82974 \\ \operatorname{colog}(s - a) &= 8.43180 \\ \log(s - b) &= 0.47712 \\ \log(s - c) &= 2.03342\end{aligned}$$

$$2 \overline{)18.77208}$$

$$\log \tan \frac{1}{2} A = 9.88604$$

$$\frac{1}{2} A = 13^\circ 40' 16''.$$

$$A = 27^\circ 20' 32''.$$

$$\begin{aligned}\operatorname{colog} s &= 7.82974 \\ \log(s - a) &= 1.56820 \\ \operatorname{colog}(s - b) &= 9.52288 \\ \log(s - c) &= 2.03342\end{aligned}$$

$$2 \overline{)20.95424}$$

$$\log \tan \frac{1}{2} B = 10.47712$$

$$\frac{1}{2} B = 71^\circ 33' 54''.$$

$$B = 143^\circ 7' 48''.$$

$$A + B = 170^\circ 28' 20'', C = 9^\circ 31' 40''.$$

4. 21, 26, 31.

$$2s = 78, s = 39.$$

$$s - a = 18, s - b = 13, s - c = 8.$$

$$\begin{aligned}\operatorname{colog} s &= 8.40894 \\ \operatorname{colog}(s - a) &= 8.74473 \\ \log(s - b) &= 1.11394 \\ \log(s - c) &= 0.90309\end{aligned}$$

$$2 \overline{)19.17070}$$

$$\log \tan \frac{1}{2} A = 9.58535$$

$$\frac{1}{2} A = 21^\circ 3' 6.3''.$$

$$A = 42^\circ 6' 13''.$$

$$\begin{aligned}\operatorname{colog} s &= 8.40894 \\ \log(s - a) &= 1.25527 \\ \operatorname{colog}(s - b) &= 8.88606 \\ \log(s - c) &= 0.90309\end{aligned}$$

$$2 \overline{)19.45336}$$

$$\log \tan \frac{1}{2} B = 9.72668$$

$$\frac{1}{2} B = 28^\circ 3' 18''.$$

$$B = 56^\circ 6' 36''.$$

$$A + B = 98^\circ 12' 49'', C = 81^\circ 47' 11''.$$

5. 19, 34, 49.

$$2s = 102, s = 51.$$

$$s - a = 32, s - b = 17, s - c = 2.$$

colog $s =$ 8.29243	colog $s =$ 8.29243
colog $(s - a) =$ 8.49485	log $(s - a) =$ 1.50515
log $(s - b) =$ 1.23045	colog $(s - b) =$ 8.76955
log $(s - c) =$ 0.30103	log $(s - c) =$ 0.30103
$2 \overline{)18.31876}$	$2 \overline{)18.86816}$
log tan $\frac{1}{2}A =$ 9.15938	log tan $\frac{1}{2}B =$ 9.43408
$\frac{1}{2}A = 8^\circ 12' 48''.$	$\frac{1}{2}B = 15^\circ 12'.$
$A = 16^\circ 25' 36''.$	$B = 30^\circ 24'.$

$$A + B = 46^\circ 49' 36'', C = 133^\circ 10' 24''.$$

6. 43, 50, 57.

$$2s = 150, s = 75$$

$$s - a = 32, s - b = 25, s - c = 18.$$

colog $s =$ 8.12494	colog $s =$ 8.12494
colog $(s - a) =$ 8.49485	log $(s - a) =$ 1.50515
log $(s - b) =$ 1.39794	colog $(s - b) =$ 8.60206
log $(s - c) =$ 1.25527	log $(s - c) =$ 1.25527
$2 \overline{)19.27300}$	$2 \overline{)19.48742}$
log tan $\frac{1}{2}A =$ 9.63650	log tan $\frac{1}{2}B =$ 9.74371
$\frac{1}{2}A = 23^\circ 24' 47.6''.$	$\frac{1}{2}B = 28^\circ 59' 52''.$
$A = 46^\circ 49' 35''.$	$B = 57^\circ 59' 44''.$

$$A + B = 104^\circ 49' 19'', C = 75^\circ 10' 41''.$$

7. 37, 58, 79.

$$2s = 174, s = 87.$$

$$s - a = 50, s - b = 29, s - c = 8.$$

colog $s =$ 8.06048	colog $s =$ 8.06048
colog $(s - a) =$ 8.30103	log $(s - a) =$ 1.69897
log $(s - b) =$ 1.46240	colog $(s - b) =$ 8.53760
log $(s - c) =$ 0.90309	log $(s - c) =$ 0.90309
$2 \overline{)18.72700}$	$2 \overline{)19.20014}$
log tan $\frac{1}{2}A =$ 9.36350	log tan $\frac{1}{2}B =$ 9.60007
$\frac{1}{2}A = 13^\circ 14.5''.$	$\frac{1}{2}B = 21^\circ 42' 40''.$
$A = 26^\circ 29''.$	$B = 43^\circ 25' 20''.$

$$A + B = 69^\circ 25' 49'', C = 110^\circ 34' 11''.$$

8. 73, 82, 91.

$$2s = 246, s = 123.$$

$$s - a = 50, s - b = 41, s - c = 32.$$

$$\begin{aligned}\text{colog } s &= 7.91009 \\ \text{colog } (s - a) &= 8.30103 \\ \log (s - b) &= 1.61278 \\ \log (s - c) &= 1.50515\end{aligned}$$

$$2 \overline{)19.32905}$$

$$\log \tan \frac{1}{2} A = 9.66453$$

$$\frac{1}{2} A = 24^\circ 47' 29''.$$

$$A = 49^\circ 34' 58''.$$

$$\begin{aligned}\text{colog } s &= 7.91009 \\ \log (s - a) &= 1.69897 \\ \text{colog } (s - b) &= 8.38722 \\ \log (s - c) &= 1.50515\end{aligned}$$

$$2 \overline{)19.50143}$$

$$\log \tan \frac{1}{2} B = 9.75072$$

$$\frac{1}{2} B = 29^\circ 23' 29''.$$

$$B = 58^\circ 46' 58''.$$

$$A + B = 108^\circ 21' 56'', C = 71^\circ 38' 4''.$$

9. $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$.

$$a = \sqrt{5} = 2.2361$$

$$b = \sqrt{6} = 2.4495$$

$$c = \sqrt{7} = 2.6458$$

$$2s = 7.3314$$

$$s = 3.6657.$$

$$s - a = 1.4296.$$

$$s - b = 1.2162.$$

$$s - c = 1.0199.$$

$$\begin{aligned}\text{colog } s &= 9.43585 - 10 \\ \text{colog } (s - a) &= 9.84478 - 10 \\ \log (s - b) &= 0.08500 \\ \log (s - c) &= 0.00856\end{aligned}$$

$$2 \overline{)19.37419 - 20}$$

$$\log \tan \frac{1}{2} A = 9.68709$$

$$\frac{1}{2} A = 25^\circ 56' 36''.$$

$$A = 51^\circ 53' 12''.$$

$$\begin{aligned}\text{colog } (s - b) &= 9.91500 - 10 \\ \log (s - c) &= 0.00856 \\ \text{colog } s &= 9.43585 - 10 \\ \log (s - a) &= 0.15522\end{aligned}$$

$$2 \overline{)19.51463 - 20}$$

$$\log \tan \frac{1}{2} B = 9.75732$$

$$\frac{1}{2} B = 29^\circ 45' 54''.$$

$$B = 59^\circ 31' 48''.$$

$$C = 68^\circ 35'.$$

10. 21, 28, 35.

$$2s = 84, s = 42.$$

$$s - a = 21, s - b = 14, s - c = 7$$

$$\begin{aligned}\text{colog } s &= 8.37675 \\ \text{colog } (s - a) &= 8.67778 \\ \log (s - b) &= 1.14613 \\ \log (s - c) &= 0.84510\end{aligned}$$

$$2 \overline{)19.04576}$$

$$\log \tan \frac{1}{2} A = 9.52288$$

$$\frac{1}{2} A = 18^\circ 26' 6''.$$

$$A = 36^\circ 52' 12''.$$

$$\begin{aligned}\text{colog } s &= 8.37675 \\ \log (s - a) &= 1.32222 \\ \text{colog } (s - b) &= 8.85378 \\ \log (s - c) &= 0.84510\end{aligned}$$

$$2 \overline{)19.39794}$$

$$\log \tan \frac{1}{2} B = 9.69897$$

$$\frac{1}{2} B = 26^\circ 33' 54''.$$

$$B = 53^\circ 7' 48''.$$

$$A + B = 90^\circ.$$

$$\therefore C = 90^\circ.$$

11. 6, 8, 10.

$$2s = 24, s = 12.$$

$$s - a = 6.$$

$$s - b = 4.$$

$$s - c = 2.$$

$$\text{colog } s = 8.92082 - 10$$

$$\text{colog } (s - a) = 9.22185 - 10$$

$$\log (s - b) = 0.60206$$

$$\log (s - c) = 0.30103$$

$$2 \overline{)19.04576 - 20}$$

$$\log \tan \frac{1}{2} A = 9.52288$$

$$\frac{1}{2} A = 18^\circ 26' 6''.$$

$$A = 36^\circ 52' 12''.$$

Since $10^2 = 6^2 + 8^2$ the triangle is a right triangle, and

$$C = 90^\circ.$$

$$\therefore B = 53^\circ 7' 48''.$$

12. 6, 6, 10.

$$2s = 22, s = 11.$$

$$s - a = 5.$$

$$s - b = 5.$$

$$s - c = 1.$$

$$\text{colog } s = 8.95861 - 10$$

$$\log (s - a) = 0.69897$$

$$\log (s - b) = 0.69897$$

$$\text{colog } (s - c) = 0.00000$$

$$2 \overline{)20.35655 - 20}$$

$$\log \tan \frac{1}{2} C = 10.17827$$

$$\frac{1}{2} C = 56^\circ 26' 33''.$$

$$C = 112^\circ 53' 6''.$$

Since this is an isosceles triangle,

$$A = B = \frac{1}{2} (180^\circ - C)$$

$$= 33^\circ 33' 27''.$$

13. 6, 6, 6.

The triangle is equilateral and is therefore also equiangular.

$$\therefore A = B = C = \frac{1}{3} \text{ of } 180^\circ = 60^\circ.$$

14. 6, 9, 12.

$$2s = 27, s = 13.5.$$

$$s - a = 7.5.$$

$$s - b = 4.5.$$

$$s - c = 1.5.$$

$$\text{colog } s = 8.86967 - 10$$

$$\text{colog } s - a = 9.12494 - 10$$

$$\log s - b = 0.65321$$

$$\log s - c = 0.17609$$

$$2 \overline{)18.82391 - 20}$$

$$\log \tan \frac{1}{2} A = 9.41196 - 10$$

$$\frac{1}{2} A = 14^\circ 28' 39''.$$

$$A = 28^\circ 57' 18''.$$

$$\text{colog } s = 8.86967 - 10$$

$$\log s - a = 0.87506$$

$$\text{colog } s - b = 9.34679 - 10$$

$$\log s - c = 0.17609$$

$$2 \overline{)19.26761 - 20}$$

$$\log \tan \frac{1}{2} B = 9.63380 - 10$$

$$\frac{1}{2} B = 23^\circ 17' 3''.$$

$$B = 46^\circ 34' 6''.$$

$$\therefore C = 104^\circ 28' 36''.$$

15. 3, 4, 5.

$$2s = 12, s = 6.$$

$$s - a = 3, s - b = 2, s - c = 1.$$

$$\text{colog } s = 9.22185$$

$$\text{colog } (s - a) = 9.52288$$

$$\log (s - b) = 0.30103$$

$$\log (s - c) = 0.00000$$

$$2 \overline{)19.04576}$$

$$\log \tan \frac{1}{2} A = 9.52288$$

$$\frac{1}{2} A = 18^\circ 26' 6''.$$

$$A = 36^\circ 52' 12''.$$

Since $3^2 + 4^2 = 5^2$ the triangle is a right triangle, and

$$C = 90^\circ.$$

$$B = 53^\circ 7' 48'',$$

16. Given $a = 14.5$, $b = 55.4$, and $c = 66.9$, find A , B , and C .

$$a = 14.5$$

$$b = 55.4$$

$$c = 66.9$$

$$2s = 136.8$$

$$s = 68.4.$$

$$s - a = 53.9.$$

$$s - b = 13.$$

$$s - c = 1.5.$$

$$\operatorname{colog} s = 8.16494$$

$$\operatorname{colog}(s - a) = 8.26841$$

$$\log(s - b) = 1.11394$$

$$\log(s - c) = 0.17609$$

$$2 \overline{) 17.72338}$$

$$\log \tan \frac{1}{2} A = 8.86169$$

$$\frac{1}{2} A = 4^\circ 9' 34.5''.$$

$$A = 8^\circ 19' 9''.$$

$$\operatorname{colog} s = 8.16494$$

$$\log(s - a) = 1.73159$$

$$\operatorname{colog}(s - b) = 8.88606$$

$$\log(s - c) = 0.17609$$

$$2 \overline{) 18.95868}$$

$$\log \tan \frac{1}{2} B = 9.47934$$

$$\frac{1}{2} B = 16^\circ 46' 48''.$$

$$B = 33^\circ 33' 36''.$$

$$A + B = 41^\circ 52' 45''.$$

$$C = 138^\circ 7' 15''.$$

17. Given $a = 2$, $b = \sqrt{6}$, and $c = \sqrt{3} - 1$, find A , B , and C .

$$a = 2.$$

$$b = \sqrt{6} = 2.44947$$

$$c = \sqrt{3} - 1 = 0.73205$$

$$2s = 5.18152$$

$$s = 2.59076.$$

$$s - a = 0.59076.$$

$$s - b = 0.14129.$$

$$s - c = 1.85871.$$

$$\log(s - a) = 9.77141 - 10$$

$$\log(s - b) = 9.15011 - 10$$

$$\log(s - c) = 0.26921$$

$$\operatorname{colog} s = 9.58657 - 10$$

$$\log r^2 = 18.77730 - 20$$

$$\log r = 9.38865 - 10.$$

$$\log \tan \frac{1}{2} A = 9.61724.$$

$$\log \tan \frac{1}{2} B = 10.23854.$$

$$\log \tan \frac{1}{2} C = 9.11944.$$

$$\frac{1}{2} A = 22^\circ 30'.$$

$$\frac{1}{2} B = 60^\circ.$$

$$\frac{1}{2} C = 7^\circ 30'.$$

$$A = 45^\circ.$$

$$B = 120^\circ.$$

$$C = 15^\circ.$$

18. Given $a = 2$, $b = \sqrt{6}$, and $c = \sqrt{3} + 1$, find A , B , and C .

$$a = 2.$$

$$b = \sqrt{6} = 2.44947$$

$$c = \sqrt{3} + 1 = 2.73205$$

$$2s = 7.18152$$

$$s = 3.59076.$$

$$s - a = 1.59076.$$

$$s - b = 1.14129.$$

$$s - c = 0.85871$$

$$\log(s - a) = 0.20160$$

$$\log(s - b) = 0.05740$$

$$\log(s - c) = 9.93385 - 10$$

$$\operatorname{colog} s = 9.44481 - 10$$

$$\log r^2 = 19.63766 - 20$$

$$\log r = 9.81883 - 10.$$

$$\log \tan \frac{1}{2} A = 9.61723.$$

$$\log \tan \frac{1}{2} B = 9.76143.$$

$$\log \tan \frac{1}{2} C = 9.88498.$$

$$\frac{1}{2} A = 22^\circ 30'.$$

$$\frac{1}{2} B = 30^\circ.$$

$$\frac{1}{2} C = 37^\circ 30'.$$

$$A = 45^\circ.$$

$$B = 60^\circ.$$

$$C = 75^\circ.$$

19. The sides of a triangle are 78.9, 65.4, and 97.3 respectively. Find the largest angle.

$$a = 78.9$$

$$b = 65.4$$

$$c = 97.3$$

$$2s = 241.6$$

$$s = 120.8.$$

$$s - a = 41.9.$$

$$s - b = 55.4.$$

$$s - c = 23.5.$$

$$\text{colog } s = 7.91793 - 10$$

$$\log(s - a) = 1.62221$$

$$\log(s - b) = 1.74351$$

$$\text{colog}(s - c) = 8.62893 - 10$$

$$2 \overline{)19.91258 - 20}$$

$$\log \tan \frac{1}{2} C = 9.95629 - 10$$

$$\frac{1}{2} C = 42^\circ 7' 17''.$$

$$C = 84^\circ 14' 34''.$$

20. The sides of a triangle are 487.25, 512.33, and 544.37 respectively. Find the smallest angle.

$$a = 487.25$$

$$b = 512.33$$

$$c = 544.37$$

$$2s = 1543.95$$

$$s = 771.975.$$

$$s - a = 284.725.$$

$$s - b = 259.645.$$

$$s - c = 227.605.$$

$$\text{colog } s = 7.11239 - 10$$

$$\text{colog}(s - a) = 7.54557 - 10$$

$$\log(s - b) = 2.41438$$

$$\log(s - c) = 2.35718$$

$$2 \overline{)19.42952 - 20}$$

$$\log \tan \frac{1}{2} A = 9.71476 - 10$$

$$\frac{1}{2} A = 27^\circ 24' 27''.$$

$$A = 54^\circ 48' 54''.$$

21. Find the angles of a triangle whose sides are $\frac{\sqrt{3}+1}{2\sqrt{2}}$, $\frac{\sqrt{3}-1}{2\sqrt{2}}$, and $\frac{\sqrt{3}}{2}$ respectively.

$$\frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$= \frac{2.44947 + 1.41421}{4}$$

$$= \frac{3.86368}{4} = 0.96592.$$

$$\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$= \frac{2.44947 - 1.41421}{4}$$

$$= \frac{1.03526}{4} = 0.25882.$$

$$\frac{\sqrt{3}}{2} = \frac{1.73205}{2} = 0.86603.$$

$$a = 0.96592$$

$$b = 0.25882$$

$$c = 0.86603$$

$$2s = 2.09077$$

$$s = 1.04538.$$

$$s - a = 0.07946.$$

$$s - b = 0.78656.$$

$$s - c = 0.17935.$$

$$\log(s - a) = 8.90015 - 10$$

$$\log(s - b) = 9.89573 - 10$$

$$\log(s - c) = 9.25370 - 10$$

$$\text{colog } s = 9.98072 - 10$$

$$\log r^2 = 18.03030 - 20$$

$$\log r = 9.01515 - 10.$$

$$\log \tan \frac{1}{2} A = 10.11500 - 10.$$

$$\log \tan \frac{1}{2} B = 9.11940 - 10.$$

$$\log \tan \frac{1}{2} C = 9.76145 - 10.$$

$$\frac{1}{2} A = 52^\circ 30'.$$

$$\frac{1}{2} B = 7^\circ 30'.$$

$$\frac{1}{2} C = 30^\circ.$$

$$A = 105^\circ.$$

$$B = 15^\circ.$$

$$C = 60^\circ.$$

22. Of three towns, A , B , and C , A is found to be 200 mi. from B and 184 mi. from C , and B is found to be 150 mi. due north from C . How many miles is A north of C ?

$$\begin{aligned} a &= 150 \\ b &= 184 \\ c &= 200 \\ 2s &= 534 \\ s &= 267. \end{aligned}$$

$$\begin{aligned} s - a &= 117. \\ s - b &= 83. \\ s - c &= 67. \end{aligned}$$

$$\begin{aligned} \operatorname{colog} s &= 7.57349 - 10 \\ \log(s - a) &= 2.06819 \\ \log(s - b) &= 1.91908 \\ \operatorname{colog}(s - c) &= 8.17393 - 10 \\ 2) 19.73469 - 20 \\ \log \tan \frac{1}{2} C &= 9.86735 \\ \frac{1}{2} C &= 36^\circ 22' 58''. \\ C &= 72^\circ 45' 56''. \end{aligned}$$

Draw a $\perp$ from A to BC . Find a' (part cut off by $\perp$ on BC from A).

$$\begin{aligned} a' &= b \cos C. \\ \log b &= 2.26482 \\ \log \cos C &= 9.47171 \\ \log a' &= 1.73653 \\ a' &= 54.516. \end{aligned}$$

Therefore A is 54.516 mi. north of C .

23. Under what visual angle is an object 7 ft. long seen by an observer whose eye is 5 ft. from one end of the object and 8 ft. from the other end?

$$\begin{aligned} a &= 5 & s - a &= 5. \\ b &= 8 & s - b &= 2. \\ c &= 7 & s - c &= 3. \\ 2s &= 20 \\ s &= 10. \end{aligned}$$

$$\begin{aligned} \operatorname{colog} s &= 9.00000 - 10 \\ \log(s - a) &= 0.69897 \\ \log(s - b) &= 0.30103 \\ \operatorname{colog}(s - c) &= 9.52288 - 10 \\ 2) 19.52288 - 20 \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2} C &= 9.76144 \\ \frac{1}{2} C &= 30^\circ. \\ C &= 60^\circ. \end{aligned}$$

24. The sides of a triangle are 14.6 in., 16.7 in., and 18.8 in. respectively. Find the length of the perpendicular from the vertex of the largest angle upon the opposite side.

$$\begin{aligned} a &= 18.8 \\ b &= 16.7 \\ c &= 14.6 \\ 2s &= 50.1 \\ s &= 25.05. \end{aligned}$$

$$\begin{aligned} s - a &= 6.25. \\ s - b &= 8.35. \\ s - c &= 10.45. \end{aligned}$$

$$\begin{aligned} \operatorname{colog} s &= 8.60119 - 10 \\ \log(s - a) &= 0.79588 \\ \operatorname{colog}(s - b) &= 9.07831 - 10 \\ \log(s - c) &= 1.01912 \\ 2) 19.49450 - 20 \\ \log \tan \frac{1}{2} B &= 9.74725 - 10 \\ \frac{1}{2} B &= 29^\circ 11' 46''. \\ B &= 58^\circ 23' 32''. \end{aligned}$$

Let AD be the perpendicular from A upon BC .

$$\frac{AD}{14.6} = \sin B.$$

$$\therefore AD = 14.6 \sin B.$$

$$\begin{aligned} \log 14.6 &= 1.16435 \\ \log \sin B &= 9.93026 \\ \log AD &= 1.09461 \\ AD &= 12.434. \end{aligned}$$

Therefore the required perpendicular is 12.434 in.

25. The distances between three cities, A , B , and C , are measured and found to be as follows: $AB = 165$ mi., $AC = 72$ mi., and $BC = 185$ mi. B is due east from A . In what direction is C from A ? What two answers are admissible?

$$a = 185$$

$$b = 72$$

$$c = 165$$

$$2s = 422$$

$$s = 211$$

$$s - a = 26.$$

$$s - b = 139.$$

$$s - c = 46.$$

$$\operatorname{colog} s = 7.67572 - 10$$

$$\operatorname{colog}(s - a) = 8.58503 - 10$$

$$\log(s - b) = 2.14301$$

$$\log(s - c) = 1.66276$$

$$2 \overline{)20.06652 - 20}$$

$$\log \tan \frac{1}{2} A = 10.03326$$

$$\frac{1}{2} A = 47^\circ 11' 31''.$$

$$A = 94^\circ 23' 2''.$$

Angle $BAC = 94^\circ 23' 2''$. Subtract 90° of the quadrant E to N , and we obtain $4^\circ 23' 2''$ W. of N .

But C may be to the southward of A . Hence two answers are admissible: W. of N . or W. of S .

26. In a quadrilateral $ABCD$, $AB = 2$ in., $BC = 3$ in., $CD = 3$ in., $DA = 4$ in., and $AC = 4$ in. Find the angles of the quadrilateral.

$$\text{In } \triangle ABC \ a = 3, b = 4, c = 2.$$

$$2s = 9, s = 4.5.$$

$$s - a = 1.5, s - b = 0.5, s - c = 2.5.$$

$$\operatorname{colog} s = 9.34679 - 10$$

$$\operatorname{colog}(s - a) = 9.82391 - 10$$

$$\log(s - b) = 9.69897 - 10$$

$$\log(s - c) = 0.39794$$

$$29.26761 - 30$$

$$2 \overline{)19.26761 - 20}$$

$$\log \tan \frac{1}{2} BAC = 9.63381 - 10$$

$$\frac{1}{2} BAC = 23^\circ 17' 3.4''.$$

$$BAC = 46^\circ 34' 7''.$$

$$\operatorname{colog} s = 9.34679 - 10$$

$$\log(s - a) = 0.17609$$

$$\operatorname{colog}(s - b) = 10.30103 - 10$$

$$\log(s - c) = 0.39794$$

$$2 \overline{)20.22185 - 20}$$

$$\log \tan \frac{1}{2} B = 10.11093 - 10$$

$$\frac{1}{2} B = 52^\circ 14' 20.7''.$$

$$B = 104^\circ 28' 41''.$$

$$ACB = 180^\circ - (BAC + B) \\ = 28^\circ 57' 12''.$$

$$\text{In the } \triangle ACD \ a = 3, b = 4, c = 4.$$

$$\therefore \triangle ACD \text{ is isosceles.}$$

$$\operatorname{colog} s = 9.25964 - 10$$

$$\operatorname{colog}(s - a) = 9.60206 - 10$$

$$\log(s - b) = 0.17609$$

$$\log(s - c) = 0.17609$$

$$2 \overline{)19.21388 - 20}$$

$$\log \tan \frac{1}{2} CAD = 9.60674 - 10$$

$$\frac{1}{2} CAD = 22^\circ 1' 28''.$$

$$CAD = 44^\circ 2' 56''.$$

$$ADC = ACD$$

$$= \frac{1}{2} (180^\circ - CAD).$$

$$\therefore D = 67^\circ 58' 32''.$$

$$A = CAD + BAC = 90^\circ 37' 3''.$$

$$B = 104^\circ 28' 41''$$

$$C = ACB + ACD = 96^\circ 55' 44''.$$

$$D = 67^\circ 58' 32''.$$

27. In a parallelogram $ABCD$, $AB = 2$ in., $AC = 3$ in., and $AD = 2.5$ in. Find $\angle CBA$.

In $\triangle ABC$ $a = 2.5$, $b = 3$, $c = 2$.

$$2s = 7.5, s = 3.75.$$

$$s - a = 1.25, s - b = 0.75, s - c = 1.75.$$

$$\text{colog } s = 9.42597 - 10$$

$$\log(s - a) = 0.09691$$

$$\text{colog}(s - b) = 10.12494 - 10$$

$$\log(s - c) = 0.24304$$

$$\log \tan \frac{1}{2} CBA = \frac{2 \overline{)19.89086 - 20}}{9.94543 - 10}$$

$$\frac{1}{2} CBA = 41^\circ 24' 35''.$$

$$CBA = 82^\circ 49' 10''.$$

28. In a rectangle $ABCD$, $AB = 3.3$ in., and $AC = 5\frac{1}{2}$ in. Find the angles that each diagonal makes with the sides.

$$\frac{AB}{AC} = \sin \angle ACB = \frac{3.3}{5.5}.$$

$$\log AB = 0.51851$$

$$\text{colog } AC = 9.25964 - 10$$

$$\log \sin \angle ACB = 9.77815 - 10$$

$$\angle ACB = 36^\circ 52' 11'',$$

$$\text{and } \angle CAD = 36^\circ 52' 11''.$$

$$\angle BAC = 90^\circ - \angle ACB = 53^\circ 7' 49'',$$

$$\text{and } \angle ACD = 53^\circ 7' 49''.$$

$\therefore$ diagonal AC makes angles of $36^\circ 52' 11''$ with AD and BC and $53^\circ 7' 49''$ with AB and CD .

As $\triangle ABC$ and ABD are similar,

$$\sin \angle ADB = \frac{3.3}{5.5}.$$

$$\angle ADB = 36^\circ 52' 11'',$$

$$\text{and } \angle ABD = 53^\circ 7' 49''.$$

$\therefore$ diagonal DB makes angles of $53^\circ 7' 49''$ with AB and CD and $36^\circ 52' 11''$ with AD and BC .

Exercise 56. Area of a Triangle (Page 128)

Find the areas of the triangles in which it is given that:

1. $a = 27$, $c = 32$, $B = 40^\circ$

$$S = \frac{1}{2} ac \sin B,$$

$$\log a = 1.43136$$

$$\log c = 1.50515$$

$$\log \sin B = 9.80807$$

$$\text{colog } 2 = 9.69897$$

$$\log S = 2.44355$$

$$S = 277.68.$$

3. $a = 4.8$, $c = 5.3$, $B = 39^\circ 27'$

$$\log a = 0.68124$$

$$\log c = 0.72428$$

$$\log \sin B = 9.80305$$

$$\text{colog } 2 = 9.69897$$

$$\log S = 0.90754$$

$$S = 8.0824.$$

2. $a = 35$, $c = 43$, $B = 37^\circ$.

$$\log a = 1.54407$$

$$\log c = 1.63347$$

$$\log \sin B = 9.77946$$

$$\text{colog } 2 = 9.69897$$

$$\log S = 2.65597$$

$$S = 452.87.$$

4. $a = 9.8$, $c = 7.6$, $B = 48.5^\circ$.

$$\log a = 0.99123$$

$$\log c = 0.88081$$

$$\log \sin B = 9.87446$$

$$\text{colog } 2 = 9.69897$$

$$\log S = 1.44547$$

$$S = 27.891.$$

5. $a = 17.3$, $b = 19.4$, $C = 56.25^\circ$. 8. $b = 423.9$, $c = 417.8$, $A = 68^\circ 27'$.

$$\log a = 1.23805$$

$$\log b = 2.62726$$

$$\log b = 1.28780$$

$$\log c = 2.62097$$

$$\log \sin C = 9.91985$$

$$\log \sin A = 9.96853$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\log S = \underline{2.14467}$$

$$\log S = \underline{4.91573}$$

$$S = 139.53.$$

$$S = 82,361.5.$$

6. $a = 48.35$, $b = 64.32$, $C = 62^\circ 37'$. 9. $b = 32.78$, $c = 29.62$,

$$\log a = 1.68440$$

$$A = 57^\circ 32' 20''.$$

$$\log b = 1.80835$$

$$\log b = 1.51561$$

$$\log \sin C = 9.94839$$

$$\log c = 1.47159$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\log \sin A = 9.92622$$

$$\log S = 3.14011$$

$$\text{colog } 2 = \underline{9.69897}$$

$$S = 1380.73.$$

$$\log S = 2.61239$$

$$S = 409.63.$$

7. $b = 127.8$, $c = 168.5$, $A = 72^\circ 43'$. 10. $b = 1487$, $c = 1634$,

$$\log b = 2.10653$$

$$A = 61^\circ 30' 30''.$$

$$\log c = 2.22660$$

$$\log b = 3.17231$$

$$\log \sin A = 9.97993$$

$$\log c = 3.21325$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\log \sin A = 9.94394$$

$$\log S = \underline{4.01203}$$

$$\text{colog } 2 = \underline{9.69897}$$

$$S = 10,280.9.$$

$$\log S = 6.02847$$

$$S = 1,067,750.$$

11. Prove that the area of a parallelogram is equal to the product of the base, the diagonal, and the sine of the angle included by them.

Let b = the base, d = the diagonal, and CAB the included angle.

The area of triangle $ABC = \frac{1}{2} bd \sin CAB$.

The area of parallelogram $= 2 \triangle ABC = 2 \cdot \frac{1}{2} bd \sin CAB = bd \sin CAB$.

12. Find the area of the quadrilateral $ABCD$, given $AB = 3$ in., $AC = 4.2$ in., $AD = 3.8$ in., $\angle BAD = 88^\circ 10'$, $\angle BAC = 36^\circ 20'$.

$$\text{Area of } \triangle ABC = \frac{1}{2} AB \cdot AC \cdot \sin BAC.$$

$$\text{Area of } \triangle ACD = \frac{1}{2} AD \cdot AC \cdot \sin CAD.$$

$$\angle CAD = \angle BAD - \angle BAC = 88^\circ 10' - 36^\circ 20' = 51^\circ 50'.$$

$$\log AB = 0.47712$$

$$\log AD = 0.57978$$

$$\log AC = 0.62325$$

$$\log AC = 0.62325$$

$$\log \sin BAC = 9.77268$$

$$\log \sin CAD = 9.89554$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\log ABC = 0.57202$$

$$\log ACD = 0.79754$$

$$ABC = 3.7327.$$

$$ACD = 6.274.$$

Area of quadrilateral $= (3.7327 + 6.274)$ sq. in. $= 10.0067$ sq. in.

13. In a quadrilateral $ABCD$, $BC = 5.1$ in., $AC = 4.8$ in., $CD = 3.7$ in., $\angle ACB = 123^\circ 42'$, and $\angle DCA = 117^\circ 26'$. Draw the figure approximately and find the area.

$$\text{Area of } \triangle ABC = \frac{1}{2} AC \cdot BC \cdot \sin ACB.$$

$$\text{Area of } \triangle ACD = \frac{1}{2} AC \cdot CD \cdot \sin ACD.$$

$$\log AC = 0.68124$$

$$\log BC = 0.70757$$

$$\log \sin ACB = 9.92010$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\log ABC = 1.00788$$

$$ABC = 10.183.$$

$$\log AC = 0.68124$$

$$\log CD = 0.56820$$

$$\log \sin ACD = 9.94819$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\log ACD = 0.89660$$

$$ACD = 7.881.$$

$$\text{Area of quadrilateral} = (10.183 + 7.881) \text{ sq. in.} = 18.064 \text{ sq. in.}$$

14. In the pentagon $ABCDE$, $AB = 3.1$ in., $AC = 4.2$ in., $AD = 3.7$ in., $AE = 2.9$ in., $\angle A = 132^\circ 18'$, $\angle BAC = 38^\circ 16'$, and $\angle DAE = 53^\circ 9'$. Find the area of the pentagon.

$$\angle CAD = \angle A - (\angle BAC + \angle DAE) = 132^\circ 18' - 91^\circ 25' = 40^\circ 53'.$$

$$\text{Area of } \triangle ABC = \frac{1}{2} AB \cdot AC \cdot \sin BAC.$$

$$\text{Area of } \triangle CAD = \frac{1}{2} AC \cdot AD \cdot \sin DAC.$$

$$\log AB = 0.49136$$

$$\log AC = 0.62325$$

$$\log AC = 0.62325$$

$$\log AD = 0.56820$$

$$\log \sin BAC = 9.79192$$

$$\log \sin DAC = 9.81592$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\log ABC = 0.60550$$

$$\log CAD = 0.70634$$

$$ABC = 4.0318.$$

$$CAD = 5.0856.$$

$$\text{Area of } \triangle ADE = \frac{1}{2} AD \cdot AE \cdot \sin DAE.$$

$$\log AD = 0.56820$$

$$\log AE = 0.46240$$

$$\log \sin DAE = 9.90320$$

$$\text{colog } 2 = \underline{9.69897}$$

$$\log ADE = 0.63277$$

$$ADE = 4.2931.$$

$$\text{Area of pentagon} = 4.032 + 5.086 + 4.293 = 13.411 \text{ sq. in.}$$

Exercise 57. Area of a Triangle (Page 129)

Find the areas of the triangles in which it is given that :

1. $a = 17$, $B = 48^\circ$, $C = 52^\circ$.

$$S = \frac{a^2 \sin B \sin C}{2 \sin(B + C)}.$$

$$\log a^2 = 2.46090$$

$$\log \sin B = 9.87107$$

$$\log \sin C = 9.89653$$

$$\text{colog } 2 = 9.69897$$

$$\text{colog } \sin(B + C) = \underline{0.00665}$$

$$\log S = \underline{1.93412}$$

$$S = 85.926.$$

2. $a = 182$, $B = 63.5^\circ$, $C = 78.4^\circ$.

$$\log a^2 = 4.52014$$

$$\log \sin B = 9.95179$$

$$\log \sin C = 9.99104$$

$$\text{colog } 2 = 9.69897$$

$$\text{colog } \sin(B + C) = \underline{0.20969}$$

$$\log S = \underline{4.37163}$$

$$S = 23,531.$$

3. $a = 298$, $B = 78.8^\circ$, $C = 95.5^\circ$.

$$\log a^2 = 4.94844$$

$$\log \sin B = 9.99165$$

$$\log \sin C = 9.99800$$

$$\text{colog } 2 = 9.69897$$

$$\text{colog } \sin(B + C) = \underline{1.00296}$$

$$\log S = \underline{5.64002}$$

$$S = 436,540.$$

4. $a = 19.8$, $B = 39^\circ 20'$, $C = 88^\circ 40'$.

$$\log a^2 = 2.59334$$

$$\log \sin B = 9.80197$$

$$\log \sin C = 9.99988$$

$$\text{colog } 2 = 9.69897$$

$$\text{colog } \sin(B + C) = \underline{0.10347}$$

$$\log S = \underline{2.19763}$$

$$S = 157.63.$$

5. $a = 2487$, $B = 87^\circ 28'$, $C = 69^\circ 32'$.

$$\log a^2 = 6.79136$$

$$\log \sin B = 9.99958$$

$$\log \sin C = 9.97168$$

$$\text{colog } 2 = 9.69897$$

$$\text{colog } \sin(B + C) = \underline{0.40812}$$

$$\log S = \underline{6.86971}$$

$$S = 7,408,200.$$

6. $b = 483.7$, $A = 84^\circ 32'$, $C = 78^\circ 49'$

$$\log b^2 = 5.36916$$

$$\log \sin A = 9.99802$$

$$\log \sin C = 9.99167$$

$$\text{colog } 2 = 9.69897$$

$$\text{colog } \sin(A + C) = \underline{0.54284}$$

$$\log S = \underline{5.60066}$$

$$S = 398,710.$$

7. $b = 527.4$, $A = 73^\circ 42'$,

$$C = 63^\circ 37'.$$

$$\log b^2 = 5.44428$$

$$\log \sin A = 9.98218$$

$$\log \sin C = 9.95223$$

$$\text{colog } 2 = 9.69897$$

$$\text{colog } \sin(A + C) = \underline{0.16880}$$

$$\log S = \underline{5.24646}$$

$$S = 176,384$$

8. $c = 296.3$, $A = 58^\circ 35'$,

$$B = 42^\circ 36'.$$

$$\log c^2 = 4.94346$$

$$\log \sin A = 9.93115$$

$$\log \sin B = 9.83051$$

$$\text{colog } 2 = 9.69897$$

$$\text{colog } \sin(A + B) = \underline{0.00833}$$

$$\log S = \underline{4.41242}$$

$$S = 25.848.$$

9. $c = 17.48$, $A = 36^\circ 27' 30''$,
 $B = 73^\circ 50'$.

$$\begin{aligned}\log c^2 &= 2.48508 \\ \log \sin A &= 9.77396 \\ \log \sin B &= 9.98248 \\ \text{colog } 2 &= 9.69897 \\ \text{colog } \sin(A + B) &= \underline{0.02782} \\ \log S &= \underline{1.96831} \\ S &= 92.963.\end{aligned}$$

10. $c = 96.37$, $A = 42^\circ 23' 35''$,
 $B = 69^\circ 52' 50''$.

$$\begin{aligned}\log c^2 &= 3.96788 \\ \log \sin A &= 9.82880 \\ \log \sin B &= 9.97265 \\ \text{colog } 2 &= 9.69897 \\ \text{colog } \sin(A + B) &= \underline{0.03368} \\ \log S &= \underline{3.50198} \\ S &= 3176.7.\end{aligned}$$

11. In a parallelogram $ABCD$ the diagonal AC makes with the sides the angles $27^\circ 10'$ and $32^\circ 43'$ respectively. AB is 2.8 in. long. What is the area of the parallelogram?

Area of $\triangle ABC$

$$\begin{aligned}&= \frac{\overline{AB}^2 \sin BAC \sin ABC}{2 \sin(BAC + ABC)} \\ \log \overline{AB}^2 &= 0.89432 \\ \log \sin BAC &= 9.65952 \\ \log \sin ABC &= 9.93702 \\ \text{colog } 2 &= 9.69897 \\ \text{colog } \sin(BAC + ABC) &= \underline{0.26722} \\ \log ABC &= \underline{0.45705} \\ ABC &= 2.8645.\end{aligned}$$

Area of parallelogram

$$= 2 ABC = 5.729 \text{ sq. in.}$$

Exercise 58. Area of a Triangle (Page 131)

Find the areas of the triangles in which it is given that:

1. $a = 3$, $b = 4$, $c = 5$.

$$\begin{aligned}2s &= 12. \\ s &= 6. \\ S &= \sqrt{s(s-a)(s-b)(s-c)} \\ \log s &= 0.77815 \\ \log(s-a) &= 0.47712 \\ \log(s-b) &= 0.30103 \\ \log(s-c) &= \underline{0.00000} \\ &2) \underline{1.55630} \\ \log S &= \underline{0.77815} \\ S &= 6.\end{aligned}$$

2. $a = 15$, $b = 20$, $c = 25$.

$$\begin{aligned}2s &= 60. \\ s &= 30. \\ \log s &= 1.47712 \\ \log(s-a) &= 1.17609 \\ \log(s-b) &= 1.00000 \\ \log(s-c) &= \underline{0.69897} \\ &2) \underline{4.35218} \\ \log S &= \underline{2.17609} \\ S &= 150.\end{aligned}$$

3. $a = 10$, $b = 10$, $c = 10$.

$$\begin{aligned}2s &= 30. \\ s &= 15. \\ \log s &= 1.17609 \\ \log(s-a) &= 0.69897 \\ \log(s-b) &= 0.69897 \\ \log(s-c) &= \underline{0.69897} \\ &2) \underline{3.27300} \\ \log S &= \underline{1.63650} \\ S &= 43.301.\end{aligned}$$

4. $a = 1.8$, $b = 3.7$, $c = 2.1$.

$$\begin{aligned}2s &= 7.6. \\ s &= 3.8. \\ \log s &= 0.57978 \\ \log(s-a) &= 0.30103 \\ \log(s-b) &= 9.00000 - 10 \\ \log(s-c) &= \underline{0.23045} \\ &2) \underline{0.11126} \\ \log S &= \underline{0.05563} \\ S &= 1.1367.\end{aligned}$$

5. $a = 5.3, b = 4.8, c = 4.6.$

$$2s = 14.7.$$

$$s = 7.35.$$

$$\log s = 0.86629$$

$$\log(s - a) = 0.31175$$

$$\log(s - b) = 0.40654$$

$$\log(s - c) = 0.43933$$

$$2 \overline{)2.02391}$$

$$\log S = 1.01196$$

$$S = 10.279.$$

6. $a = 7.1, b = 5.3, c = 6.4.$

$$2s = 18.8.$$

$$s = 9.4.$$

$$\log s = 0.97313$$

$$\log(s - a) = 0.36173$$

$$\log(s - b) = 0.61278$$

$$\log(s - c) = 0.47712$$

$$2 \overline{)2.42476}$$

$$\log S = 1.21238$$

$$S = 16.307.$$

7. There is a triangular piece of land with sides 48.5 rd., 52.3 rd., and 61.4 rd. Find the area in square rods; in acres.

$$2s = 162.2.$$

$$s = 81.1.$$

$$\log s = 1.90902$$

$$\log(s - a) = 1.51322$$

$$\log(s - b) = 1.45939$$

$$\log(s - c) = 1.29447$$

$$2 \overline{)6.17610}$$

$$\log S = 3.08805$$

$$S = 1224.8 \text{ sq. rd.} = 7.655 \text{ A.}$$

Find the areas of the triangles in which it is given that:

8. $a = 2.4, b = 3.2, c = 4, R = 2.$

10. $a = 3.9, b = 5.2, c = 6.5,$

$$R = 3.25.$$

$$S = \frac{abc}{4R}.$$

$$\log a = 0.59106$$

$$\log a = 0.38021$$

$$\log b = 0.71600$$

$$\log b = 0.50515$$

$$\log c = 0.81291$$

$$\log c = 0.60206$$

$$\colog 4R = 8.88606$$

$$\colog 4R = 9.09691$$

$$\log S = 1.00603$$

$$\log S = 0.58433$$

$$S = 10.14.$$

$$S = 3.84.$$

11. $a = 12, b = 12, c = 12,$

$$R = 6.928.$$

9. $a = 2.7, b = 3.6, c = 4.5, R = 2.25.$

$$\log a = 0.43136$$

$$\log a = 1.07918$$

$$\log b = 0.55630$$

$$\log b = 1.07918$$

$$\log c = 0.65321$$

$$\log c = 1.07918$$

$$\colog 4R = 9.04576$$

$$\colog 4R = 8.55733$$

$$\log S = 0.68663$$

$$\log S = 1.79487$$

$$S = 4.8599.$$

$$S = 62.354.$$

12. Given $a = 60$, $B = 40^\circ 35' 12''$, area = 12, find the radius of the inscribed circle.

$$\frac{1}{2} ac \sin B = 12.$$

$$c = \frac{24}{a \sin B}.$$

$$\log 24 = 1.38021$$

$$\text{colog } a = 8.22185 - 10$$

$$\text{colog } \sin B = \frac{0.18669}{}$$

$$\log c = \frac{9.78875 - 10}{}$$

$$c = 0.61483.$$

$$a - c = 59.38517.$$

$$a + c = 60.61483.$$

$$A + C = 139^\circ 24' 48''.$$

$$\frac{1}{2}(A + C) = 69^\circ 42' 24''.$$

$$\log(a - c) = 1.77368$$

$$\text{colog}(a + c) = 8.21742 - 10$$

$$\log \tan \frac{1}{2}(A + C) = \frac{10.43206}{}$$

$$\log \tan \frac{1}{2}(A - C) = \frac{10.42316}{}$$

$$\frac{1}{2}(A - C) = 69^\circ 19' 19''.$$

$$\therefore A = 139^\circ 1' 43''.$$

$$b = \frac{a \sin B}{\sin A}.$$

$$\log a = 1.77815$$

$$\log \sin B = 9.81331$$

$$\text{colog } \sin A = \frac{0.18331}{}$$

$$\log b = \frac{1.77477}{}$$

$$b = 59.534.$$

$$a = 60.$$

$$b = 59.534.$$

$$c = \frac{0.61483}{}$$

$$2s = \frac{120.14883}{}$$

$$s = 60.07442.$$

$$S = rs.$$

$$\therefore r = \frac{S}{s} = \frac{12}{60.07442}.$$

$$\log S = 1.07918$$

$$\text{colog } s = 8.22131 - 10$$

$$\log r = \frac{9.30049 - 10}{}$$

$$r = 0.19975.$$

Find the areas of the triangles in which it is given that :

13. $a = 40$, $b = 13$, $c = 37$.

$$s = 45.$$

$$s - a = 5.$$

$$s - b = 32.$$

$$s - c = 8.$$

$$\log s = 1.65321$$

$$\log(s - a) = 0.69897$$

$$\log(s - b) = 1.50515$$

$$\log(s - c) = \frac{0.90309}{}$$

$$2) \overline{4.76042}$$

$$\log S = \frac{2.38021}{}$$

$$S = 240.$$

14. $a = 408$, $b = 41$, $c = 401$.

$$s = 425.$$

$$s - a = 17.$$

$$s - b = 384.$$

$$s - c = 24.$$

$$\log s = 2.62839$$

$$\log(s - a) = 1.23045$$

$$\log(s - b) = 2.58433$$

$$\log(s - c) = \frac{1.38021}{}$$

$$2) \overline{7.82338}$$

$$\log S = \frac{3.91169}{}$$

$$S = 8160.$$

15. $a = 624$, $b = 205$, $c = 445$.

$$s = 637.$$

$$s - a = 13.$$

$$s - b = 432.$$

$$s - c = 192.$$

$$\log s = 2.80414$$

$$\log(s - a) = 1.11394$$

$$\log(s - b) = 2.63548$$

$$\log(s - c) = 2.28330$$

$$\underline{2)8.83686}$$

$$\log S = 4.41843$$

$$S = 26,208.$$

16. $b = 8$, $c = 5$, $A = 60^\circ$.

$$S = \frac{1}{2}bc \sin A$$

$$= \frac{1}{2} \times 8 \times 5 \times 0.86603$$

$$= 20 \times 0.86603$$

$$= 17.3206.$$

17. $a = 7$, $c = 3$, $A = 60^\circ$.

$$\log c = 0.47712$$

$$\log \sin A = 9.93753$$

$$\text{colog } a = 9.15490 - 10$$

$$\log \sin C = 9.56955$$

$$C = 21^\circ 47' 12''.$$

$$\therefore B = 98^\circ 12' 48''.$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 0.84510$$

$$\log c = 0.47712$$

$$\log \sin B = 9.99552$$

$$\log S = 1.01671$$

$$S = 10.392.$$

18. $b = 21.66$, $c = 36.94$,

$$A = 66^\circ 4' 19''.$$

$$S = \frac{1}{2}bc \sin A.$$

$$\log b = 1.33566$$

$$\log c = 1.56750$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log \sin A = 9.96097$$

$$\log S = 2.56310$$

$$S = 365.68.$$

19. $a = 215.9$, $c = 307.7$,

$$A = 25^\circ 9' 31''.$$

$$a < c \text{ and } > c \sin A.$$

$$A < 90^\circ. \therefore \text{two solutions.}$$

$$\log c = 2.48813$$

$$\log \sin A = 9.62852$$

$$\text{colog } a = 7.66575 - 10$$

$$\log \sin C = 9.78240$$

$$C = 37^\circ 17' 38'',$$

$$\therefore B = 117^\circ 32' 51'';$$

or $C' = 142^\circ 42' 22'',$

$$\therefore B' = 12^\circ 8' 7''.$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 2.33425$$

$$\log c = 2.48813$$

$$\log \sin B = 9.94774$$

$$\log S = 4.46909$$

$$S = 29,450.$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 2.33425$$

$$\log c = 2.48813$$

$$\log \sin B' = 9.32268$$

$$\log S' = 3.84403$$

$$S' = 6982.8.$$

20. $b = 149$, $A = 70^\circ 42' 30''$,

$$B = 39^\circ 18' 28''.$$

$$A = 70^\circ 42' 30''.$$

$$B = 39^\circ 18' 28''.$$

$$\therefore C = 69^\circ 59' 2''.$$

$$\log b = 2.17319$$

$$\log \sin A = 9.97490$$

$$\text{colog } \sin B = 0.19827$$

$$\log a = 2.34636$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log a = 2.34636$$

$$\log b = 2.17319$$

$$\log \sin C = 9.97294$$

$$\log S = 4.19146$$

$$S = 15,540.$$

$$21. a = 4474.5, b = 2164.5, \\ C = 116^\circ 30' 20''.$$

$$S = \frac{1}{2} ab \sin C.$$

$$\log a = 3.65075$$

$$\log b = 3.33536$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log \sin C = \frac{9.95177}{}$$

$$\log S = 6.63685$$

$$S = 4,333,600.$$

$$22. a = 510, c = 173, B = 162^\circ 30' 28''.$$

$$\log a = 2.70757$$

$$\log c = 2.23805$$

$$\log \sin B = 9.47795$$

$$\text{colog } 2 = \frac{9.69897 - 10}{}$$

$$\log S = 4.12254$$

$$S = 13,260.$$

23. If a is the side of an equilateral triangle, show that the area is $\frac{1}{4} a^2 \sqrt{3}$.

$$S = \frac{1}{2} bc \sin A$$

$$= \frac{1}{2} a^2 \sin 60^\circ$$

$$= \frac{1}{2} a^2 \times \frac{1}{2} \sqrt{3}$$

$$= \frac{1}{4} a^2 \sqrt{3}.$$

24. Two sides of a triangle are 12.38 ch. and 6.78 ch., and the included angle is $46^\circ 24'$. Find the area.

$$\text{Area} = \frac{1}{2} \times 12.38 \times 6.78 \sin 46^\circ 24'.$$

$$\log 6.19 = 0.79169$$

$$\log 6.78 = 0.83123$$

$$\log \sin 46^\circ 24' = \frac{9.85984}{}$$

$$\log \text{area} = 1.48276$$

$$\text{Area} = 30.392.$$

$$30.392 \text{ sq. ch.} = 3 \text{ A. } 0.392 \text{ sq. ch.}$$

25. Two sides of a triangle are 18.37 ch. and 13.44 ch., and they form a right angle. Find the area.

$$\text{Area} = \frac{1}{2} \times 18.37 \times 13.44.$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log 18.37 = 1.26411$$

$$\log 13.44 = \frac{1.12840}{}$$

$$\log \text{area} = 2.39251$$

$$\text{Area} = 123.45.$$

$$123.45 \text{ sq. ch.} = 12 \text{ A. } 3.45 \text{ sq. ch.}$$

26. Two angles of a triangle are $76^\circ 54'$ and $57^\circ 33' 12''$, and the included side is 9 ch. Find the area.

$$\text{Area} = \frac{9^2 \sin 76^\circ 54' \sin 57^\circ 33' 12''}{2 \sin 134^\circ 27' 12''}.$$

$$\log 81 = 1.90849$$

$$\log \sin 76^\circ 54' = 9.98855$$

$$\log \sin 57^\circ 33' 12'' = 9.92629$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log \sin 134^\circ 27' 12'' = \frac{0.14641}{}$$

$$\log \text{area} = 1.66871$$

$$\text{Area} = 46.634.$$

$$46.634 \text{ sq. ch.} = 4 \text{ A. } 6.634 \text{ sq. ch.}$$

27. The three sides of a triangle are 49 ch., 50.25 ch., and 25.69 ch. Find the area.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$s = \frac{1}{2} (49 + 50.25 + 25.69)$$

$$= 62.47.$$

$$s - a = 13.47.$$

$$s - b = 12.22.$$

$$s - c = 36.78.$$

$$\log 62.47 = 1.79567$$

$$\log 13.47 = 1.12937$$

$$\log 12.22 = 1.08707$$

$$\log 36.78 = \frac{1.56561}{}$$

$$2 \overline{) 5.57772}$$

$$\log \text{area} = 2.78886$$

$$\text{Area} = 614.97.$$

$$614.97 \text{ sq. ch.} = 61 \text{ A. } 4.97 \text{ sq. ch.}$$

28. The three sides of a triangle are 10.64 ch., 12.28 ch., and 9 ch. Find the area.

$$s = \frac{1}{2}(10.64 + 12.28 + 9) \\ = 15.96.$$

$$s - a = 5.32.$$

$$s - b = 3.68.$$

$$s - c = 6.96.$$

$$\log 15.96 = 1.20303$$

$$\log 5.32 = 0.72591$$

$$\log 3.68 = 0.56585$$

$$\log 6.96 = 0.84261$$

$$2 \overline{) 3.33740}$$

$$\log \text{area} = 1.66870$$

$$\text{Area} = 46.633.$$

$$46.633 \text{ sq. ch.} = 4 \text{ A. } 6.633 \text{ sq. ch.}$$

29. The sides of a triangular field, of which the area is 14 A., are proportional to 3, 5, 7. Find the sides.

Let the sides, measured in chains, be $3x$, $5x$, $7x$.

$$14 \text{ A.} = 140 \text{ sq. ch.}$$

$$\text{Then } s = \frac{1}{2}(3x + 5x + 7x) \\ = 7.5x.$$

$$s - a = 4.5x.$$

$$s - b = 2.5x.$$

$$s - c = 0.5x.$$

$$140 = \sqrt{7.5x \times 4.5x \times 2.5x \times 0.5x}$$

$$= \frac{x^2}{4} \sqrt{15 \times 9 \times 5}$$

$$= \frac{15x^2}{4} \sqrt{3}.$$

$$\therefore x^2 = \frac{4 \times 140}{15\sqrt{3}} = \frac{112}{3\sqrt{3}}.$$

$$\log 112 = 2.04922$$

$$\text{colog } 3\sqrt{3} = 9.28432 - 10$$

$$2 \overline{) 1.33354}$$

$$\log x = 0.66677$$

$$x = 4.6427.$$

$$3x = 13.9281.$$

$$5x = 23.2135.$$

$$7x = 32.4989.$$

Sides are 13.93 ch., 23.21 ch., 32.50 ch.

30. Two sides of a triangle are 19.74 ch. and 17.34 ch. The first bears N. $82^\circ 30'$ W.; the second S. $24^\circ 15'$ E. Find the area.

$$\text{Included angle} = 121^\circ 45'.$$

$$\text{Area} = \frac{1}{2} \times 19.74 \times 17.34 \sin 121^\circ 45'.$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log 19.74 = 1.29535$$

$$\log 17.34 = 1.23905$$

$$\log \sin 121^\circ 45' = 9.92960$$

$$\log \text{area} = 2.16297$$

$$\text{Area} = 145.54.$$

$$145.54 \text{ sq. ch.} = 14 \text{ A. } 5.54 \text{ sq. ch.}$$

31. The base of an isosceles triangle is 20, and its area is $100 \div \sqrt{3}$; find its angles.

$$a = b.$$

$$c = 20.$$

$$S = 100 \div \sqrt{3}.$$

$$\frac{1}{2} ch = \frac{100}{\sqrt{3}}$$

$$10h = \frac{100}{\sqrt{3}}$$

$$h = \frac{10}{\sqrt{3}}.$$

$$\frac{h}{\frac{1}{2}c} = \tan A.$$

$$\log h = 0.76144$$

$$\text{colog } \frac{1}{2}c = 9.00000 - 10$$

$$\log \tan A = 9.76144$$

$$A = 30^\circ.$$

$$\therefore B = 30^\circ.$$

$$\therefore C = 120^\circ.$$

32. Two sides and the included angle of a triangle are 2416 ft., 1712 ft., and 30° ; and two sides and the included angle of another triangle are 1948 ft., 2848 ft., and 150° . Find the sum of their areas.

Let $a = 2416$, $c = 1712$, $B = 30^\circ$.

$$S = \frac{1}{2} ac \sin B.$$

$$\log a = 3.38310$$

$$\log c = 3.23350$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log \sin B = \frac{9.69897}{\text{-----}}$$

$$\log S = 6.01454$$

$$S = 1,034,000 \text{ sq. ft.}$$

Let $a' = 1948$, $c' = 2848$, $B' = 150^\circ$.

$$S' = \frac{1}{2} a' c' \sin B'.$$

$$\log a' = 3.28959$$

$$\log c' = 3.45454$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log \sin B' = \frac{9.69897}{\text{-----}}$$

$$\log S' = 6.14207$$

$$S' = 1,387,000 \text{ sq. ft.}$$

$$\therefore S + S' = 2,421,000 \text{ sq. ft.}$$

33. Two adjacent sides of a rectangle are 52.25 ch. and 38.24 ch. Find the area.

$$\text{Area} = 52.25 \times 38.24.$$

$$\log 52.25 = 1.71809$$

$$\log 38.24 = \frac{1.58252}{\text{-----}}$$

$$\log \text{area} = 3.30061$$

$$\text{Area} = 1998.$$

$$1998 \text{ sq. ch.} = 199 \text{ A. } 8 \text{ sq. ch.}$$

34. Two adjacent sides of a parallelogram are 59.8 ch. and 37.05 ch., and the included angle is $72^\circ 10'$. Find the area.

$$\text{Area} = 59.8 \times 37.05 \sin 72^\circ 10'.$$

$$\log 59.8 = 1.77670$$

$$\log 37.05 = 1.56879$$

$$\log \sin 72^\circ 10' = \frac{9.97861}{\text{-----}}$$

$$\log \text{area} = 3.32410$$

$$\text{Area} = 2109.1.$$

$$2109.1 \text{ sq. ch.} = 210 \text{ A. } 9.1 \text{ sq. ch.}$$

35. Two adjacent sides of a parallelogram are 15.36 ch. and 11.46 ch., and the included angle is $47^\circ 30'$. Find the area.

$$\text{Area} = 15.36 \times 11.46 \sin 47^\circ 30'.$$

$$\log 15.36 = 1.18639$$

$$\log 11.46 = 1.05918$$

$$\log \sin 47^\circ 30' = \frac{9.86763}{\text{-----}}$$

$$\log \text{area} = 2.11320$$

$$\text{Area} = 129.78.$$

$$129.78 \text{ sq. ch.} = 12 \text{ A. } 9.78 \text{ sq. ch.}$$

36. Show that the area of a quadrilateral is equal to one half the product of its diagonals into the sine of the included angle.

Let the lengths of the diagonals be denoted by a and b , and the included angle by C . Let the lengths of the segments made by their point of intersection at C be denoted by a_1 , a_2 , and b_1 , b_2 , respectively.

The area of the quadrilateral is equal to the sum of the areas of the four triangles.

$$\begin{aligned} \therefore S &= \frac{1}{2} a_1 b_2 \sin C + \frac{1}{2} a_2 b_2 \sin C \\ &\quad + \frac{1}{2} a_2 b_1 \sin C + \frac{1}{2} a_1 b_1 \sin C \\ &= \frac{1}{2} (a_1 b_2 + a_2 b_2 + a_2 b_1 + a_1 b_1) \sin C \\ &= \frac{1}{2} (a_1 + a_2) (b_1 + b_2) \sin C \\ &= \frac{1}{2} ab \sin C. \end{aligned}$$

37. The diagonals of a quadrilateral are 34 ft. and 56 ft., intersecting at an angle of 67° . Find the area.

$$\text{Area} = \frac{1}{2} \times 34 \times 56 \times \sin 67^\circ.$$

$$\log 17 = 1.23045$$

$$\log 56 = 1.74819$$

$$\log \sin 67^\circ = \underline{9.96403}$$

$$\log \text{area} = \underline{2.94267}$$

$$\text{Area} = 876.34 \text{ sq. ft.}$$

38. The diagonals of a quadrilateral are 75 ft. and 49 ft., intersecting at an angle of 42° . Find the area.

$$\text{colog } 2 = 9.69897 - 10$$

$$\log 75 = 1.87506$$

$$\log 49 = 1.69020$$

$$\log \sin 42^\circ = \underline{9.82551}$$

$$\log \text{area} = \underline{3.08974}$$

$$\text{Area} = 1229.5 \text{ sq. ft.}$$

39. In the quadrilateral $ABCD$ we have AB , 17.22 ch.; AD , 7.45 ch.; CD , 14.10 ch.; BC , 5.25 ch.; and the diagonal AC , 15.04 ch. Find the area.

In the triangle ABC ,

$$s = \frac{1}{2}(17.22 + 5.25 + 15.04) = 18.755.$$

$$s - a = 1.535.$$

$$s - b = 13.505.$$

$$s - c = 3.715.$$

$$\log 18.755 = 1.27312$$

$$\log 1.535 = 0.18611$$

$$\log 13.505 = 1.13049$$

$$\log 3.715 = \underline{0.56996}$$

$$2) \underline{3.15968}$$

$$\log \text{area} = \underline{1.57984}$$

$$\text{Area} = 38.005 \text{ sq. ch.}$$

In the triangle ACD ,

$$s = \frac{1}{2}(15.04 + 14.10 + 7.45) = 18.295.$$

$$s - a = 3.255.$$

$$s - b = 4.195.$$

$$s - c = 10.845.$$

$$\log 18.295 = 1.26233$$

$$\log 3.255 = 0.51255$$

$$\log 4.195 = 0.62273$$

$$\log 10.845 = 1.03523$$

$$2) \underline{3.43284}$$

$$\log \text{area} = \underline{1.71642}$$

$$\text{Area} = 52.050 \text{ sq. ch.}$$

$$\text{Area } ABC = 38.005$$

$$\text{Area } ACD = 52.050$$

$$\text{Area } ABCD = 90.055 \text{ sq. ch.}$$

$$90.055 \text{ sq. ch.} = 9 \text{ A. } 0.055 \text{ sq. ch.}$$

40. Show that the area of a regular polygon of n sides, of which one side is a , is $\frac{na^2}{4} \cot \frac{180^\circ}{n}$.

Lines joining the vertices to the center divide the polygon into n equal isosceles triangles, the bases of which are a , and the vertical angles $\frac{360^\circ}{n}$. The altitude of each triangle is

$$h = \frac{a}{2} \cot \frac{180^\circ}{n};$$

and the area of each is

$$\frac{1}{2} ah = \frac{a^2}{4} \cot \frac{180^\circ}{n}.$$

Hence, the area of the polygon is

$$\frac{na^2}{4} \cot \frac{180^\circ}{n}.$$

41. One side of a regular pentagon is 25. Find the area.

$$\begin{aligned}\text{Area} &= \frac{5 \times 25^2}{4} \cot \frac{180^\circ}{5} \\ &= 781.25 \cot 36^\circ.\end{aligned}$$

$$\log 781.25 = 2.89279$$

$$\log \cot 36^\circ = 10.13874$$

$$\log \text{area} = 3.03153$$

$$\text{Area} = 1075.3.$$

42. One side of a regular hexagon is 32. Find the area.

$$\begin{aligned}\text{Area} &= \frac{6 \times 32^2}{4} \cot \frac{180^\circ}{6} \\ &= 1536 \cot 30^\circ.\end{aligned}$$

$$\log 1536 = 3.18639$$

$$\log \cot 30^\circ = 10.23856$$

$$\log \text{area} = 3.42495$$

$$\text{Area} = 2660.4.$$

43. One side of a regular decagon is 46. Find the area.

$$\text{Area} = \frac{10 \times 46^2}{4} \cot \frac{180^\circ}{10} = 5290 \cot 18^\circ.$$

$$\log 5290 = 3.72346$$

$$\log \cot 18^\circ = 10.48822$$

$$\log \text{area} = 4.21168$$

$$\text{Area} = 16,281.$$

44. If r is the radius of a circle, show that the area of the regular circumscribed polygon of n sides is $nr^2 \tan \frac{180^\circ}{n}$, and the area of the regular inscribed polygon is $\frac{n}{2} r^2 \sin \frac{360^\circ}{n}$.

Lines drawn from the vertices to the center divide the polygon into n equal isosceles triangles, the bases of which are the sides of the polygon and the vertical angles $\frac{360^\circ}{n}$.

In the circumscribed polygon, each side $= 2r \tan \frac{180^\circ}{n}$, and the altitude of each triangle is r . Hence, the area of each triangle is $r^2 \tan \frac{180^\circ}{n}$, and the area of the polygon $nr^2 \tan \frac{180^\circ}{n}$.

In the inscribed polygon, each side $= 2r \sin \frac{180^\circ}{n}$, and the altitude of each triangle is $r \cos \frac{180^\circ}{n}$. Hence, the area of each triangle is $r^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n} = \frac{r^2}{2} \sin \frac{360^\circ}{n}$, and the area of the polygon is $\frac{nr^2}{2} \sin \frac{360^\circ}{n}$.

45. Obtain a formula for the area of a parallelogram in terms of two adjacent sides and the included angle.

By Geometry, area of parallelogram = base $\times$ height.

In this case,

$$\text{area} = bh.$$

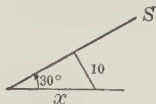
But

$$h = a \sin A.$$

$$\therefore \text{area of } \square = ab \sin A.$$

Exercise 59. Right Triangles (Page 133)

1. If the sun's altitude is 30° , find the length of the longest shadow which can be cast on a horizontal plane by a stick 10 ft. in length.



The longest shadow is cast when the stick is $\perp$ to the sun's rays.

$$\sin 30^\circ = \frac{1}{2} = \frac{10}{x}.$$

$$\therefore x = \frac{10}{\frac{1}{2}} = 20.$$

The longest shadow cast by the stick is 20 ft.

2. A flagstaff 90 ft. high, on a horizontal plane, casts a shadow of 117 ft. Find the altitude of the sun.

Given $a = 90$ ft., $b = 117$ ft.; required A .

$$\tan A = \frac{a}{b}.$$

$$\log a = 1.95424$$

$$\text{colog } b = 7.93181 - 10$$

$$\log \tan A = 9.88605$$

$$A = 37^\circ 34' 5''.$$

Altitude of sun is $37^\circ 34' 5''$.

3. If the sun's altitude is 60° , what angle must a stick make with the horizon in order that its shadow in a horizontal plane may be the longest possible?

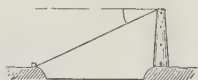
Elevation of sun $= 60^\circ$.

In order to cast the longest shadow possible, the stick must be $\perp$ to the rays of the sun.

$$90^\circ - 60^\circ = 30^\circ.$$

The stick is at an angle of 30° with the horizon.

4. A tower 93.97 ft. high is situated on the bank of a river. The angle of depression of an object on the opposite bank is $25^\circ 12'$. Find the breadth of the river.



$$a = b \tan A.$$

$$\log b = 1.97299$$

$$\log \tan A = 10.32738$$

$$\log a = 2.30037$$

$$a = 199.70$$

The breadth of the river is 199.70 ft.

5. The angle of elevation of the top of a tower is $48^\circ 19'$, and the distance of the base from the point of observation is 95 ft. Find the height of the tower and the distance of its top from the point of observation.

$$a = b \tan A.$$

$$A = 48^\circ 19'$$

$$b = 95 \text{ ft.}$$

$$\log b = 1.97772$$

$$\log \tan A = 10.05039$$

$$\log a = 2.02811$$

$$a = 106.69$$

$$c = \frac{b}{\cos A}.$$

$$\log b = 1.97772$$

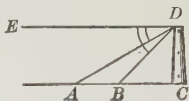
$$\text{colog } \cos A = 0.17717$$

$$\log c = 2.15489$$

$$c = 142.85$$

The height of the tower is 106.69 ft., and the distance of its top from the point of observation is 142.85 ft.

6. From a tower 58 ft. high the angles of depression of two objects situated in the same horizontal line with the base of the tower, and on the same side, are $30^\circ 13'$ and $45^\circ 46'$. Find the distance between these two objects.



Given $\angle ADE = 30^\circ 13'$,

$\angle BDE = 45^\circ 46'$;

then $\angle BDC = 90^\circ - (45^\circ 46')$
 $= 44^\circ 14'$;

and $\angle ADC = 90^\circ - (30^\circ 13')$
 $= 59^\circ 47'$.

$$BC = 58 \tan BDC.$$

$$\log 58 = 1.76343$$

$$\log \tan 44^\circ 14' = 9.98838$$

$$\log BC = 1.75181$$

$$BC = 56.468.$$

$$AC = 58 \tan ADC.$$

$$\log 58 = 1.76343$$

$$\log \tan 59^\circ 47' = 10.23478$$

$$\log AC = 1.99821$$

$$AC = 99.588.$$

$AC - BC = 43.12$ ft., distance between the two objects.

7. From one edge of a ditch 36 ft. wide the angle of elevation of the top of a wall on the opposite edge is $62^\circ 39'$. Find the length of a ladder that will just reach from the point of observation to the top of the wall.

Given $b = 36$,

$A = 62^\circ 39'$; required c

$$\frac{b}{c} = \cos A.$$

$$c = \frac{36}{\cos A}.$$

$$\log 36 = 1.55630$$

$$\log \cos A = 0.33779$$

$$\log c = 1.89409$$

$$c = 78.36.$$

Length of the ladder = 78.36 ft.

8. The top of a flagstaff has been partly broken off and touches the ground at a distance of 15 ft. from the foot of the staff. If the length of the broken part is 39 ft., find the length of the whole staff.

Given $c = 39$, $b = 15$; required $c + a$.

$$a = \sqrt{(c + b)(c - b)}$$

$$= \sqrt{(39 + 15)(39 - 15)}$$

$$= \sqrt{54 \times 24}$$

$$= \sqrt{6^2 \times 9 \times 4}$$

$$= 36.$$

$$c + a = 39 + 36 = 75.$$

Whole length of flagstaff = 75 ft.

9. From a balloon which is directly above one town the angle of depression of another town is observed to be $10^\circ 14'$. The towns being 8 mi. apart, find the height of the balloon.

Given $A = 90^\circ - (10^\circ 14')$
 $= 79^\circ 46'$,
 $a = 8$; required b .
 $b = a \cot A$.
 $\log a = 0.90309$
 $\log \cot A = 9.25655$
 $\log b = 0.15964$
 $b = 1.4442$.

The height of the balloon is 1.4442 mi.

10. A ladder 40 ft. long reaches a window 33 ft. high, on one side of a street. Being turned over upon its foot, the ladder reaches another window 21 ft. high, on the opposite side of the street. Find the width of the street.

Width of the one part of the street
 $= \sqrt{40^2 - 33^2}$
 $= \sqrt{511}$
 $= 22.605$.

Width of other part
 $= \sqrt{40^2 - 21^2}$
 $= \sqrt{1159}$
 $= 34.044$.

$$22.605 + 34.044 = 56.649.$$

Total width of the street = 56.649 ft.

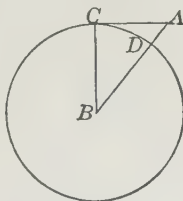
11. From a mountain 1000 ft. high the angle of depression of a ship is $27^\circ 35' 11''$. Find the distance of the ship from the summit of the mountain.

Given $a = 1000$,
 $A = 27^\circ 35' 11''$; required c .
 $\sin A = \frac{1000}{c}, c = \frac{1000}{\sin A}$.

$$\begin{aligned}\log 1000 &= 3.00000 \\ \text{colog } \sin A &= 0.33434 \\ \log c &= 3.33434 \\ c &= 2159.5.\end{aligned}$$

Distance of ship from summit = 2159.5 ft.

12. From the top of a mountain 3 mi. high the angle of depression of the most distant object which is visible on the earth's surface is found to be $2^\circ 13' 50''$. Find the diameter of the earth.



Let A be the top of the mountain, C the object observed, B the center of the earth. Then given $B = 90^\circ - A = 2^\circ 13' 50''$, $AD = 3$; required a . $BC = AB \cos B$.

$$\begin{aligned}a &= (a + 3) \cos B \\ \therefore a(1 - \cos B) &= 3 \cos B \\ a &= \frac{3 \cos B}{1 - \cos B}; \\ \text{by [34],} \quad &= \frac{3 \cos B}{2 \sin^2 \frac{B}{2}}.\end{aligned}$$

$$\begin{aligned}\log \frac{3}{2} &= 0.17609 \\ \log \cos B &= 9.99967 \\ \text{colog } \sin^2 \frac{B}{2} &= 3.42154 \\ \log a &= 3.59730 \\ a &= 3956.4. \\ 2a &= 7912.8.\end{aligned}$$

Diameter of earth = 7912.8 mi.

13. A lighthouse 54 ft. high is situated on a rock. The angle of elevation of the top of the lighthouse, as observed from a ship, is $4^{\circ} 52'$, and the angle of elevation of the top of the rock is $4^{\circ} 2'$. Find the height of the rock and its distance from the ship.

Let h = height of rock.

a = distance of ship.

$$\text{Then } \frac{h + 54}{h} = \frac{\tan 4^{\circ} 52'}{\tan 4^{\circ} 2'}.$$

$$1 + \frac{54}{h} = \frac{\tan 4^{\circ} 52'}{\tan 4^{\circ} 2'}.$$

$$\frac{54}{h} = \frac{\tan 4^{\circ} 52' - \tan 4^{\circ} 2'}{\tan 4^{\circ} 2'}.$$

$$h = 54 \frac{\tan 4^{\circ} 2'}{\tan 4^{\circ} 52' - \tan 4^{\circ} 2'}$$

$$= 54 \frac{\cos 4^{\circ} 52' \sin 4^{\circ} 2'}{\sin (4^{\circ} 52' - 4^{\circ} 2')}$$

$$= 54 \frac{\cos 4^{\circ} 52' \sin 4^{\circ} 2'}{\sin 50'}.$$

$$\log 54 = 1.73239$$

$$\log \cos 4^{\circ} 52' = 9.99843$$

$$\log \sin 4^{\circ} 2' = 8.84718$$

$$\text{colog } \sin 50' = 1.83732$$

$$\log h = 2.41532$$

$$h = 260.21.$$

Also $a = h \cot 4^{\circ} 2'.$

$$\log h = 2.41532$$

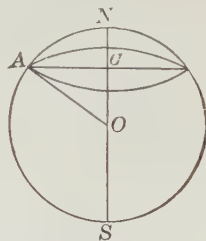
$$\log \cot 4^{\circ} 2' = 11.15174$$

$$\log a = 3.56706$$

$$a = 3690.3.$$

Height of rock = 260.21 ft.; distance of ship = 3690.3 ft.

14. The latitude of Cambridge, Massachusetts, is $42^{\circ} 22' 49''$. What is the length of the radius of that parallel of latitude?



Let O be the center of the earth, NS the axis, NAS the meridian of Cambridge, A the position of Cambridge, and C the center of its parallel of latitude. Then, in the right triangle OAC , given $O = 90^{\circ} - 42^{\circ} 22' 49'' = 47^{\circ} 37' 11''$, $OA = 3956.2$ mi.; required AC .

$$AC = AO \sin O.$$

$$\log AO = 3.59728$$

$$\log \sin O = 9.86846$$

$$\log AC = 3.46574$$

$$AC = 2922.4.$$

Radius of parallel of latitude = 2922.4 mi.

15. At what latitude is the circumference of the parallel of latitude equal to half the equator?

The radius of the parallel is half the radius of the earth.

In the figure of Example 14, given $AC = \frac{1}{2} AO$; required $90^{\circ} -$ angle O , that is, angle A .

$$\cos A = \frac{AC}{AO} = \frac{1}{2}.$$

$$\therefore A = 60^{\circ}.$$

The required latitude is 60° .

16. In a circle with a radius of 6.7 is inscribed a regular polygon of thirteen sides. Find the length of one of its sides.

Let O be the center of the circle, AB a side of the polygon, and C the middle point of AB . Then in the right triangle OCB , given

$$O = \frac{360^\circ}{26} = 13^\circ 50' 46'', \quad OB = 6.7; \quad \text{required } AB,$$

that is, $2 CB$.

$$CB = OB \sin BOC.$$

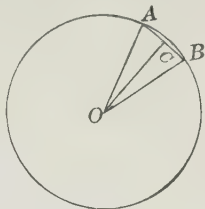
$$\log OB = 0.82607$$

$$\log \sin BOC = 9.37897$$

$$\log CB = 0.20504$$

$$CB = 1.6034.$$

$$AB = 3.2068.$$



Length of a side of the polygon = 3.2068.

17. A regular heptagon, one side of which is 5.73, is inscribed in a circle. Find the radius of the circle.

In the figure of Example 16, given $BC = \frac{1}{2} \times 5.73 = 2.865$ and angle

$$BOC = \frac{360^\circ}{14} = 25^\circ 42' 51''; \quad \text{required } OB.$$

$$OB = BC \csc BOC.$$

$$\log BC = 0.45712$$

$$\log \csc BOC = 0.36263$$

$$\log OB = 0.81975$$

$$OB = 6.6031.$$

Radius of circle = 6.6031.

18. When the moon is setting at any place, the angle at the moon subtended by the earth's radius passing through that place is $57' 3''$. If the earth's radius is 3956.2 mi., what is the moon's distance from the earth's center?

Let C represent the place, A the moon, and B the earth's center. Then in the right triangle ABC , given $A = 57' 3''$, $a = 3956.2$ mi.; required c .

$$c = a \csc A.$$

$$\log a = 3.59728$$

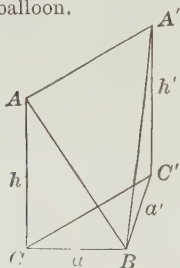
$$\log \csc A = 1.78004$$

$$\log c = 5.37732$$

$$c = 238,410.$$

Moon's distance = 238,410 mi.

19. A man in a balloon observes the angle of depression of an object on the ground, bearing south, to be $35^\circ 30'$; the balloon drifts $2\frac{1}{2}$ mi. east at the same height, when the angle of depression of the same object is $23^\circ 14'$. Find the height of the balloon.



Let A and A' be the first and second positions of the balloon, respectively, C and C' the points on the ground directly under A and A' , and B the object observed.

$$\begin{aligned}\text{Then} \quad A &= 54^\circ 30'. \\ A' &= 66^\circ 46'. \\ CC' = AA' &= 2\frac{1}{2}. \\ a &= h \tan A. \\ a' &= h \tan A'.\end{aligned}$$

$$\begin{aligned}a'^2 - a^2 &= (2\frac{1}{2})^2, \\ h^2 \tan^2 A' - h^2 \tan^2 A &= (2\frac{1}{2})^2.\end{aligned}$$

$$h^2 = \frac{(2\frac{1}{2})^2}{\tan^2 A' - \tan^2 A}.$$

$$h = \frac{2\frac{1}{2}}{\sqrt{\tan^2 A' - \tan^2 A}}.$$

$$\begin{aligned}\text{But } \tan^2 A' - \tan^2 A &= (\tan A' + \tan A)(\tan A' - \tan A) \\ &= \frac{\sin(A' + A)}{\cos A' \cos A} \times \frac{\sin(A' - A)}{\cos A' \cos A} \\ &= \frac{\sin(A' + A) \sin(A' - A)}{\cos^2 A' \cos^2 A}.\end{aligned}$$

$$\text{Hence } h = \frac{2\frac{1}{2} \cos A' \cos A}{\sqrt{\sin(A' + A) \sin(A' - A)}}.$$

$$\begin{aligned}\log 2\frac{1}{2} &= 0.39794 \\ \log \cos A' &= 9.59602 \\ \log \cos A &= 9.76395 \\ \text{colog } \sqrt{\sin(A' + A)} &= 0.03408 \\ \text{colog } \sqrt{\sin(A' - A)} &= 0.33636 \\ \log h &= 0.12835 \\ h &= 1.3438.\end{aligned}$$

Height of balloon = 1.3438 miles.

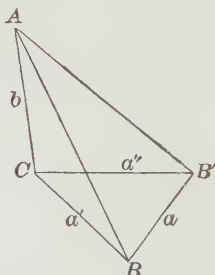
20. The angle at the earth's center subtended by the sun's radius is $16' 2''$, and the sun's distance is 92,400,000 mi. Find the sun's diameter in miles.

Let A represent the center of the earth, B that of the sun, and C a point on the edge of the sun's disk. Then in the right triangle ABC , given $A = 16' 2''$, $c = 92,400,000$ mi.; required $2a$.

$$\begin{aligned}a &= c \sin A. \\ \log c &= 7.96567 \\ \log \sin A &= 7.66874 \\ \log a &= 5.63441 \\ a &= 430,930. \\ 2a &= 861,860.\end{aligned}$$

Sun's diameter = 861,860 mi.

21. A man standing south of a tower and on the same horizontal plane observes its angle of elevation to be $54^\circ 16'$; he goes east 100 yd. and then finds its angle of elevation is $50^\circ 8'$. Find the height of the tower.



Let AC be the tower, B and B' the first and second positions of the observer.

Then $BB' = 100$.

$$a' = b \cot ABC.$$

$$a'' = b \cot AB'C.$$

$$a''^2 - a'^2 = a^2.$$

$$b^2 (\cot^2 AB'C - \cot^2 ABC) = 100^2.$$

$$b = \frac{100}{\sqrt{\cot^2 50^\circ 8' - \cot^2 54^\circ 16'}} \\ = \frac{100 \sin 54^\circ 16' \sin 50^\circ 8'}{\sqrt{\sin 104^\circ 24' \sin 4^\circ 8'}}.$$

$$\log 100 = 2.00000$$

$$\log \sin 54^\circ 16' = 9.90942$$

$$\log \sin 50^\circ 8' = 9.88510$$

$$\text{colog } \sqrt{\sin 104^\circ 24'} = 0.00693$$

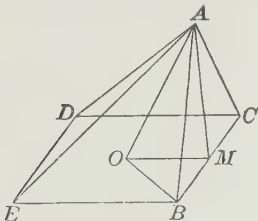
$$\text{colog } \sqrt{\sin 4^\circ 8'} = 0.57110$$

$$\log b = 2.37255$$

$$b = 235.81.$$

Height of tower = 235.81 yd.

22. A regular pyramid, with a square base, has a lateral edge 150 ft. long, and the side of the base is 200 ft. Find the inclination of the face of the pyramid to the base.



Let A be the vertex of the pyramid, $BCDE$ its base, O the center of the base, and M the middle point of the side BC . Required the angle AMO .

In the right triangle AOB ,

$$AB = 150,$$

$$OB = \frac{1}{2} BD = 100\sqrt{2}.$$

$$\therefore AO = \sqrt{AB^2 - OB^2} = 50.$$

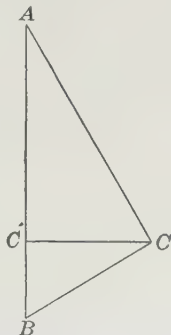
In the right triangle AOB ,

$$\tan OMA = \frac{AO}{OM} = \frac{50}{100} = 0.5.$$

$$OMA = 26^\circ 34'.$$

Inclination of face of pyramid to base = $26^\circ 34'$.

23. The height of a house subtends a right angle at a window on the other side of the street, and the angle of elevation of the top of the house from the same point is 60° . The street is 30 ft. wide. How high is the house?



Given $CC' = 30$, $ACC' = 60^\circ$, $BCC' = 30^\circ$; required AB .

$$CAB = 30^\circ.$$

$$\therefore AC = 2 \times CC' = 60.$$

$$\cos A = \frac{AC}{AB}.$$

$$\therefore AB = \frac{AC}{\cos A} = \frac{60}{\frac{1}{2}\sqrt{3}} \\ = 40\sqrt{3} = 69.282.$$

Height of house = 69.282 ft.

24. The perpendicular from the vertex of the right angle of a right triangle divides the hypotenuse into two segments 364.3 ft. and 492.8 ft. in length respectively. Find the acute angles of the triangle.

Let D be the point of division in AB the given hypotenuse.

Then CD is the mean proportional between the segments of AB .

$$\overline{DC}^2 = 492.8 \times 364.3.$$

$$\log 492.8 = 2.69267$$

$$\log 364.3 = 2.56146$$

$$\log \overline{DC}^2 = 5.25413$$

$$\log DC = 2.62706.$$

$$\tan A = \frac{DC}{492.8}.$$

$$\log DC = 2.62706$$

$$\text{colog } 492.8 = 7.30733 - 10$$

$$\log \tan A = 9.93439$$

$$A = 40^\circ 41' 18''.$$

$$B = 49^\circ 18' 42''.$$

25. The bisector of the right angle of a right triangle divides the hypotenuse into two segments 431.9 ft. and 523.8 ft. in length respectively. Find the acute angles of the triangle.

Given DC bisector of rt. $\angle C$.

$$AD = 523.8,$$

$$BD = 431.9.$$

From Plane Geometry,

$$\frac{AD}{BD} = \frac{AC}{CB}.$$

$$\tan B = \frac{AC}{CB} = \frac{AD}{BD}.$$

$$\log AD = 2.71917$$

$$\text{colog } BD = \frac{7.36462 - 10}{10.08379}$$

$$\log \tan B = 10.08379$$

$$B = 50^\circ 29' 35''.$$

$$A = 39^\circ 30' 25''.$$

The acute $\angle$ s of the $\triangle$ are

$$50^\circ 29' 35'' \text{ and } 39^\circ 30' 25''.$$

26. Find the number of degrees, minutes, and seconds in an arc of a circle, knowing that the chord which subtends it is 238.25 ft., and that the radius is 196.27 ft.

Given

$$AB = 238.25,$$

$$OB = 196.27.$$

Find arc AB in circular measure.

$OD \perp$ to AB bisects AB .

$$DB = \frac{1}{2} AB = 119.125.$$

$$\sin BOD = \frac{119.125}{196.27}.$$

$$\log 119.125 = 2.07600$$

$$\text{colog } 196.27 = 7.70715 - 10$$

$$\log \sin BOD = 9.78315$$

$$BOD = 37^\circ 22' 7''.$$

$$AOB = 74^\circ 44' 14''.$$

$$\therefore \text{arc } AB = 74^\circ 44' 14''.$$

27. Calculate to the nearest hundredth of an inch the chord which subtends an arc of $37^{\circ} 43'$ in a circle having a radius of 542.35 in.

$$\text{Given } OB = 542.35,$$

$$\angle BOC = 37^{\circ} 43'.$$

$$\text{Then } \angle BOD = 18^{\circ} 51' 30''.$$

Find BC , to the nearest hundredth of an inch.

$$\sin BOD = \frac{BD}{OB}.$$

$$BO = OB \sin BOD.$$

$$BD = 542.35 \sin BOD.$$

$$BC = 2 BD.$$

$$\log 542.35 = 2.73428$$

$$\log \sin BOD = 9.50951$$

$$\log 2 = 0.30103$$

$$\log BC = 2.54482$$

$$BC = 350.607.$$

$$\text{Length of chord} = 350.61 \text{ in.}$$

28. Calculate to the nearest hundredth of an inch the chord which subtends an arc of 14° in a circle having a radius of 475.23 in.

$$\text{Given arc } DB = 14^{\circ},$$

$$r = 475.23 \text{ in.}$$

Find chord DB .

$$\angle O = 7^{\circ}.$$

$$\sin O = \frac{BC}{r}.$$

$$\text{chord } DB = 2 BC.$$

$$BC = r \sin O.$$

$$\log r = 2.67691$$

$$\log \sin O = 9.08589$$

$$\log 2 = 0.30103$$

$$\log DB = 2.06383$$

$$DB = 115.833.$$

$$\text{Length of chord} = 115.83 \text{ in.}$$

29. In an isosceles triangle ABC the base AB is 1235 in., and $\angle A = \angle B = 64^{\circ} 22'$. Find the radius of the inscribed circle.

$$\text{Given } AB = 1235 \text{ in.,}$$

$$\angle A = \angle B = 64^{\circ} 22'.$$

Required OD .

ODB is a rt. $\triangle$.

Since OB bisects $\angle B$,

$$\angle OBD = 32^{\circ} 11'.$$

$$BD = \frac{1}{2} \text{ of } 1235 = 617.5.$$

$$\tan OBD = \frac{OD}{BD}.$$

$$OD = BD \tan OBD.$$

$$\log BD = 2.79064$$

$$\log \tan OBD = 9.79888$$

$$\log OD = 2.58952$$

$$OD = 388.62 \text{ in., radius of the inscribed circle.}$$

30. Find the number of degrees, minutes, and seconds in an arc of a circle, knowing that the chord which subtends it is two thirds of the diameter.

Given the diameter AB , and chord $AC = \frac{2}{3} AB$.

Find arc AC .

Angle C is a rt. $\angle$, being subtended by a semicircumference.

$$\therefore \sin B = \frac{2}{3}.$$

$$\sin B = 0.6666\frac{2}{3}.$$

$$B = 41^{\circ} 48' 50''.$$

Since B is measured by $\frac{1}{2}$ arc AC ,

$$\text{arc } AC = 83^{\circ} 37' 40''.$$

31. Find the number of degrees, minutes, and seconds in an arc of a circle, knowing that the chord which subtends it is three fourths of the diameter.

Given chord $AC = \frac{3}{4}AB$.

Find AC .

$$\sin B = \frac{3}{4} = 0.75.$$

$$B = 48^\circ 35' 30''.$$

Since $\angle B$ is measured by $\frac{1}{2}$ the arc AC ,

$$\begin{aligned}\text{arc } AC &= 2 \times 48^\circ 35' 30'' \\ &= 97^\circ 11' .\end{aligned}$$

32. The radius of a circle being 2548.36 in., and the length of a chord BC being 3609.02 in., find the angle BAC made by two tangents drawn at B and C respectively.

Given $R = 2548.36$ in.,

$$BC = 3609.02 \text{ in.}$$

Find $\angle BAC$, formed by tangents AB and AC .

Let BD be the diameter of the circle.

$$BD = 2 \times 2548.36 = 5096.72 \text{ in.}$$

$$\sin BDC = \frac{BC}{BD}.$$

$$\log BC = 3.55739$$

$$\text{colog } BD = 6.29271 - 10$$

$$\log \sin BDC = 9.85010$$

$$BDC = 45^\circ 4' 51''.$$

$$\angle BDC = \angle ACB = \angle ABC,$$

being measured by $\frac{1}{2}$ arc BC .

$$\begin{aligned}\therefore \angle BAC &= 180^\circ - 2(45^\circ 4' 51'') \\ &= 89^\circ 50' 18'' .\end{aligned}$$

33. Find the ratio of a chord to the diameter, knowing that the chord subtends an arc $27^\circ 48'$. If

the diameter is 8 in., how long is the chord? If the chord is 8 in., how long is the diameter?

(i) Given

$$\text{arc } AB = 27^\circ 48',$$

$$\angle C = 13^\circ 54'.$$

$$\frac{\text{chord } AB}{AC} = \sin C.$$

$$\sin C = 0.2402.$$

$\therefore$ ratio of the chord to the diameter = 0.2402.

(ii) Given diameter = 8 in.

Find chord.

$$\text{chord } AB = AC \sin C$$

$$= 8 \times 0.2402$$

$$= 1.9216 \text{ in.}$$

$$\text{chord } AB = 1.92 \text{ in.}$$

(iii) Given chord $AB = 8$ in.

Find diameter.

$$AC = \frac{\text{chord } AB}{\sin C}.$$

$$\log 8 = 0.90309$$

$$\text{colog } 0.2402 = 0.61943$$

$$\log \text{diameter} = 1.52252$$

$$\text{Diameter} = 33.306 \text{ in.}$$

34. Find the length of the diameter of a regular pentagon of which the side is 1 in., and the length of the side of a regular pentagon of which the diameter is 1 in.

Given $AB = 1$ in., the side of a regular pentagon.

$$\angle AOB = 72^\circ.$$

$$DB = .5 \text{ in.}$$

$$\angle DOB = 36^\circ.$$

$$\sin DOB = \frac{DB}{OB}.$$

$$OB = \frac{DB}{\sin DOB},$$

$$\log DB = 9.69897 - 10$$

$$\text{colog } \sin DOB = 0.23078$$

$$\log OB = 9.92975 - 10$$

$$OB = 0.85064.$$

$$CB = 2OB = 1.70128.$$

$\therefore$ diameter of regular pentagon
= 1.7 in.

Let $CB = 1$ in.

Find AB . $OB = 0.5$.

$$DB = OB \sin 36^\circ.$$

$$\log OB = 9.69897 - 10$$

$$\log \sin 36^\circ = 9.76922 - 10$$

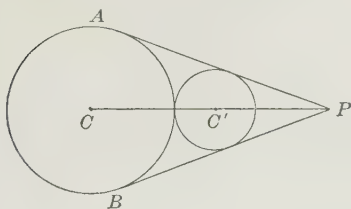
$$\log DB = 9.46819 - 10$$

$$DB = 0.2939.$$

$$AB = 0.5878.$$

The side of a regular pentagon whose diameter is 1 in. is 0.588 in.

35. Two circles of radii a and b are externally tangent. The common tangents AP , BP , and the line of centers $CC'P$ are drawn as shown in the figure. Find $\sin APC$.



Draw radii CA and $C'A'$ to points of tangency and $C'D \parallel$ to AP .

$$\angle APC = \angle DC'C.$$

$$CA \text{ is } \parallel \text{ to } C'A',$$

and $AD = C'A' = b.$

$$CD = a - b.$$

$$CC' = a + b.$$

$$\sin DC'C = \frac{a - b}{a + b}.$$

$$\therefore \sin APC = \frac{a - b}{a + b}.$$

36. In Ex. 35 find $\angle APC$, knowing that $a = 3b$.

$$\text{Since } a = 3b$$

$$\sin APC = \frac{3b - b}{3b + b} = \frac{2b}{4b} = \frac{1}{2}.$$

$$\sin 30^\circ = \frac{1}{2}.$$

$$\therefore \angle APC = 30^\circ.$$

37. In $\triangle ABC$, $\angle A = 68^\circ 26' 27''$, $\angle B = 75^\circ 8' 23''$, and altitude h , from C , is 148.17 in. Required the lengths of the three sides.

$$\frac{h}{AC} = \sin A$$

$$\frac{AC}{h} = \frac{1}{\sin A}.$$

$$\log h = 2.17076$$

$$\text{colog } \sin A = 0.03150$$

$$\log AC = 2.20226$$

$$AC = 159.31.$$

$$\frac{h}{BC} = \sin B.$$

$$\frac{BC}{h} = \frac{1}{\sin B}.$$

$$\log h = 2.17076$$

$$\text{colog } \sin B = 0.01478$$

$$\log BC = 2.18554$$

$$BC = 153.3.$$

$$AB = AD + DB.$$

$$AD = \frac{h}{\tan A}.$$

$$\log h = 2.17076$$

$$\text{colog } \tan A = 9.59672 - 10$$

$$\log AD = 1.76748$$

$$AD = 58.544.$$

$$DB = \frac{h}{\tan B}.$$

$$\log h = 2.17076$$

$$\text{colog } \tan B = 9.42385 - 10$$

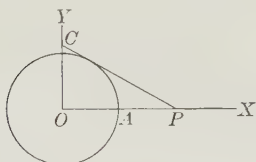
$$\log DB = 1.59461$$

$$DB = 39.32.$$

$$AB = 58.54 + 39.32 = 97.86.$$

The three sides are 97.9 in., 153.3 in., and 159.3 in.

38. Two axes, OX and OY , form a right angle at O , the center of a circle of radius 1091 ft. Through P , a point on OX 1997 ft. from O , a tangent is drawn, meeting OY at C . Required OC and the angle CPO .



$$\sin CPO = \frac{1091}{1997}.$$

$$\log 1091 = 3.03782$$

$$\text{colog } 1997 = 6.69962 - 10$$

$$\log \sin CPO = 9.73744$$

$$CPO = 33^\circ 6' 51''.$$

$$\tan CPO = \frac{OC}{1997}.$$

$$OC = 1997 \tan CPO.$$

$$\log 1997 = 3.30038$$

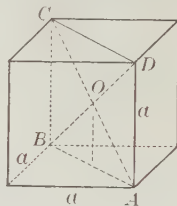
$$\log \tan CPO = 9.81441$$

$$\log OC = 3.11479$$

$$OC = 1302.545 \text{ ft.}$$

$$\angle CPO = 33^\circ 6' 51''.$$

39. Find the sine of the angle formed by the intersection of the diagonals of a cube.



$ABCD$ is a $\square$.

$$\therefore AC = BD.$$

$\triangle ABD$ and BAC are equal.

$$\overline{AB}^2 = a^2 + a^2.$$

$$AB = a\sqrt{2}.$$

$\triangle ABO$ is isosceles

and $\angle BAO = \angle ABO$.

$$\angle BAO = \angle BAC.$$

$$\tan BAC = \frac{a}{a\sqrt{2}} = \frac{1}{2}\sqrt{2}.$$

$$\tan BAC = 0.7071.$$

$$\angle BAC = 35^\circ 15' 48''.$$

$$\angle AOB = 180^\circ - 2(35^\circ 15' 48'') \\ = 109^\circ 28' 24''.$$

$$\sin AOB = 0.94275$$

$$= 0.9428.$$

40. The angle of elevation of the top of a tower observed at a place A , south of it, is 30° ; and at a place B , west of A , and at a distance of a from it, the angle of elevation is 18° . Show that the height of the tower is

$$\frac{a}{\sqrt{2+2\sqrt{5}}}, \text{ the tangent of } 18^\circ \text{ being } \frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}.$$

With the figure and notation of Ex. 21,

$$b = \frac{a}{\sqrt{\cot^2 18^\circ - \cot^2 30^\circ}}.$$

$$\text{But } \cot^2 18^\circ = \frac{10+2\sqrt{5}}{6-2\sqrt{5}} = \frac{(10+2\sqrt{5})(6+2\sqrt{5})}{6^2 - (2\sqrt{5})^2} = 5+2\sqrt{5},$$

$$\text{and } \cot^2 30^\circ = 3.$$

$$\text{Hence } b = \frac{a}{\sqrt{2+2\sqrt{5}}}.$$

41. Standing directly in front of one corner of a flat-roofed house, which is 150 ft. in length, I observe that the horizontal angle which the length subtends has for its cosine $\sqrt{\frac{1}{5}}$, and that the vertical angle subtended by its height has for its sine $\frac{3}{\sqrt{34}}$. What is the height of the house?

Let a = distance of observer from house,

b = height of house,

B = horizontal angle subtended by length of house,

B' = vertical angle subtended by height of house.

Then $a = 150 \cot B$.
 $b = a \tan B'$
 $= 150 \cot B \tan B'$.

But $\cos B = \sqrt{\frac{1}{5}}$;
 hence $\sin B = \sqrt{1 - \frac{1}{5}} = \frac{2}{\sqrt{5}}$.
 $\therefore \cot B = \frac{\cos B}{\sin B} = \frac{1}{2}$.

Also $\sin B' = \frac{3}{\sqrt{34}}$.

$\therefore \cos B' = \frac{5}{\sqrt{34}}$.

$\therefore \tan B' = \frac{3}{5}$.

Hence $b = 150 \times \frac{1}{2} \times \frac{3}{5} = 45$.

Height of house = 45 ft.

42. At a distance a from the foot of a tower, the angle of elevation A of the top of the tower is the complement of the angle of elevation of a flagstaff on top of it. Show that the length of the staff is $2a \cot 2A$.

Let h = height of tower,
 and l = length of staff.

Then $h = a \tan A$.

$h + l = a \cot A$.

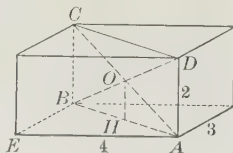
$l = a (\cot A - \tan A)$

$= a \frac{\cot^2 A - 1}{\cot A}$;

$= 2a \cot 2A$.

by [33],

43. A rectangular solid is 4 in. long, 3 in. wide, and 2 in. high. Calculate the tangent of the angle formed by the intersection of any two of the diagonals.



$$AB = \sqrt{4^2 + 3^2} = 5.$$

As in Ex. 39,

$$\tan \frac{1}{2} AOB = \frac{AH}{OH} = \frac{\frac{5}{2}}{1} = \frac{5}{2} = 2.5.$$

$$\frac{1}{2} AOB = 68^\circ 11' 54''.$$

$$AOB = 136^\circ 23' 48''.$$

$$\therefore \tan AOB = -0.9524.$$

44. Calculate the tangent as in Ex. 43, the solid being l units long, w units wide, and h units high.

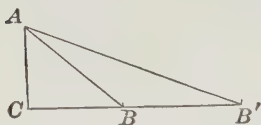
$EA = l$, $EB = w$, $CB = h$.

$$\begin{aligned} \tan \frac{1}{2} AOB &= \frac{\frac{1}{2} AB}{\frac{1}{2} BC} = \frac{\frac{1}{2} \sqrt{l^2 + w^2}}{\frac{1}{2} h} \\ &= \frac{\sqrt{l^2 + w^2}}{h}. \end{aligned}$$

$$\begin{aligned} \text{By [31], } \tan AOB &= \frac{2 \tan \frac{1}{2} AOB}{1 - \tan^2 \frac{1}{2} AOB} \\ &= \frac{2 \frac{\sqrt{l^2 + w^2}}{h}}{1 - \frac{l^2 + w^2}{h^2}} \\ &= \frac{2h \sqrt{l^2 + w^2}}{h^2 - l^2 - w^2}. \end{aligned}$$

Exercise 60. Oblique Triangles (Page 137)

1. Show how to determine the height of an inaccessible object situated on a horizontal plane by observing its angles of elevation at two points in the same line with its base and measuring the distance between these two points.

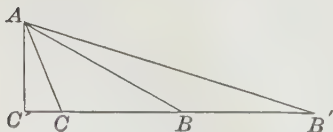


Let AC be the object, B and B' the two points of observation. Then given the angles B' and ABC , and the side BB' ; required AC .

$$\begin{aligned} AB &= BB' \frac{\sin B'}{\sin BAB'} \\ &= BB' \frac{\sin B'}{\sin (ABC - B')} \end{aligned}$$

$$\begin{aligned} AC &= AB \sin ABC \\ &= BB' \frac{\sin B' \sin ABC}{\sin (ABC - B')} \end{aligned}$$

2. Show how to determine the height of an inaccessible object standing on an inclined plane.



Let CBB' be the inclined plane, AC the object, B and B' two points

of observation, AC' the perpendicular from A on CBB' . Then given CB , BB' , and the angles ABC , B' ; required AC .

From the solution of Ex. 1,

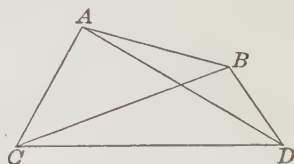
$$AC' = BB' \frac{\sin B' \sin ABC}{\sin (ABC - B')},$$

$$\text{and } C'B = BB' \frac{\sin B' \cos ABC}{\sin (ABC - B')}.$$

$$\text{Then } C'C = C'B - CB,$$

$$\text{and } AC = \sqrt{AC'^2 + C'C^2}.$$

3. Show how to determine the distance between two inaccessible objects by observing angles at the ends of a line of known length.



Let A and B be the inaccessible objects, C and D the ends of the given line. Then, given CD , $\angle ACD$, $\angle BCD$, $\angle ADC$, and $\angle BDC$; required AB .

$$AC = CD \frac{\sin ADC}{\sin CAD}.$$

$$BC = CD \frac{\sin BDC}{\sin CBD}.$$

Then in the triangle CAB , two sides and the included angle are known, and the third side can be computed as usual.

4. The angle of elevation of the top of an inaccessible tower standing on a horizontal plain is $63^{\circ} 26'$; at a point 500 ft. farther from the base of the tower the angle of elevation of the top is $32^{\circ} 14'$. Find the height of the tower.

From the solution of Ex. 1,

$$\begin{aligned} AC &= 500 \frac{\sin 32^{\circ} 14' \sin 63^{\circ} 26'}{\sin (63^{\circ} 26' - 32^{\circ} 14')} \\ &= 500 \frac{\sin 32^{\circ} 14' \sin 63^{\circ} 26'}{\sin 31^{\circ} 12'}. \end{aligned}$$

$$\log 500 = 2.69897$$

$$\log \sin 32^{\circ} 14' = 9.72703 - 10$$

$$\log \sin 63^{\circ} 26' = 9.95154 - 10$$

$$\text{colog} \sin 31^{\circ} 12' = \underline{0.28565}$$

$$\log AC = \underline{2.66319}$$

$$AC = 460.46.$$

Height of the tower = 460.46 ft.

5. A tower stands on the bank of a river. From the opposite bank the angle of elevation of the top of the tower is $60^{\circ} 13'$, and from a point 40 ft. further off the angle of elevation is $50^{\circ} 19'$. Find the width of the river.

In the figure for the solution of Ex. 1,

$$CB = AB \cos ABC$$

$$= BB' \frac{\sin B' \cos ABC}{\sin (ABC - B')}.$$

Hence

$$CB = 40 \frac{\sin 50^{\circ} 19' \cos 60^{\circ} 13'}{\sin 9^{\circ} 54'}.$$

$$\log 40 = 1.60206$$

$$\log \sin 50^{\circ} 19' = 9.88626 - 10$$

$$\log \cos 60^{\circ} 13' = 9.69611 - 10$$

$$\text{colog} \sin 9^{\circ} 54' = \underline{0.76465}$$

$$\log CB = \underline{1.94908}$$

$$CB = 88.936.$$

Breadth of river = 88.936 ft.

6. At the distance of 40 ft. from the foot of a vertical tower on an inclined plane, the tower subtends an angle of $41^{\circ} 19'$; at a point 60 ft. farther away the angle subtended by the tower is $23^{\circ} 45'$. Find the height of the tower.

From the solution of Ex. 2,

$$AC' = 60 \frac{\sin 23^{\circ} 45' \sin 41^{\circ} 19'}{\sin 17^{\circ} 34'}.$$

$$C'B = 60 \frac{\sin 23^{\circ} 45' \cos 41^{\circ} 19'}{\sin 17^{\circ} 34'}.$$

$$\log 60 = 1.77815$$

$$\log \sin 23^{\circ} 45' = 9.60503 - 10$$

$$\log \sin 41^{\circ} 19' = 9.81969 - 10$$

$$\text{colog} \sin 17^{\circ} 34' = \underline{0.52026}$$

$$\log AC' = \underline{1.72313}$$

$$AC' = 52.860.$$

$$\log 60 = 1.77815$$

$$\log \sin 23^{\circ} 45' = 9.60503 - 10$$

$$\log \cos 41^{\circ} 19' = 9.87568 - 10$$

$$\text{colog} \sin 17^{\circ} 34' = \underline{0.52026}$$

$$\log C'B = \underline{1.77912}$$

$$C'B = 60.134.$$

$$C'C = 60.134 - 40$$

$$= 20.134.$$

$$\tan ACC' = \frac{AC'}{C'C}.$$

$$AC = AC' \csc ACC'$$

$$\log AC' = 1.72313$$

$$\text{colog} C'C = \underline{8.69607 - 10}$$

$$\log \tan ACC' = \underline{0.41920}$$

$$ACC' = 69^{\circ} 8' 55''.$$

$$\log AC' = 1.72313$$

$$\log \csc ACC' = \underline{0.02941}$$

$$\log AC = \underline{1.75254}$$

$$AC = 56.564.$$

Height of tower = 56.564 ft.

7. A building makes an angle of $113^\circ 12'$ with the inclined plane on which it stands; at a distance of 89 ft. from its base, measured down the plane, the angle subtended by the building is $23^\circ 27'$. Find the height of the building.

In the triangle ACB , given $CB = 89$ feet, $C = 113^\circ 12'$, $B = 23^\circ 27'$; required AC .

$$A = 180^\circ - (B + C) \\ = 43^\circ 21'.$$

$$AC = CB \frac{\sin B}{\sin A}.$$

$$\log 89 = 1.94939$$

$$\log \sin 23^\circ 27' = 9.59983 - 10$$

$$\text{colog } \sin 43^\circ 21' = 0.16339$$

$$\log AC = 1.17261$$

$$AC = 51.595.$$

Height of building = 51.595 ft.

8. A person goes 70 yd. up a slope of 1 in $3\frac{1}{2}$ from the bank of a river and observes the angle of depression of an object on the opposite bank to be $2\frac{1}{4}^\circ$. Find the width of the river.

Let A and B be the original and final positions of the observer, and C the object observed. Then given $c = 70$, $C = 2\frac{1}{4}^\circ$, $A = 180^\circ$ minus the angle whose tangent is $\frac{1}{3\frac{1}{2}}$; required b .

$$\frac{1}{3\frac{1}{2}} = \tan 15^\circ 56' 40''.$$

$$A = 180^\circ - 15^\circ 56' 40'' \\ = 164^\circ 3' 20''.$$

$$B = 180^\circ - (A + C) \\ = 13^\circ 41' 40'',$$

$$b = c \frac{\sin B}{\sin C}.$$

$$\log 70 = 1.84510$$

$$\log \sin 13^\circ 41' 40'' = 9.37428 - 10$$

$$\text{colog } \sin 2^\circ 15' = 1.40605$$

$$\log b = 2.62543$$

$$b = 422.11.$$

Breadth of river = 422.11 yd.

9. A tree stands on a declivity inclined 15° to the horizon. A man ascends the declivity 80 ft. from the foot of the tree and finds the angle then subtended by the tree to be 30° . Find the height of the tree.

Let A and B be the top and bottom of the tree, and C the position of observation. Then given $a = 80$, $B = 75^\circ$, $C = 30^\circ$; required c .

$$A = 180^\circ - (B + C) = 75^\circ.$$

$$c = \frac{a \sin C}{\sin A} \\ = \frac{80 \sin 30^\circ}{\sin 75^\circ}.$$

$$\log 80 = 1.90309$$

$$\log \sin 30^\circ = 9.69897 - 10$$

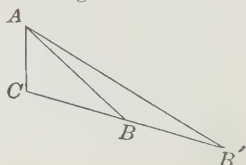
$$\text{colog } \sin 75^\circ = 0.01506$$

$$\log c = 1.61712$$

$$c = 41.411.$$

Height of tree = 41.411 ft.

10. The angle subtended by a tree on an inclined plane is, at a certain point, $42^\circ 17'$, and 325 ft. further down it is $21^\circ 47'$. The inclination of the plane is $8^\circ 53'$. Find the height of the tree.



$$\begin{aligned}
 AB &= BB' \frac{\sin B'}{\sin BAB'} \\
 &= BB' \frac{\sin B'}{\sin (B - B')} \\
 AC &= \frac{AB \sin B}{\sin C} \\
 &= BB' \frac{\sin B \sin B'}{\sin C \sin (B - B')} \\
 &= 325 \frac{\sin 42^\circ 17' \sin 21^\circ 47'}{\sin 98^\circ 53' \sin 20^\circ 30'}
 \end{aligned}$$

$$\log 325 = 2.51188$$

$$\log \sin 42^\circ 17' = 9.82788 - 10$$

$$\log \sin 21^\circ 47' = 9.56949 - 10$$

$$\text{colog} \sin 98^\circ 53' = 0.00524$$

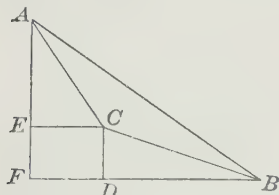
$$\text{colog} \sin 20^\circ 30' = 0.45567$$

$$\log AC = 2.37016$$

$$AC = 234.51.$$

Height of tree = 234.51 ft.

11. From a point B at the foot of a mountain, the angle of elevation of the top A is 60° . After ascending the mountain one mile, at an inclination of 30° to the horizon, and reaching a point C , an observer finds that the angle ACB is 135° . Find the number of feet in the height of the mountain.



$$\begin{aligned}
 CD &= CB \sin CBD \\
 &= 5280 \times \frac{1}{2} = 2640.
 \end{aligned}$$

$$AC = \frac{CB \sin CBA}{\sin CAB}.$$

$$\begin{aligned}
 AE &= AC \sin ECA \\
 &= \frac{CB \sin CBA \sin ECA}{\sin CAB} \\
 &= \frac{5280 \sin 30^\circ \sin 75^\circ}{\sin 15^\circ} \\
 &= \frac{5280 \times \frac{1}{2} \cos 15^\circ}{\sin 15^\circ} \\
 &= 2640 \cot 15^\circ.
 \end{aligned}$$

$$\log 2640 = 3.42160$$

$$\log \cot 15^\circ = 10.57195 - 10$$

$$\log AE = 3.99355$$

$$AE = 9852.6.$$

$$AF = AE + CD = 12,492.6.$$

Height of the mountain

$$= 12,492.6 \text{ ft.}$$

12. The length of a lake subtends, at a certain point, an angle of $46^\circ 24'$, and the distances from this point to the two ends of the lake are 346 ft. and 290 ft. Find the length of the lake.

Given $A = 46^\circ 24'$, $b = 346$, $c = 290$:

required a .

$$\begin{aligned}
 \tan \frac{1}{2}(B - C) &= \frac{b - c}{b + c} \tan \frac{1}{2}(B + C) \\
 &= \frac{56}{636} \tan 66^\circ 48'.
 \end{aligned}$$

$$\log 56 = 1.74819$$

$$\text{colog} 636 = 7.19654 - 10$$

$$\log \tan 66^\circ 48' = 10.36795 - 10$$

$$\log \tan \frac{1}{2}(B - C) = 9.31268$$

$$\frac{1}{2}(B - C) = 11^\circ 36' 33''.$$

$$B - C = 23^\circ 13' 6''.$$

$$B + C = 133^\circ 36'.$$

$$\therefore B = 78^\circ 24' 33''.$$

$$a - b \frac{\sin A}{\sin B} = 346 \frac{\sin 46^\circ 24'}{\sin 78^\circ 24' 33''}.$$

$$\log 346 = 2.53908$$

$$\log \sin 46^\circ 24' = 9.85984 - 10$$

$$\text{colog} \sin 78^\circ 24' 33'' = 0.00895$$

$$\log a = 2.40787$$

$$a = 255.78.$$

Length of lake = 255.78 ft.

13. Along the bank of a river is drawn a base line of 500 ft. The angular distance of one end of this line from an object on the opposite side of the river, as observed from the other end of the line, is 53° ; that of the second extremity from the same object, observed at the first, is $79^\circ 12'$. Find the width of the river.

Given $B = 53^\circ$, $C = 79^\circ 12'$, $a = 500$; required p , the perpendicular from A on a .

$$A = 180^\circ - (B + C)$$

$$= 47^\circ 48'.$$

$$b = a \frac{\sin B}{\sin A}.$$

$$p = b \sin C = a \frac{\sin B \sin C}{\sin A}$$

$$= 500 \frac{\sin 53^\circ \sin 79^\circ 12'}{\sin 47^\circ 48'}.$$

$$\log 500 = 2.69897$$

$$\log \sin 53^\circ = 9.90235 - 10$$

$$\log \sin 79^\circ 12' = 9.99224 - 10$$

$$\text{colog } \sin 47^\circ 48' = 0.13030 \quad \text{_____}$$

$$\log p = 2.72386$$

$$p = 529.49.$$

$$\text{Width of river} = 529.49 \text{ ft.}$$

14. Two observers, stationed on opposite sides of a cloud, observe its angles of elevation to be $44^\circ 56'$ and $36^\circ 4'$. Their distance from each other is 700 ft. What is the height of the cloud?

Given $A = 44^\circ 56'$, $B = 36^\circ 4'$, $c = 700$; required the perpendicular p from C on c .

$$C = 180^\circ - (A + B) = 99^\circ.$$

$$b = c \frac{\sin B}{\sin C}.$$

$$p = b \sin A = c \frac{\sin B \sin A}{\sin C}$$

$$= 700 \frac{\sin 36^\circ 4' \sin 44^\circ 56'}{\sin 99^\circ}.$$

$$\log 700 = 2.84510$$

$$\log \sin 36^\circ 4' = 9.76991$$

$$\log \sin 44^\circ 56' = 9.84898$$

$$\text{colog } \sin 99^\circ = 0.00538$$

$$\log p = 2.46937$$

$$p = 294.69.$$

$$\text{Height of cloud} = 294.69 \text{ ft.}$$

15. From the top of a house 42 ft. high the angle of elevation of the top of a pole is $14^\circ 13'$; at the bottom of the house it is $23^\circ 19'$. Find the height of the pole.

Let A be the top of the pole, B and B' the top and bottom of the house, and C the foot of the perpendicular from A on BB' ; required $B'C$.

From the solutions of Examples 1 and 4,

$$CB = BB' \frac{\sin A B' C \cos A B C}{\sin (A B C - A B' C)}$$

$$= 42 \frac{\sin 66^\circ 41' \cos 75^\circ 47'}{\sin 9^\circ 6'}.$$

$$\log 42 = 1.62325$$

$$\log \sin 66^\circ 41' = 9.96300$$

$$\log \cos 75^\circ 47' = 9.39021$$

$$\text{colog } \sin 9^\circ 6' = 0.80091$$

$$\log CB = 1.77737$$

$$CB = 59.892.$$

$$B'C = CB + BB'$$

$$= 59.892 + 42$$

$$= 101.892.$$

$$\text{Height of pole} = 101.892 \text{ ft.}$$

16. From a window on a level with the bottom of a steeple the angle of elevation of the top of the steeple is 40° , and from a second window 18 ft. higher the angle of elevation is $37^\circ 30'$. Find the height of the steeple.

Let A and B be the windows, and C the top of the steeple. Then given $c = 18$, $A = 50^\circ$, $B = 127^\circ 30'$; required height of steeple.

$$h = b \sin 40^\circ.$$

$$C = 180^\circ - (A + B) = 2^\circ 30'.$$

$$b = c \frac{\sin B}{\sin C} = 18 \frac{\sin 127^\circ 30'}{\sin 2^\circ 30'}.$$

$$h = 18 \frac{\sin 127^\circ 30' \sin 40^\circ}{\sin 2^\circ 30'}.$$

$$\log 18 = 1.25527$$

$$\log \sin 127^\circ 30' = 9.89947$$

$$\log \sin 40^\circ = 9.80807$$

$$\text{colog} \sin 2^\circ 30' = 1.36032$$

$$\log h = 2.32313$$

$$h = 210.44.$$

Height of steeple = 210.44 ft.

17. The sides of a triangle are 17, 21, 28. Prove that the length of a line bisecting the longest side and drawn from the opposite angle is 13.

Here $a = 28$, $b = 21$, $c = 17$.

By § 111, $c^2 = a^2 + b^2 - 2ab \cos C$.

$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{28^2 + 21^2 - 17^2}{2 \times 28 \times 21}$$

$$= \frac{936}{1176} = \frac{39}{49} = 0.7959.$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Here $a = 14$, $b = 21$, $\cos C = 0.7959$.

$$\therefore c = \sqrt{14^2 + 21^2 - 2 \times 14 \times 21 \times 0.7959}$$

$$= \sqrt{196 + 441 - 467.9892}$$

$$= \sqrt{169.0108} = 13.0004.$$

18. The sum of the sides of a triangle is 100. The angle at A is double that at B , and the angle at B is double that at C . Determine the sides.

$$B = 2C.$$

$$A = 2B = 4C.$$

$$A + B + C = 7C = 180^\circ$$

$$\therefore C = 25^\circ 42' 51\frac{3}{7}''.$$

$$B = 51^\circ 25' 42\frac{6}{7}''.$$

$$A = 102^\circ 51' 25\frac{5}{7}''.$$

$$\frac{a}{c} = \frac{\sin A}{\sin C}.$$

$$\log \sin A = 9.98897$$

$$\text{colog} \sin C = \underline{0.36263}$$

$$\log \frac{a}{c} = 0.35160$$

$$\frac{a}{c} = 2.247.$$

$$a = 2.247c.$$

$$\frac{b}{c} = \frac{\sin B}{\sin C}.$$

$$\log \sin B = 9.89311$$

$$\text{colog} \sin C = \underline{0.36263}$$

$$\log \frac{b}{c} = 0.25574$$

$$\frac{b}{c} = 1.802.$$

$$b = 1.802c.$$

$$a + b + c = (2.247 + 1.802 + 1)c = 5.049c.$$

$$\therefore c = \frac{100}{5.049} = 19.806$$

$$a = 2.247c = 44.504$$

$$b = 1.802c = 35.690$$

$$a + b + c = 100.000$$

The sides are 19.8, 35.7, 44.5.

19. A ship sailing north sees two lighthouses 8 mi. apart in a line due west; after an hour's sailing, one lighthouse bears S.W., and the other S. $22^{\circ} 30'$ W. ($22^{\circ} 30'$ west of south). Find the ship's rate.

In the figure for the solution of Example 1, let B and B' be the lighthouses, C the original position of the ship, and A its final position. Then $CAB = 22^{\circ} 30'$ and $CAB' = 45^{\circ}$; hence $ABC = 67^{\circ} 30'$ and $B' = 45^{\circ}$.

$$AC = 8 \frac{\sin 45^{\circ} \sin 67^{\circ} 30'}{\sin 22^{\circ} 30'}$$

$$= 8 \sin 45^{\circ} \cot 22^{\circ} 30'.$$

$$\log 8 = 0.90309$$

$$\log \sin 45^{\circ} = 9.84949$$

$$\log \cot 22^{\circ} 30' = 10.38278$$

$$\log AC = 1.13533$$

$$AC = 13.657.$$

Ship's rate = 13.657 mi. per hour.

20. A ship, 10 mi. S. W. of a harbor, sees another ship sail from the harbor in a direction S. 80° E., at a rate of 9 mi. an hour. In what direction and at what rate must the first ship sail in order to catch up with the second ship in $1\frac{1}{2}$ hr.?

Let A be the harbor, B the original position of the first ship, and C the point where the vessels are to meet. Then $A = 125^{\circ}$, $b = 13\frac{1}{2}$, $c = 10$; required B and $\frac{a}{1\frac{1}{2}}$.

$$\tan \frac{1}{2}(B - C) = \frac{b - c}{b + c} \tan \frac{1}{2}(B + C)$$

$$= \frac{3.5}{23.5} \tan 27^{\circ} 30'$$

$$= \frac{7}{47} \tan 27^{\circ} 30'.$$

$$\log 7 = 0.84510$$

$$\text{colog } 47 = 8.32790 - 10$$

$$\log \tan 27^{\circ} 30' = 9.71648$$

$$\log \tan \frac{1}{2}(B - C) = 8.88948$$

$$\frac{1}{2}(B - C) = 4^{\circ} 26'.$$

$$B - C = 8^{\circ} 52'.$$

$$B + C = 55^{\circ}.$$

$$\therefore B = 31^{\circ} 56'.$$

$$a = b \frac{\sin A}{\sin B}$$

$$= 13.5 \frac{\sin 125^{\circ}}{\sin 31^{\circ} 56'}.$$

$$\log 13.5 = 1.13033$$

$$\log \sin 125^{\circ} = 9.91336$$

$$\text{colog } \sin 31^{\circ} 56' = 0.27660$$

$$\log a = 1.32029$$

$$a = 20.907.$$

$$\frac{a}{1\frac{1}{2}} = 13.938.$$

First ship's course = $31^{\circ} 56'$ E. of N.E., or N. $76^{\circ} 56'$ E.; rate = 13.938 mi. per hour.

21. Two ships are a mile apart. The angular distance of the first ship from a lighthouse on shore, as observed from the second ship, is $35^{\circ} 14' 10''$; the angular distance of the second ship from the lighthouse, observed from the first ship, is $42^{\circ} 11' 53''$. Find the distance in feet from each ship to the lighthouse.

Given $B = 35^\circ 14' 10''$, $C = 42^\circ 11' 53''$, $a = 5280$; required b and c .

$$A = 180 - (B + C) \\ = 102^\circ 33' 57''.$$

$$b = a \frac{\sin B}{\sin A}.$$

$$\log 5280 = 3.72263$$

$$\log \sin 35^\circ 14' 10'' = 9.76114$$

$$\text{colog } \sin 102^\circ 33' 57'' = 0.01053$$

$$\log b = 3.49430$$

$$b = 3121.1.$$

$$c = a \frac{\sin C}{\sin A}.$$

$$\log 5280 = 3.72263$$

$$\log \sin 42^\circ 11' 53'' = 9.82717$$

$$\text{colog } \sin 102^\circ 33' 57'' = 0.01053$$

$$\log c = 3.56033$$

$$c = 3633.5.$$

Distance of first ship from lighthouse = 3121.1 ft.; distance of second ship from lighthouse = 3633.5 ft.

22. A lighthouse bears N. $11^\circ 15'$ E., as seen from a ship. The ship sails northwest 30 mi., and then the lighthouse bears east. How far is the lighthouse from the second point of observation?

Let A be the lighthouse, B and C the first and second positions of the ship. Then given $B = 56^\circ 15'$, $C = 45^\circ$, $a = 30$; required b .

$$A = 180^\circ - (B + C) = 78^\circ 45'.$$

$$b = \frac{a \sin B}{\sin A} = \frac{30 \sin 56^\circ 15'}{\sin 78^\circ 45'}.$$

$$\log 30 = 1.47712$$

$$\log \sin 56^\circ 15' = 9.91985$$

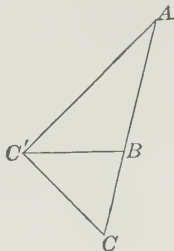
$$\text{colog } \sin 78^\circ 45' = 0.00843$$

$$\log b = 1.40540$$

$$b = 25.433.$$

Distance of lighthouse from second point of observation = 25.433 mi.

23. Two rocks are seen in the same straight line with a ship, bearing N. 15° E. After the ship has sailed N.W. 5 mi., the first rock bears E., and the second N.E. Find the distance between the rocks.



Let A and B be the two rocks, C and C' the first and second positions of the ship. Then given $C = 60^\circ$, $CC'B = 45^\circ$, $CC'A = 90^\circ$, $CC' = 5$; required AB .

$$AC = CC' \sec C = 5 \times 2 = 10.$$

$$BC = CC' \frac{\sin BC'C}{\sin CBC'} = 5 \frac{\sin 45^\circ}{\sin 75^\circ}.$$

$$\log 5 = 0.69897$$

$$\log \sin 45^\circ = 9.84949$$

$$\text{colog } \sin 75^\circ = 0.01506$$

$$\log BC = 0.56352$$

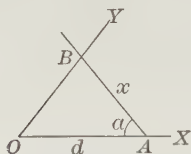
$$BC = 3.6603.$$

$$AB = AC - BC$$

$$= 6.3397.$$

Distance between rocks = 6.3397 mi.

24. On the side OX of a given angle XOY a point A is taken such that $OA = d$. Deduce a formula for the length AB of a line from A to OY that makes a given angle a with OX . From this formula, x is a minimum when what sine is the maximum? Under those circumstances what is the sum of O and a ? Then what is the size of $\angle B$? State the conclusion as to the size of $\angle a$ in order that x shall be the minimum.



$$\frac{\sin [180^\circ - (O + a)]}{\sin O} = \frac{d}{x}$$

$$x = \frac{d \sin O}{\sin [180^\circ - (O + a)]}$$

x is the minimum when $\sin 180^\circ - (O + a)$ is the maximum.

Under this condition,

$$O + a = 90^\circ,$$

$$\text{and } B = 180^\circ - (O + a) = 90^\circ.$$

By Geometry, in order that x may be the minimum,

OB must be $\perp$ to AB .

Then $\angle a = 90^\circ - O$.

25. Three points, A , B , and C , form the vertices of an equilateral triangle, AB being 500 ft. Each of the two sides AB and AC is seen

from a point P under an angle of 120° ; that is, $\angle APB = 120^\circ = \angle CPA$. Find the length of AP .

$$\sin 60^\circ = \frac{250}{AP}$$

$$AP = \frac{250}{\sin 60^\circ}$$

$$\log 250 = 2.39794$$

$$\text{colog } \sin 60^\circ = 0.06247$$

$$\log AP = 2.46041$$

$$AP = 288.67 \text{ ft.}$$

26. A lighthouse facing south sends out its rays extending in a quadrant from S.E. to S.W. A steamer sailing due east first sees the light when 6 mi. away from the lighthouse and continues to see it for 45 min. At what rate is the ship sailing?

As the ship sails due E., the line of sailing is $\perp$ to N. and S. The $\triangle$ formed by AB (the course of the ship), and the rays of light is an isosceles right $\triangle$.

$$\angle C = 90^\circ.$$

$$\angle A = 45^\circ = \angle B.$$

$$\sin A = \frac{CB}{AB}$$

$$AB = \frac{BC}{\sin A}$$

$$\log CB = 0.77815$$

$$\text{colog } \sin A = 0.15052$$

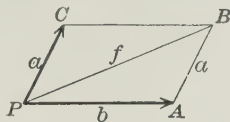
$$\log AB = 0.92867$$

$$\text{colog } \frac{3}{4} = 0.12494$$

$$\log \text{ of rate of sailing} = 1.05363$$

Rate of sailing = 11.314 mi. per hour.

27. If two forces, represented in intensity by the lengths a and b , pull from P in the directions PC and PA , respectively, and if $\angle APC$ is known, the resultant force is represented in intensity and direction by f , the diagonal of the parallelogram $ABCP$. Show how to find f and $\angle APB$, given a , b , and $\angle APC$.



In $\triangle APB$, $\angle APB$ and ABP are equal to $\angle APC$.

$$\text{By [49],} \quad \frac{b-a}{b+a} = \frac{\tan \frac{1}{2}(ABP - APB)}{\tan \frac{1}{2}(ABP + APB)}.$$

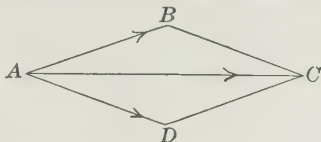
From this equation we can find $\angle ABP$ and APB .

$$\angle A = 180^\circ - \angle APC.$$

$$\text{By [47],} \quad \frac{b}{f} = \frac{\sin ABP}{\sin A}. \quad f = \frac{b \sin A}{\sin ABP}.$$

28. Two forces, one of 410 lb. and the other of 320 lb., make an angle of $51^\circ 37'$. Find the intensity and the direction of their resultant.

Let AB and AD represent the forces, and AC their resultant. Then, in the triangle ABC , given $c = 410$, $a = 320$, $B = 180^\circ - 51^\circ 37' = 128^\circ 23'$; required b and A .



$$\tan \frac{1}{2}(C - A) = \frac{c - a}{c + a} \tan \frac{1}{2}(C + A) = \frac{90}{730} \tan 25^\circ 48' 30''.$$

$$\log 90 = 1.95424$$

$$\log \tan 25^\circ 48' 30'' = 9.68448$$

$$\text{colog } 730 = 7.13668 - 10$$

$$\tan \frac{1}{2}(C - A) = 8.77540$$

$$\frac{1}{2}(C - A) = 3^\circ 24' 43''$$

$$\frac{1}{2}(C + A) = 25^\circ 48' 30''$$

$$A = 22^\circ 23' 47''$$

$$b = a \frac{\sin B}{\sin A} = 320 \frac{\sin 128^\circ 23'}{\sin 22^\circ 23' 47''}.$$

$$\log 320 = 2.50515$$

$$\log \sin 128^\circ 23' = 9.89425$$

$$\text{colog } \sin 22^\circ 23' 47'' = 0.41906$$

$$\log b = 2.81846$$

$$b = 658.36.$$

Intensity of resultant = 658.36 lb.; angle between resultant and first force = $22^\circ 23' 47''$.

29. An unknown force combined with one of 128 lb. produces a resultant of 200 lb., and this resultant makes an angle of $18^\circ 24'$ with the known force. Find the intensity and direction of the unknown force.

In the figure for the solution of Example 28, given, in the triangle ABC , $c = 128$, $A = 18^\circ 24'$, $b = 200$; required a and B .

$$\begin{aligned}\tan \frac{1}{2}(B - C) &= \frac{b - c}{b + c} \tan \frac{1}{2}(B + C) \\ &= \frac{72}{328} \tan 80^\circ 48' .\end{aligned}$$

$$\begin{aligned}\log 72 &= 1.85733 \\ \log \tan 80^\circ 48' &= 10.79058 \\ \text{colog } 328 &= \frac{7.48413 - 10}{\log \tan \frac{1}{2}(B - C) = 10.13204}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}(B - C) &= 53^\circ 34' 44'' \\ \frac{1}{2}(B + C) &= 80^\circ 48' \\ B &= 134^\circ 22' 44''\end{aligned}$$

$$180^\circ - B = 45^\circ 37' 16'' .$$

$$a = \frac{b \sin A}{\sin B} = 200 \frac{\sin 18^\circ 24'}{\sin 134^\circ 22' 44''} .$$

$$\begin{aligned}\log 200 &= 2.30103 \\ \log \sin 18^\circ 24' &= 9.49920 \\ \text{colog } \sin 134^\circ 22' 44'' &= \frac{0.14586}{\log a = 1.94609} \\ a &= 88.326 .\end{aligned}$$

Intensity of unknown force = 88.326 lb.; angle between known and unknown forces = $45^\circ 37' 16''$.

30. Wishing to determine the distance between a church A and a tower B , on the opposite side of a river, a man measured a line CD

along the river (C being nearly opposite A), and observed the angles ACB , $58^\circ 20'$; ACD , $95^\circ 20'$; ADB , $53^\circ 30'$; BDC , $98^\circ 45'$. CD is 600 ft. What is the distance required?

From the solution of Example 3,

$$\begin{aligned}AC &= CD \frac{\sin ADC}{\sin CAD} \\ &= 600 \frac{\sin 45^\circ 15'}{\sin 39^\circ 25'} .\end{aligned}$$

$$\begin{aligned}BC &= CD \frac{\sin BDC}{\sin CBD} \\ &= 600 \frac{\sin 98^\circ 45'}{\sin 44^\circ 15'} .\end{aligned}$$

$$\begin{aligned}\log 600 &= 2.77815 \\ \log \sin 45^\circ 15' &= 9.85137 \\ \text{colog } \sin 39^\circ 25' &= \frac{0.19726}{\log AC = 2.82678}\end{aligned}$$

$$AC = 671.09 .$$

$$\begin{aligned}\log 600 &= 2.77815 \\ \log \sin 98^\circ 45' &= 9.99492 \\ \text{colog } \sin 44^\circ 15' &= \frac{0.15627}{\log BC = 2.92934} \\ BC &= 849.84 .\end{aligned}$$

$$\begin{aligned}&\tan \frac{1}{2}(CAB - CBA) \\ &= \frac{BC - AC}{BC + AC} \tan \frac{1}{2}(CAB + CBA) \\ &= \frac{178.75}{1520.93} \tan 60^\circ 50' .\end{aligned}$$

$$\begin{aligned}\log 178.75 &= 2.25224 \\ \text{colog } 1520.93 &= \frac{6.81789 - 10}{\log \tan \frac{1}{2}(CAB - CBA) = 10.25327}\end{aligned}$$

$$\begin{aligned}&= 9.32340 \\ \frac{1}{2}(CAB - CBA) &= 11^\circ 53' 28'' \\ \frac{1}{2}(CAB + CBA) &= 60^\circ 50' \\ CAB &= 72^\circ 43' 28''\end{aligned}$$

$$AB = BC \frac{\sin ACB}{\sin CAB}$$

$$= 849.84 \frac{\sin 58^\circ 20'}{\sin 72^\circ 43' 28''}.$$

$$\log 849.84 = 2.92934$$

$$\log \sin 58^\circ 20' = 9.92999$$

$$\text{colog } \sin 72^\circ 43' 28'' = 0.02005$$

$$\log AB = 2.87938$$

$$AB = 757.50.$$

Required distance = 757.50 ft.

31. Wishing to find the height of a summit A , a man measured a horizontal base line CD , 440 yd. At C the angle of elevation of A is $37^\circ 18'$, and the horizontal angle between D and the summit is $76^\circ 18'$; at D the horizontal

angle between C and the summit is $67^\circ 14'$. Find the height.

Let A' be the point directly under A , in the same horizontal plane with CD . Then, in the triangle $A'CD$,

$$A'C = CD \frac{\sin D}{\sin A'}$$

$$= 440 \frac{\sin 67^\circ 14'}{\sin 36^\circ 28'}.$$

$$AA' = A'C \tan A'CA'$$

$$= 440 \frac{\sin 67^\circ 14'}{\sin 36^\circ 28'} \tan 37^\circ 18'.$$

$$\log 440 = 2.64345$$

$$\log \sin 67^\circ 14' = 9.96477$$

$$\log \tan 37^\circ 18' = 9.88184$$

$$\text{colog } \sin 36^\circ 28' = 0.22595$$

$$\log AA' = 2.71601$$

$$AA' = 520.01.$$

Height = 520.01 yd.

32. A balloon is observed from two stations 3000 ft. apart. At the first station the horizontal angle of the balloon and the other station is $75^\circ 25'$, and the angle of elevation of the balloon is 18° . The horizontal angle of the first station and the balloon, measured at the second station, is $64^\circ 30'$. Find the height of the balloon.

Let B be the first station, C the second, A the position of the balloon, and A' the point directly under A , in the same horizontal plane as BC . Then

$$AA' = A'B \tan A'BA = BC \frac{\sin A'CB}{\sin BA'C} \tan A'BA$$

$$= 3000 \frac{\sin 64^\circ 30'}{\sin 40^\circ 5'} \tan 18^\circ.$$

$$\log 3000 = 3.47712$$

$$\log \sin 64^\circ 30' = 9.95549$$

$$\log \tan 18^\circ = 9.51178$$

$$\text{colog } \sin 40^\circ 5' = 0.19118$$

$$\log AA' = 3.13557$$

$$AA' = 1366.4.$$

Height of balloon = 1366.4 ft.

33. At two stations the height of a kite subtends the same angle A . The angle which the line joining one station and the kite subtends at the other station is B ; and the distance between the two stations is a . Show that the height of the kite is $\frac{1}{2}a \sin A \sec B$.

Let C be the position of the kite, D and E the stations, and C' the point directly under C in the same horizontal plane with DE .

Since the elevation of the kite is the same at D and E , the triangle CDE is isosceles, and

$$CD = CE = \frac{1}{2}a \sec B.$$

Also

$$CC' = CD \sin A = \frac{1}{2}a \sin A \sec B.$$

34. Two towers on a horizontal plane are 120 ft. apart. A person standing successively at their bases observes that the angle of elevation of one is double that of the other; but when he is halfway between the towers, the angles of elevation are complementary. Prove that the heights of the towers are 90 ft. and 40 ft.

Let A and B be the tops of the towers, A' and B' their bases, and C the point halfway between them. Then the triangles $AA'C$ and $BB'C$ are similar, and

$$\frac{AA'}{B'C} = \frac{A'C}{BB'}.$$

$$AA' \times BB' = B'C \times A'C = 3600.$$

Also

$$AB'A' = 2 BA'B'.$$

$$\therefore \tan AB'A' = \frac{2 \tan BA'B'}{1 - \tan^2 BA'B'},$$

or

$$\frac{AA'}{120} = \frac{2 \frac{BB'}{120}}{1 - \frac{BB'^2}{120^2}} = \frac{240 BB'}{120^2 - BB'^2}.$$

$$AA'(120^2 - BB'^2) = 120 \times 240 BB'.$$

$$\frac{3600}{BB'}(120^2 - BB'^2) = 120 \times 240 BB'.$$

$$120^2 - BB'^2 = 8 BB'^2.$$

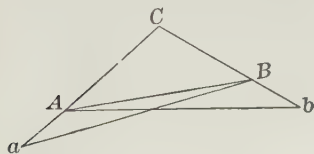
$$BB'^2 = 40^2.$$

$$BB' = 40.$$

$$AA' = 90.$$

Heights of towers = 90 ft. and 40 ft.

35. To find the distance of an inaccessible point C from either of two points A and B , having no instruments to measure angles. Prolong CA to a , and CB to b , and draw AB , Ab , and Ba . Measure AB , 500 ft.; aA , 100 ft.; aB , 560 ft.; bB , 100 ft.; and Ab , 550 ft. Compute the distances AC and BC .



In the triangle aAB ,

$$s = \frac{1}{2}(500 + 100 + 560) = 580.$$

$$\begin{aligned}\tan \frac{1}{2} aAB &= \sqrt{\frac{80 \times 480}{580 \times 20}} \\ &= \sqrt{\frac{96}{29}}.\end{aligned}$$

$$\log 96 = 1.98227$$

$$\begin{array}{r} \text{colog } 29 = 8.53760 - 10 \\ 2) \quad 0.51987 \end{array}$$

$$\log \tan \frac{1}{2} aAB = 10.25993$$

$$\frac{1}{2} aAB = 61^\circ 12' 20''.$$

$$aAB = 122^\circ 24' 40''.$$

$$CAB = 57^\circ 35' 20''.$$

In the triangle bAB ,

$$s = \frac{1}{2}(500 + 550 + 100) = 575.$$

$$\begin{aligned}\tan \frac{1}{2} bBA &= \sqrt{\frac{75 \times 475}{575 \times 25}} \\ &= \sqrt{\frac{57}{23}}.\end{aligned}$$

$$\log 57 = 1.75587$$

$$\begin{array}{r} \text{colog } 23 = 8.63827 - 10 \\ 2) \quad 0.39414 \end{array}$$

$$\log \tan \frac{1}{2} bBA = 10.19707$$

$$\frac{1}{2} bBA = 57^\circ 34' 30''.$$

$$bBA = 115^\circ 9'.$$

$$CBA = 64^\circ 51'.$$

In the triangle ABC ,

$$A = 57^\circ 35' 20'',$$

$$B = 64^\circ 51',$$

$$C = 57^\circ 33' 40''.$$

$$\begin{aligned}BC &= AB \frac{\sin A}{\sin C} \\ &= 500 \frac{\sin 57^\circ 35' 20''}{\sin 57^\circ 33' 40''}.\end{aligned}$$

$$\begin{aligned}AC &= AB \frac{\sin B}{\sin C} \\ &= 500 \frac{\sin 64^\circ 51'}{\sin 57^\circ 33' 40''}.\end{aligned}$$

$$\log 500 = 2.69897$$

$$\log \sin 57^\circ 35' 20'' = 9.92646$$

$$\text{colog } \sin 57^\circ 33' 40'' = 0.07368$$

$$\log BC = 2.69911$$

$$BC = 500.16.$$

$$\log 500 = 2.69897$$

$$\log \sin 64^\circ 51' = 9.95674$$

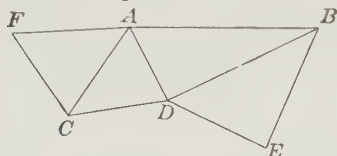
$$\text{colog } \sin 57^\circ 33' 40'' = 0.07368$$

$$\log AC = 2.72939$$

$$AC = 536.28.$$

Distance of C from A = 536.28 ft.;
distance of C from B = 500.16 ft.

36. To compute the horizontal distance between two inaccessible points A and B when no point can be found whence both can be seen. Take two points C and D , distant 200 yd., so that A can be seen from C , and B from D . From C measure CF , 200 yd. to F , whence A can be seen; and from D measure DE , 200 yd. to E , whence B can be seen. Measure AFC , 83° ; ACD , $53^\circ 30'$; ACF , $54^\circ 31'$; BDE , $54^\circ 30'$; BDC , $156^\circ 25'$; DEB , $88^\circ 30'$. Compute the distance AB .



$$AC = CF \frac{\sin AFC}{\sin CAF} = 200 \frac{\sin 83^\circ}{\sin 42^\circ 29'}.$$

$$\log 200 = 2.30103$$

$$\log \sin 83^\circ = 9.99675$$

$$\text{colog } \sin 42^\circ 29' = 0.17045$$

$$\log AC = 2.46823$$

$$AC = 293.92.$$

$$BD = DE \frac{\sin BED}{\sin DBE} = 200 \frac{\sin 88^\circ 30'}{\sin 37^\circ}.$$

$$\log 200 = 2.30103$$

$$\log \sin 88^\circ 30' = 9.99985$$

$$\text{colog } \sin 37^\circ = 0.22054$$

$$\log BD = 2.52142$$

$$BD = 332.22.$$

$$\tan \frac{1}{2}(ADC - CAD)$$

$$= \frac{AC - CD}{AC + CD} \tan \frac{1}{2}(ADC + CAD)$$

$$= \frac{93.92}{493.92} \tan 63^\circ 15'.$$

$$\log 93.92 = 1.97276$$

$$\text{colog } 493.92 = 7.30634 - 10$$

$$\log \tan 63^\circ 15' = 10.29753 - 10$$

$$\log \tan \frac{1}{2}(ADC - CAD) = 9.57663$$

$$\frac{1}{2}(ADC - CAD) = 20^\circ 40' 8''$$

$$\frac{1}{2}(ADC + CAD) = 63^\circ 15'$$

$$ADC = 83^\circ 55' 8''$$

$$AD = AC \frac{\sin ACD}{\sin ADC}$$

$$= 293.92 \frac{\sin 53^\circ 30'}{\sin 83^\circ 55' 8''}$$

$$\log 293.92 = 2.46823$$

$$\log \sin 53^\circ 30' = 9.90518 - 10$$

$$\text{colog } \sin 83^\circ 55' 8'' = 0.00245$$

$$\log AD = 2.37586$$

$$AD = 237.61.$$

$$BDA = BDC - ADC$$

$$= 156^\circ 25' - 83^\circ 55' 8''$$

$$= 72^\circ 29' 52''.$$

$$\tan \frac{1}{2}(DAB - DBA)$$

$$= \frac{BD - AD}{BD + AD} \tan \frac{1}{2}(DAB + DBA)$$

$$= \frac{94.61}{569.83} \tan 53^\circ 45' 4''.$$

$$\log 94.61 = 1.97594$$

$$\text{colog } 569.83 = 7.24426 - 10$$

$$\log \tan 53^\circ 45' 4'' = 10.13478 - 10$$

$$\log \tan \frac{1}{2}(DAB - DBA) = 9.35498$$

$$\frac{1}{2}(DAB - DBA) = 12^\circ 45' 35''$$

$$\frac{1}{2}(DAB + DBA) = 53^\circ 45' 4''$$

$$DAB = 66^\circ 30' 39''$$

$$AB = BD \frac{\sin ADB}{\sin BAD}$$

$$= 332.22 \frac{\sin 72^\circ 29' 52''}{\sin 66^\circ 30' 39''}.$$

$$\log 332.22 = 2.52142$$

$$\log \sin 72^\circ 29' 52'' = 9.97941 - 10$$

$$\text{colog } \sin 66^\circ 30' 39'' = 0.03757$$

$$\log AB = 2.53840$$

$$AB = 345.46.$$

$$\text{Distance } AB = 345.46 \text{ yd.}$$

37. A column in the north temperate zone is S. $67^{\circ} 30'$ E. of an observer, and at noon the extremity of its shadow is northeast of him. The shadow is 80 ft. in length, and the elevation of the column at the observer's station is 45° . Find the height of the column.

Let A be the observer's position, B the extremity of the shadow, and O the base of the column. Then given $A = 67^{\circ} 30'$, $C = 67^{\circ} 30'$, $a = 80$; required b .

$$b = a \frac{\sin B}{\sin A} = 80 \frac{\sin 45^{\circ}}{\sin 67^{\circ} 30'}$$

$$\log 80 = 1.90309$$

$$\log \sin 45^{\circ} = 9.84949 - 10$$

$$\text{colog } \sin 67^{\circ} 30' = \underline{0.03438}$$

$$\log b = 1.78696$$

$$b = 61.23.$$

Let B' be the top of the column. Then $\triangle AB'C$ is isosceles since $A = B' = 45^{\circ}$.

Therefore the height of the column is 61.23 ft.

Exercise 61. Miscellaneous Applications (Page 141)

1. In the trapezoid $ABCD$ it is known that $\angle A = 90^{\circ}$, $\angle B = 32^{\circ} 25'$, $AB = 324.35$ ft., and $CD = 208.15$ ft. Find the area.

From C draw the altitude $CK = h$.
 $KB = 324.35 - 208.15 = 116.2$ ft.

$$\tan B = \frac{h}{116.2}$$

$$h = 116.2 \tan B.$$

$$\log 116.2 = 2.06521$$

$$\log \tan B = 9.80279 - 10$$

$$\log h = 1.86800$$

$$h = 73.79.$$

$$S = \frac{1}{2} h (208.15 + 324.35)$$

$$= \frac{1}{2} h \times 532.5.$$

$$\log h = 1.86800$$

$$\log 532.5 = 2.72632$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log S = 4.29329$$

$$S = 19,647 \text{ sq. ft.}$$

2. Find the area of a regular pentagon of which each side is 4 in.

Let AB be the side of the pentagon. Draw OA and OB , radii of the circumscribed circle, and h the altitude of the $\triangle$ thus formed.

$$\angle AOB = 72^{\circ}.$$

$$\cot 36^{\circ} = \frac{h}{\frac{1}{2} AB}$$

$$h = \frac{1}{2} AB \cot 36^{\circ}$$

$$= 2 \cot 36^{\circ}.$$

$$S = \frac{1}{2} hp$$

$$= 10 h$$

$$= 20 \cot 36^{\circ}.$$

$$\log 20 = 1.30103$$

$$\log \cot 36^{\circ} = 0.13874$$

$$\log S = 1.43977$$

$$S = 27.527 \text{ sq. in.}$$

3. Find the area of a regular inscribed and circumscribed regular hexagon of which each side is 4 in. polygons of half the number of sides.

Let $AB = 4$ in., the side of a regular hexagon; OA and OB radii of the circumscribed circle; h the altitude of the $\triangle AOB$.

Then $\angle AOB = 60^\circ$.

$$\frac{\frac{1}{2}AB}{h} = \tan 30^\circ.$$

$$h = \frac{2}{\tan 30^\circ} = 2\sqrt{3}.$$

$$S = \frac{1}{2}hp = 24\sqrt{3}.$$

$$\log \sqrt{3} = 0.23856$$

$$\log 24 = 1.38021$$

$$\log S = 1.61877$$

$$S = 41.569 \text{ sq. in.}$$

4. The area of a regular polygon inscribed in a circle is to that of the circumscribed regular polygon of the same number of sides as 3 to 4. Find the number of sides.

$$\frac{n}{2}r^2 \sin \frac{360^\circ}{n} : nr^2 \tan \frac{180^\circ}{n} = 3 : 4.$$

$$2nr^2 \sin \frac{360^\circ}{n} = 3nr^2 \tan \frac{180^\circ}{n}.$$

$$2 \sin \frac{360^\circ}{n} = 3 \tan \frac{180^\circ}{n}.$$

$$4 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n} = 3 \frac{\sin \frac{180^\circ}{n}}{\cos \frac{180^\circ}{n}}.$$

$$4 \cos^2 \frac{180^\circ}{n} = 3.$$

$$\cos \frac{180^\circ}{n} = \frac{1}{2}\sqrt{3}.$$

$$\frac{180^\circ}{n} = 30^\circ.$$

$$n = 6.$$

The number of sides is 6

5. The area of a regular polygon inscribed in a circle is the geometric mean between the areas of the

inscribed and circumscribed regular polygons of half the number of sides.

Area of inscribed polygon of $2n$ sides

$$= nr^2 \sin \frac{180^\circ}{n}.$$

Area of inscribed polygon of n sides

$$= \frac{n}{2}r^2 \sin \frac{360^\circ}{n}.$$

Area of circumscribed polygon of n sides

$$= nr^2 \tan \frac{180^\circ}{n}.$$

$$\frac{n}{2}r^2 \sin \frac{360^\circ}{n} \times nr^2 \tan \frac{180^\circ}{n}$$

$$= \frac{n^2 r^4}{2} \sin \frac{360^\circ}{n} \tan \frac{180^\circ}{n}$$

$$= n^2 r^4 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n} \frac{\sin \frac{180^\circ}{n}}{\cos \frac{180^\circ}{n}}$$

$$= n^2 r^4 \sin^2 \frac{180^\circ}{n}$$

$$= \left(nr^2 \sin \frac{180^\circ}{n} \right)^2.$$

6. Find the ratio of a square inscribed in a circle to a square circumscribed about the same circle. Find the ratio of the perimeters.

Let a be the apothem of the inscribed square, r will be the apothem of the circumscribed square.

$2a$ = side of inscribed square.

$2r$ = side of circumscribed square

$$\therefore \frac{AB^2}{A'B'^2} = \frac{4a^2}{4r^2} = \frac{a^2}{r^2}.$$

$$\frac{a}{r} = \cos 45^\circ = \frac{1}{2}\sqrt{2}.$$

$$\therefore \frac{AB^2}{A'B'^2} = \frac{1}{2} = .5.$$

The inscribed square : circumscribed square = 1 : 2.

$$\frac{8a}{8r} = \frac{a}{r} = \frac{1}{2}\sqrt{2}.$$

$\therefore$ the ratio of the perimeters = $\frac{1}{2}\sqrt{2}$.

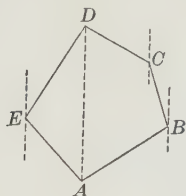
7. The square circumscribed about a circle is four thirds the inscribed regular dodecagon.

$$\text{Area of square} = 4r^2.$$

By Prob. 44, Ex. 58

$$\begin{aligned}\text{Area of dodecagon} &= \frac{12}{2} r^2 \sin \frac{360^\circ}{12} \\ &= 6r^2 \sin 30^\circ \\ &= 3r^2.\end{aligned}$$

8. In finding the area of a field $ABCDE$ a surveyor measured the lengths of the sides and the angle which each side makes with the meridian (north and south) line through its extremities. AD happened to be a meridian line. Show how he could compute the area.



Draw AC .

$\angle E = 180^\circ - \text{sum of } \angle \text{ made by } DE \text{ and } AE \text{ with meridian.}$

In $\triangle AED$,

$$S = \frac{1}{2} ED \times EA \sin E.$$

$$AD = \frac{AE \sin E}{\sin EDA}.$$

In $\triangle ADC$,

$$S = \frac{1}{2} AD \times DC \sin ADC.$$

In $\triangle ABC$,

$$S = \frac{1}{2} AB \times BC \sin B.$$

$\angle B = 180^\circ - \text{sum of } \angle \text{ made by } BC \text{ and } BA \text{ with meridian.}$

The sum of the three Δ = area of field.

9. Two sides of a triangle are 3 and 12, and the included angle is 30° . Find the hypotenuse of the isosceles right triangle of equal area.

$$b = 3, \quad c = 12.$$

Let x = side of equivalent isosceles rt. Δ .

$$\text{Then } \frac{x^2}{2} = \text{area,}$$

and $x\sqrt{2}$ = hypotenuse.

$$\frac{h}{3} = \sin 30^\circ.$$

$$h = 3 \times \frac{1}{2} = 1\frac{1}{2}.$$

$$S = \frac{1}{2} ch = 9.$$

$$\frac{x^2}{2} = 9.$$

$$x^2 = 18.$$

$$x = 3\sqrt{2}$$

but $x\sqrt{2}$ = hypotenuse.

Substituting,

$$3\sqrt{2} \times \sqrt{2} = \text{hypotenuse.}$$

$\therefore$ hypotenuse = 6.

10. In the quadrilateral $ABCD$ we have given AB , BC , $\angle A$, $\angle B$, and $\angle C$. Show how to find the area of the quadrilateral.

$$\angle D = 360^\circ - (A + B + C).$$

Draw diagonal AC .

$$\frac{AB - BC}{AB + BC} = \frac{\tan \frac{1}{2} (ACB - BAC)}{\tan \frac{1}{2} (ACB + BAC)}.$$

From this formula we can find $\angle ACB$ and BAC .

By [47],

$$AC = \frac{AB \sin B}{\sin ACB}.$$

$$\angle DCA = \angle C - \angle ACB.$$

$$\angle DAC = \angle A - \angle BAC.$$

$$S \text{ of } \triangle ABC = \frac{1}{2} BC \times AB \sin B.$$

$$S \text{ of } \triangle ADC = \frac{AC^2 \sin ACD \sin CAD}{2 \sin (ACD + CAD)}.$$

Add areas of the triangles for area of the quadrilateral.

11. In Ex. 10, suppose $AB = 175$ ft., $BC = 198$ ft., $\angle A = 95^\circ$, $\angle B = 92^\circ 15'$, and $\angle C = 96^\circ 45'$. What is the area?

$$\angle D = 360^\circ - (A + B + C) = 76^\circ.$$

Draw diagonal AC .

$$BAC + ACB = 180^\circ - B = 87^\circ 45'.$$

$$\frac{BC - AB}{BC + AB} = \frac{\tan \frac{1}{2}(BAC - ACB)}{\tan \frac{1}{2}(BAC + ACB)}.$$

$$\tan \frac{1}{2}(BAC - ACB) = \frac{(BC - AB) \tan \frac{1}{2}(BAC + ACB)}{BC + AB}.$$

$$\log (BC - AB) = 1.36173$$

$$\log \tan \frac{1}{2}(BAC + ACB) = 9.98294 - 10$$

$$\text{colog } (BC + AB) = 7.42829 - 10$$

$$\log \tan \frac{1}{2}(BAC - ACB) = 8.77296$$

$$\frac{1}{2}(BAC - ACB) = 3^\circ 28' 34''.$$

$$BAC - ACB = 6^\circ 47' 8''$$

$$BAC + ACB = 87^\circ 45'$$

$$2BAC = 94^\circ 32' 8''$$

$$BAC = 47^\circ 16' 4''.$$

$$ACB = 40^\circ 28' 56''.$$

$$AC = \frac{AB \sin B}{\sin ACB}.$$

$$\log AB = 2.24304$$

$$\log \sin B = 9.99967 - 10$$

$$\text{colog } \sin ACB = 0.18761$$

$$\log AC = 2.43032$$

$$AC = 269.35.$$

$$\angle ACD = 96^\circ 45' - 40^\circ 28' 56'' \quad S \text{ of } \triangle ADC = \frac{\overline{AC}^2 \sin ACD \sin CAD}{2 \sin (ACD + CAD)}$$

$$= 56^\circ 16' 4''.$$

$$\angle CAD = 95^\circ - 47^\circ 16' 4''$$

$$= 47^\circ 43' 56''.$$

$$\log \overline{AC}^2 = 4.86064$$

$$\log \sin ACD = 9.91994 - 10$$

$$\log \sin CAD = 9.86923 - 10$$

$$\text{colog } 2 = 9.69897 - 10$$

$$S \text{ of } \triangle ABC = \frac{1}{2} AB \times BC \sin B.$$

$$\log AB = 2.24304$$

$$\log BC = 2.29667$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log \sin B = 9.99967 - 10$$

$$\log S = 4.23835$$

$$\text{colog } \sin (ACD + CAD) = 0.01310$$

$$\log S \text{ of } \triangle ADC = 4.36188$$

$$S \text{ of } \triangle ADC = 23,008.$$

$$S \text{ of } ABCD$$

$$= S \text{ of } \triangle ABC + S \text{ of } \triangle ADC$$

$$= 17,312 + 23,008 = 40,320 \text{ sq. ft.}$$

$$\text{Area of } ABCD = 40,320 \text{ sq. ft.}$$

$$S \text{ of } \triangle ABC = 17,312 \text{ sq. ft.}$$

Exercise 62. Area of a Field (Page 142)

1. Find the number of acres in a triangular field of which the sides are 14 ch., 16 ch., and 20 ch.

$$\begin{aligned}
 s &= \frac{1}{2}(14 + 16 + 20) = 25. \\
 S &= \sqrt{s(s-a)(s-b)(s-c)} \\
 &= \sqrt{25 \times 11 \times 9 \times 5} \\
 &= \sqrt{275 \times 45}. \\
 \log 275 &= 2.43933 \\
 \log 45 &= 1.65321 \\
 \log S^2 &= 4.09254 \\
 \log S &= 2.04627 \\
 S &= 111.2425 \text{ sq. ch.} \\
 &= 11.124 \text{ A.}
 \end{aligned}$$

2. Find the number of acres in a triangular field having two sides 16 ch. and 30 ch., and the included angle $64^\circ 15'$.

$$\begin{aligned}
 S &= \frac{1}{2}bc \sin A. \\
 S &= \frac{1}{2}(16 \times 30 \sin A). \\
 \log 16 &= 1.20412 \\
 \log 30 &= 1.47712 \\
 \log \sin A &= 9.95458 - 10 \\
 \text{colog } 2 &= 9.69897 - 10 \\
 \log S &= 2.33479 \\
 S &= 216.173 \text{ sq. ch.} \\
 &= 21.617 \text{ A.}
 \end{aligned}$$

3. Find the number of acres in a triangular field having two angles 68.4° and 47.2° , and the included side 20 ch.

By [56],

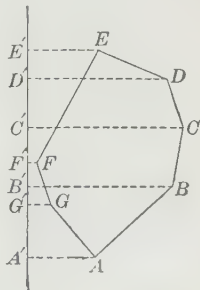
$$\begin{aligned}
 S &= \frac{c^2 \sin A \sin B}{2 \sin(A+B)}. \\
 \log c^2 &= 2.60206 \\
 \log \sin A &= 9.96838 - 10 \\
 \log \sin B &= 9.86554 - 10 \\
 \text{colog } 2 &= 9.69897 - 10 \\
 \text{colog } \sin(A+B) &= 0.04487 \\
 \log S &= 2.17982 \\
 S &= 151.29 \text{ sq. ch.}
 \end{aligned}$$

Area of field = 15.129 A.

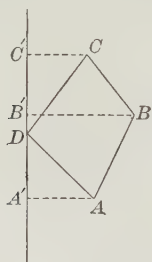
4. Required the area of the field described in § 123, knowing that $AA' = 8$ ch., $BB' = 12$ ch., $CC' = 13$ ch., $DD' = 12$ ch., $EE' = 8$ ch., $FF' = 1$ ch., $GG' = 2$ ch., $A'G' = 6$ ch., $G'B' = 1.5$ ch., $B'F' = 2.3$ ch., $F'C' = 3$ ch., $C'D' = 4$ ch., $D'E' = 2.9$ ch.

$$\begin{aligned}
 S \text{ of } D'E &= \frac{1}{2}(8 + 12) \times 2.9 = 29 \text{ sq. ch.} \\
 S \text{ of } C'D &= \frac{1}{2}(12 + 13) \times 4 = 50 \text{ sq. ch.} \\
 S \text{ of } B'C &= \frac{1}{2}(13 + 12) \times 5.3 = 66.25 \text{ sq. ch.} \\
 S \text{ of } A'B &= \frac{1}{2}(12 + 8) \times (7.5) = 75 \text{ sq. ch.} \\
 S \text{ of } ABCDEE'A' &= 220.25 \text{ sq. ch.} \\
 S \text{ of } F'E &= \frac{1}{2}(8 + 1) \times (9.9) = 44.55 \text{ sq. ch.} \\
 S \text{ of } G'F &= \frac{1}{2}(1 + 2) \times (3.8) = 5.7 \text{ sq. ch.} \\
 S \text{ of } A'G &= \frac{1}{2}(2 + 8) \times 6 = 30 \text{ sq. ch.} \\
 S \text{ of } AGFEE'A' &= 80.25 \text{ sq. ch.}
 \end{aligned}$$

Area of field = $220.25 - 80.25 = 140$ sq. ch. = 14 A.



5. In a quadrangular field $ABCD$, AB runs N. 27° E. 12.5 ch., BC runs N. 30° W. 10 ch., CD runs S. 37° W. 15 ch., and DA runs S. 47° E. 11.5 ch. Find the area.



Through D draw the meridian line $A'C'$. Perpendicular to the meridian draw $C'C$, $B'B$, and $A'A$.

$$\begin{aligned}\angle DCC' &= 90^\circ - 37^\circ \\ &= 53^\circ.\end{aligned}$$

$$C'C = CD \cos 53^\circ.$$

$$\log 15 = 1.17609$$

$$\log \cos 53^\circ = \underline{9.77946 - 10}$$

$$\log C'C = 0.95555$$

$$C'C = 9.0272.$$

$$B'B = C'C + CB \cos 60^\circ$$

$$= 9.0272 + 5$$

$$= 14.0272.$$

$$B'C' = CB \sin 60^\circ = 8.66.$$

$$\begin{aligned}S \text{ of } B'BCC' &= \frac{1}{2}(B'B + C'C) \times B'C' \\ &= 11.527 \times 8.66.\end{aligned}$$

$$\log 11.527 = 1.06172$$

$$\log 8.66 = 0.93752$$

$$\log S = \underline{1.99924}$$

$$S = 99.825.$$

$$A'A = DA \sin 47^\circ.$$

$$\log 11.5 = 1.06070$$

$$\log \sin 47^\circ = \underline{9.86413 - 10}$$

$$\log A'A = 0.92483$$

$$A'A = 8.4106.$$

$$A'B' = AB \cos 27^\circ.$$

$$\log 12.5 = 1.09691$$

$$\log \cos 27^\circ = \underline{9.94988 - 10}$$

$$\log A'B' = 1.04679$$

$$A'B' = 11.137.$$

$$\begin{aligned}S \text{ of } A'ABB' &= \frac{1}{2}(A'A + B'B) \times A'B' \\ &= 11.219 \times 11.137.\end{aligned}$$

$$\log 11.219 = 1.04995$$

$$\log 11.137 = \underline{1.04677}$$

$$\log S = 2.09672$$

$$S = 124.95.$$

$$\begin{aligned}\text{Area of } A'ABCC' &= 124.95 + 99.825 \\ &= 224.775 \text{ sq. ch.}\end{aligned}$$

$$S \text{ of } \triangle CC'D = \frac{1}{2} DC' \times C'C.$$

$$DC' = CD \cos 37^\circ.$$

$$\log C'C = 0.95555$$

$$\log 15 = 1.17609$$

$$\log \cos 37^\circ = \underline{9.90235 - 10}$$

$$\text{colog } 2 = \underline{9.69897 - 10}$$

$$\log S = 1.73296$$

$$S = 54.07.$$

$$S \text{ of } \triangle D'AA = \frac{1}{2} A'A \times A'D.$$

$$A'D = DA \cos 47^\circ.$$

$$\log A'A = 0.92483$$

$$\log 11.5 = 1.06070$$

$$\log \cos 47^\circ = \underline{9.83378 - 10}$$

$$\text{colog } 2 = \underline{9.69897 - 10}$$

$$\log S = 1.51828$$

$$S = 32.982.$$

$$\begin{aligned}\text{Area of } ABCD &= 224.775 \\ &\quad - (54.07 + 32.982) \\ &= 137.723 \text{ sq. ch.} \\ &= 13.7723 \text{ A.}\end{aligned}$$

$$\text{Area of the field} = 13.77 \text{ A.}$$

6. In a triangular field ABC , AB runs N. 10° E. 30 ch., BC runs S. 30° W. 19 ch., and CA runs S. 30° E. 16 ch. Find the area.

$$s = \frac{1}{2}(30 + 19 + 16) = 32.5.$$

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$s - a = 13.5.$$

$$s - b = 16.5.$$

$$s - c = 2.5.$$

$$\log 32.5 = 1.51188$$

$$\log 13.5 = 1.13033$$

$$\log 16.5 = 1.21748$$

$$\log 2.5 = \underline{0.39794}$$

$$\log S^2 = 4.25763$$

$$\log S = 2.12882.$$

$$S = 134.53 \text{ sq. ch.}$$

$$= 13.453 \text{ A.}$$

Area of the field = 13.453 A.

7. In a field $ABCD$, AB runs E. 10 ch., BC runs N. 12 ch., CD runs S. $68^\circ 12'$ W. 10.77 ch., and DA runs S. 8 ch. Find the area.

$ABCD$ is a trapezoid.

$$\angle A = 90^\circ, \angle B = 90^\circ.$$

AD is $\parallel$ to BC .

$$S = \frac{1}{2}(AD + BC) \times AB$$

$$= \frac{1}{2}(8 + 12) \times 10$$

$$= 100 \text{ sq. ch.}$$

$$= 10 \text{ A.}$$

Area of the field = 10 A.

8. A field is in the form of a right triangle, of which the sides are 15 ch., 20 ch., and 25 ch. From the vertex

of the right angle a line is run to the hypotenuse, making an angle of 30° with the side that is 15 ch. long. Find the area of each of the triangles into which the field is divided.

Draw BD making an angle of 30° with BC ; find area of $\triangle ABD$ and BDC .

$$S \text{ of } \triangle ABC = \frac{1}{2}(AB \times BC) \\ = 150 \text{ sq. ch.}$$

By [56], S of $\triangle ABD$

$$= \frac{\overline{AB}^2 \sin A \sin B}{2 \sin(A + B)}.$$

$$\sin A = \frac{BC}{AC} = .8.$$

$$A = 53^\circ 8'.$$

$$\log \overline{AB}^2 = 2.35218$$

$$\log \sin A = 9.90311 - 10$$

$$\log \sin B = 9.69897 - 10$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\text{colog } \sin(A + B) = \underline{0.00313}$$

$$\log S = 1.65636$$

$$S = 45.327 \text{ sq. ch.}$$

S of $\triangle BCD$

$$= S \text{ of } \triangle ABC - S \text{ of } \triangle ABD$$

$$= 150 - 45.327$$

$$= 104.673 \text{ sq. ch.}$$

$$\text{Area of } \triangle ABD = 4.535 \text{ A.}$$

$$\text{Area of } \triangle BDC = 10.47 \text{ A.}$$

Using a protractor, draw to scale the fields referred to in the following examples, and find the areas:

9. AB , N. 72° E. 18 ch., BC , N. 10° E. 12.5 ch., CD , N. 68° W. 21 ch., DA , S. 12° E. 26.3 ch.

$$\angle A = 72^\circ + 12^\circ = 84^\circ.$$

$$\angle C = 180^\circ - (68^\circ + 10^\circ) = 102^\circ.$$

Draw diagonal DB .

$$S \text{ of } \triangle ABD = \frac{1}{2} AB \times DA \sin A$$

$$= 9 \times 26.3 \sin A.$$

$$\log 9 = 0.95424$$

$$\log 26.3 = 1.41996$$

$$\log \sin A = \underline{9.99761 - 10}$$

$$\log S = 2.37181$$

$$S = 235.4 \text{ sq. ch.}$$

$$S \text{ of } \triangle BCD = \frac{1}{2} BC \times CD \sin C.$$

$$\log 12.5 = 1.09691$$

$$\log 21 = 1.32222$$

$$\log \sin C = 9.99040 - 10$$

$$\text{colog } 2 = \underline{9.69897 - 10}$$

$$\log S = 2.10850$$

$$S = 128.379 \text{ sq. ch.}$$

$$S \text{ of } ABCD = 235.4 + 128.379$$

$$= 363.779 \text{ sq. ch.}$$

$$\text{Area of field} = 36.38 \text{ A.}$$

10. AB , N. 45° E. 10 ch.,
 BC , S. 75° E. 11.55 ch.,
 CD , S. 15° W. 18.21 ch.,
 DA , N. 45° W. 19.11 ch.

Draw diagonal BD .

$$\angle A = 45^\circ + 45^\circ = 90^\circ.$$

$$\angle C = 75^\circ + 15^\circ = 90^\circ.$$

$$\begin{aligned} S \text{ of } \triangle BAD &= \frac{1}{2} AB \times AD \\ &= 5 \times 19.11 \\ &= 95.55 \text{ sq. ch.} \end{aligned}$$

$$\begin{aligned} S \text{ of } \triangle BCD &= \frac{1}{2} BC \times CD \\ &= \frac{1}{2} (11.55 \times 18.21). \end{aligned}$$

$$\log BC = 1.06258$$

$$\log CD = 1.26031$$

$$\text{colog } 2 = \frac{9.69897 - 10}{}$$

$$\log S = 2.02186$$

$$S = 105.16 \text{ sq. ch.}$$

$$\begin{aligned} S \text{ of } ABCD &= 95.55 + 105.16 \\ &= 200.71 \text{ sq. ch.} \end{aligned}$$

Area of field = 20.07 A.

11. AB , N. $5^\circ 30'$ W. 6.08 ch.,
 BC , S. $82^\circ 30'$ W. 6.51 ch.,
 CD , S. 3° E. 5.33 ch.,
 DA , E. 6.72 ch.

Draw diagonal AC .

$$\begin{aligned} \angle B &= 82^\circ 30' + 5^\circ 30' \\ &= 88^\circ. \end{aligned}$$

$$\begin{aligned} \angle D &= 90^\circ + 3^\circ \\ &= 93^\circ. \end{aligned}$$

$$S \text{ of } \triangle ABC = \frac{1}{2} AB \times BC \sin B.$$

$$\log AB = 0.78390$$

$$\log BC = 0.81358$$

$$\log \sin B = 9.99974 - 10$$

$$\text{colog } 2 = \frac{9.69897 - 10}{}$$

$$\log S = 1.29619$$

$$S = 19.7782 \text{ sq. ch.}$$

$$S \text{ of } \triangle ACD = \frac{1}{2} CD \times DA \sin D.$$

$$\log CD = 0.72673$$

$$\log DA = 0.82737$$

$$\log \sin D = 9.99940 - 10$$

$$\text{colog } 2 = \frac{9.69897 - 10}{}$$

$$\log S = 1.25247$$

$$S = 17.884 \text{ sq. ch.}$$

$$\begin{aligned} S \text{ of } ABCD &= 19.778 + 17.884 \\ &= 37.662 \text{ sq. ch.} \\ &= 3.766 \text{ A.} \end{aligned}$$

Area of field = 3.77 A.

12. AB , N. $6^\circ 15'$ W. 6.31 ch.,
 BC , S. $81^\circ 50'$ W. 4.06 ch.,
 CD , S. 5° E. 5.86 ch.,
 DA , N. $88^\circ 30'$ E. 4.12 ch.

Draw diagonal AC .

$$\begin{aligned} \angle B &= 81^\circ 50' + 6^\circ 15' \\ &= 88^\circ 5'. \end{aligned}$$

$$\begin{aligned} \angle D &= 88^\circ 30' + 5^\circ \\ &= 93^\circ 30'. \end{aligned}$$

$$S \text{ of } \triangle ADC = \frac{1}{2} DA \times CD \sin D.$$

$$\log DA = 0.61490$$

$$\log CD = 0.76790$$

$$\log \sin D = 9.99919 - 10$$

$$\text{colog } 2 = \frac{9.69897 - 10}{}$$

$$\log S = 1.08096$$

$$S = 12.049 \text{ sq. ch.}$$

$$S \text{ of } \triangle ABC = \frac{1}{2} AB \times BC \sin B.$$

$$\log AB = 0.80003$$

$$\log BC = 0.60853$$

$$\log \sin B = 9.99976 - 10$$

$$\text{colog } 2 = \frac{9.69897 - 10}{}$$

$$\log S = 1.10729$$

$$S = 12.802 \text{ sq. ch.}$$

$$\begin{aligned} S \text{ of } ABCD &= 12.049 + 12.802 \\ &= 24.851 \text{ sq. ch.} \\ &= 2.4851 \text{ A.} \end{aligned}$$

Area of field = 2.485 A.

13. A farm is bounded and described as follows: Beginning at the southwest corner of lot No. 13, thence N. $1\frac{1}{4}^{\circ}$ E. 132 rods and 23 links to a stake in the west boundary line of said lot; thence S. 89° E. 32 rods and $15\frac{4}{10}$ links to a stake; thence N. $1\frac{1}{4}^{\circ}$ E. 29 rods and 15 links to a stake in the north boundary line of said lot; thence S. 89° E. 61 rods and $18\frac{6}{10}$ links to a stake; thence S. $32\frac{1}{2}^{\circ}$ W. 54 rods to a stake; thence S. $35\frac{1}{4}^{\circ}$ E. 22 rods and 4 links to a stake; thence S. 48° E. 33 rods and 2 links to a stake; thence S. $7\frac{1}{2}^{\circ}$ W. 76 rods and 20 links to a stake in the south boundary line of said lot; thence N. 89° W. 96 rods and 10 links to the place of beginning. Containing 85.65 acres, more or less. Verify the area given and plot the farm.

Let $ABEK$ represent the farm. Extend AB and ED to B' .

Divide the figure into trapezoids by the lines $F'F$, $G'G$, and $H'H \parallel$ to AK , which is $\parallel$ to $B'E$.

Draw the $\perp$ s FP , OF , NG , and KM .

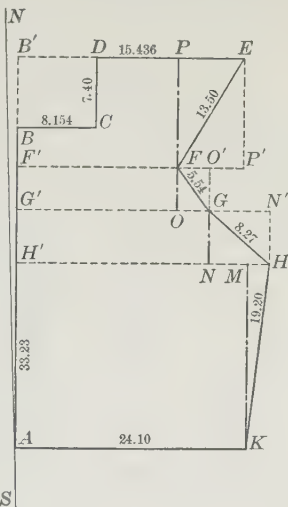
Extend $F'F$ to P' and draw $EP' \parallel$ to AB .

Draw $GO' \parallel$ to AB .

Extend $G'G$ to N' and draw $HN' \parallel$ to AB .

Reduce all dimensions to chains.

In the rt. $\triangle FPE$, $\angle PEF = 58^\circ 30'$. S



$$\frac{PF}{EF} = \sin 58^\circ 30'.$$

$$PF = 13.50 \sin 58^\circ 30'.$$

$$\log 13.50 = 1.13033$$

$$\log \sin 58^\circ 30' = 9.93077$$

$$\log PF = \overline{1.06110}$$

In the $\triangle FP'E$, $\angle FP'E = 90^\circ 15'$ and $\angle FEP' = 31^\circ 15'$.

$$\frac{FP'}{EF} = \frac{\sin FEP'}{\sin FP'E},$$

$$FP' = \frac{13.50 \sin 31^\circ 15'}{\sin 90^\circ 15'}$$

$$\log 13.50 = 1.13033$$

$$\log \sin 31^{\circ} 15' = 9.71498$$

$$\text{colog } \sin 90^\circ 15' = 0.00000$$

$$\log FP' = \overline{0.84531}$$

$$FP' = 7.0034$$

$$\angle GHN = 41^\circ.$$

$$GN = 8.27 \sin 41^\circ.$$

$$\log 8.27 = 0.91751$$

$$\log \sin 41^\circ = 9.81694$$

$$\log GN = 0.73445$$

$$\angle GN'H = 89^\circ 45'.$$

$$\angle GHN' = 49^\circ 15'.$$

$$\frac{GN'}{GH} = \frac{\sin 49^\circ 15'}{\sin 89^\circ 45'}.$$

$$GN' = GH \sin 49^\circ 15'.$$

$$\log 8.27 = 0.91751$$

$$\log \sin 49^\circ 15' = 9.87942$$

$$\log GN' = 0.79693$$

$$GN' = 6.2651.$$

$$H'H = G'G + GN' = 19.8819 + 6.2651 = 26.147$$

$$\begin{aligned} \text{Area of } H'HGG' &= \frac{G'G + H'H}{2} \times GN \\ &= \frac{19.8819 + 26.1470}{2} \times GN \\ &= 23.0145 \times GN. \end{aligned}$$

$$\log 23.0145 = 1.36201$$

$$\log GN = 0.73445$$

$$\log \text{area } H'HGG' = 2.09646$$

$$\text{Area } H'HGG' = 124.871 \text{ sq. ch.}$$

$$\angle MHK = 83^\circ 30'.$$

$$KM = 19.20 \sin 83^\circ 30'.$$

$$\log 19.20 = 1.28330$$

$$\log \sin 83^\circ 30' = 9.99720$$

$$\log KM = 1.28050$$

$$\begin{aligned} \text{Area of } AKHH' &= \frac{H'H + AK}{2} \times KM \\ &= \frac{26.147 + 24.10}{2} \times KM \\ &= 25.124 \times KM. \end{aligned}$$

$$\log 25.124 = 1.40009$$

$$\log KM = 1.28050$$

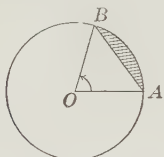
$$\log \text{area } AKHH' = 2.68059$$

$$\text{Area } AKHH' = 479.278 \text{ sq. ch.}$$

$$\begin{aligned} \text{Area of farm} &= 170.8864 + 81.465 + 124.871 + 479.278 \\ &= 856.5004 \text{ sq. ch.} \\ &= 85.65 \text{ A.} \end{aligned}$$

Exercise 63. The Circle (Page 144)

1. A sector of a circle of radius 8 in. has an angle of 62.5° . A chord joining the extremities of the radii forming the sector cuts off a segment. What is the area of this segment?



$$\angle BOA = 62.5^\circ.$$

$$\text{Area of a circle} = \pi r^2.$$

$$\begin{aligned} \text{Area of the sector} &= \frac{62.5}{360} \times \pi r^2. \\ \log 62.5 &= 1.79588 \\ \text{colog } 360 &= 7.44370 - 10 \\ \log \pi &= 0.49715 \\ \log r^2 &= \frac{1.80618}{} \\ \log S &= \frac{1.54291}{} \end{aligned}$$

$$\log 62.5 = 1.79588$$

$$\text{colog } 360 = 7.44370 - 10$$

$$\log \pi = 0.49715$$

$$\log r^2 = \frac{1.80618}{}$$

$$\log S = \frac{1.54291}{}$$

$$\text{Area of sector} = 34.907 \text{ sq. in.}$$

$$\begin{aligned} S \text{ of } \triangle BOA &= \frac{1}{2} \times 8 \times 8 \times \sin 62.5^\circ \\ &= 32 \sin 62.5^\circ. \end{aligned}$$

$$\log 32 = 1.50515$$

$$\log \sin 62^\circ 30' = 9.94793 - 10$$

$$\log S = 1.45308$$

$$S = 28.3844 \text{ sq. in.}$$

$$\text{Area of segment}$$

$$= 34.9067 - 28.3844 = 6.5223 \text{ sq. in.}$$

2. A sector of a circle of diameter 9.2 in. has an angle of $29^\circ 42'$. A chord joining the extremities of the radii forming the sector cuts off a segment. What is the area of the remainder of the circle?

$$\text{Given } r = 4.6 \text{ in.,}$$

$$\angle O = 29^\circ 42'.$$

$$\text{Area of a circle} = \pi r^2.$$

$$\log \pi = 0.49715$$

$$\log r^2 = 1.32552$$

$$\log S = 1.82267$$

$$S = 66.4767 \text{ sq. in.}$$

$$\text{Area of } \triangle OBA = \frac{1}{2} r^2 \sin O.$$

$$\log r^2 = 1.32552$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log \sin O = 9.69501 - 10$$

$$\log S = 0.71950$$

$$S = 5.2420 \text{ sq. in.}$$

$$\text{Area of sector} = \pi r^2 \times \frac{29.7}{360}.$$

$$\log \pi r^2 = 1.82267$$

$$\log 29.7 = 1.47276$$

$$\text{colog } 360 = 7.44370 - 10$$

$$\log S = 0.73913$$

$$S = 5.4844 \text{ sq. in.}$$

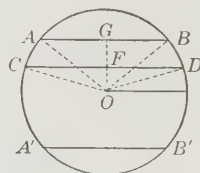
$$\text{Area of segment} = 5.4844 - 5.2420$$

$$= 0.2424 \text{ sq. in.}$$

$$\text{Area of remainder of circle}$$

$$= 66.4767 - 0.2424 = 66.2343 \text{ sq. in.}$$

3. In a circle of radius 3.5 in., what is the area included between two parallel chords of 6 in. and 5 in. respectively? (Give two answers.)



$$\text{Given } r = 3.5 \text{ in.,}$$

$$AB = 5 \text{ in.,}$$

$$A'B' = 5 \text{ in.,}$$

$$CD = 6 \text{ in.}$$

Required the area of $ABCD$ and $A'B'CD$.

$$\sin GOB = \frac{2.5}{3.5}.$$

$$\log 2.5 = 0.39794$$

$$\text{colog } 3.5 = 9.45593 - 10$$

$$\log \sin GOB = 9.85387 - 10$$

$$\angle GOB = 45^\circ 35' 4.6''.$$

$$\angle AOB = 91^\circ 10' 9''.$$

$$\sin FOD = \frac{3}{3.5}.$$

$$\log 3 = 0.47712$$

$$\text{colog } 3.5 = 9.45593 - 10$$

$$\log \sin FOD = 9.93305 - 10$$

$$\angle FOD = 58^\circ 59' 45''.$$

$$\angle COD = 117^\circ 59' 30''.$$

Area of sector AOB

$$= \pi r^2 \times \frac{91^\circ 10' 9''}{360}$$

$$= \pi r^2 \times \frac{91.169}{360}.$$

$$\log \pi = 0.49715$$

$$\log r^2 = 1.08814$$

$$\log 91.169 = 1.95984$$

$$\text{colog } 360 = 7.44370 - 10$$

$$\log S = 0.98883$$

$$S = 9.7463 \text{ sq. in.}$$

$$\text{Area of } \triangle AOB = \frac{1}{2} r^2 \sin AOB.$$

$$\log r^2 = 1.08814$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log \sin AOB = 9.99991 - 10$$

$$\log S = 0.78702$$

$$S = 6.1238 \text{ sq. in.}$$

Area of segment AB

$$= 9.7463 - 6.1238$$

$$= 3.6225 \text{ sq. in.}$$

Area of sector COD

$$= \pi r^2 \times \frac{117^\circ 59' 45''}{360^\circ}$$

$$= \pi r^2 \times \frac{117.996}{360}.$$

$$\log \pi = 0.49715$$

$$\log r^2 = 1.08814$$

$$\log 117.996 = 2.07186$$

$$\text{colog } 360 = 7.44370 - 10$$

$$\log S = 1.10085$$

Area of sector COD

$$= 12.6138 \text{ sq. in.}$$

Area of $\triangle COD$

$$= \frac{1}{2} r^2 \sin 117^\circ 59' 45''.$$

$$\log r^2 = 1.08814$$

$$\text{colog } 2 = 9.69897 - 10$$

$$\log \sin 117^\circ 59' 45'' = 9.94595 - 10$$

$$\log S = 0.73306$$

$$\text{Area of } \triangle COD = 5.4083 \text{ sq. in.}$$

$$\text{Area of } ABCD = \text{sector } COD - (\triangle COD + \text{segment } AB)$$

$$= 12.6138 - (5.4083 + 3.6225)$$

$$= 3.583 \text{ sq. in.}$$

$$\text{Area of } A'B'CD = \text{area of circle} - (\text{segment } A'B' + \text{segment } CABD).$$

$$\text{Area of segment } CABD$$

$$= \text{sector } COD - \triangle COD$$

$$= 12.6138 - 5.4083$$

$$= 7.2055 \text{ sq. in.}$$

$$\text{Area of circle} = \pi r^2$$

$$\log \pi = 0.49715$$

$$\log r^2 = 1.08814$$

$$\log S = 1.58529$$

$$\text{Area of circle} = 38.4845 \text{ sq. in.}$$

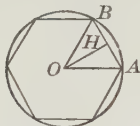
$$\text{Area of } A'B'CD$$

$$= 38.4845 - (3.6225 + 7.2055)$$

$$= 27.6565 \text{ sq. in.}$$

Area of part between parallel chords is 3.583 sq. in. when chords are on same side of diameter and 27.6565 sq. in. when they are on opposite sides.

4. A regular hexagon is inscribed in a circle of radius 4 in. What is the area of that part of the circle not covered by the hexagon?



$$\begin{aligned}\text{Area of } \triangle AOB &= \frac{1}{2} AB \times OH \\ &= 4\sqrt{3}.\end{aligned}$$

$$\begin{aligned}\text{Area of hexagon} &= 6 \times 4\sqrt{3} \\ &= 24\sqrt{3}.\end{aligned}$$

$$\begin{aligned}\log 24 &= 1.38021 \\ \log \sqrt{3} &= 0.23856 \\ \log S &= 1.61877\end{aligned}$$

$$\text{Area of hexagon} = 41.569 \text{ sq. in.}$$

$$\text{Area of circle} = \pi r^2.$$

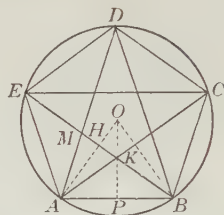
$$\begin{aligned}\log \pi &= 0.49715 \\ \log r^2 &= 1.20412 \\ \log S &= 1.70127\end{aligned}$$

$$\text{Area of circle} = 50.2655 \text{ sq. in.}$$

$$\begin{aligned}\text{Area of the six segments} \\ &= 50.2655 - 41.569 \\ &= 8.6965 \text{ sq. in.}\end{aligned}$$

$$\begin{aligned}\text{Area of part of circle not covered} \\ \text{by the hexagon} &= 8.6965 \text{ sq. in.}\end{aligned}$$

5. In a circle of radius 10 in. a regular five-pointed star is inscribed. What is the area of the star? What is the area of that part of the circle not covered by the star?



$$\begin{aligned}OA &= 10 \text{ in.} \\ OP &= 10 \cos 36^\circ. \\ \angle AOB &= 72^\circ. \\ \angle AOP &= 36^\circ. \\ OP &= \text{apothem.}\end{aligned}$$

$$\begin{aligned}\log 10 &= 1.00000 \\ \log \cos 36^\circ &= 9.90796 - 10 \\ \log OP &= 0.90796\end{aligned}$$

$$AP = 10 \sin 36^\circ.$$

$$\begin{aligned}\log 10 &= 1.00000 \\ \log \sin 36^\circ &= 9.76922 - 10 \\ \log AP &= 0.76922\end{aligned}$$

$$\text{Area of pentagon}$$

$$\begin{aligned}&= 5 \times \frac{1}{2} (AB \times OP) \\ &= AP \times OP \times 5.\end{aligned}$$

$$\begin{aligned}\log AP &= 0.76922 \\ \log OP &= 0.90796 \\ \log 5 &= 0.69897 \\ \log S &= 2.37615\end{aligned}$$

$$S = 237.767 \text{ sq. in.}$$

$$\text{Area of circle} = \pi r^2.$$

$$\begin{aligned}\log \pi &= 0.49715 \\ \log r^2 &= 2.00000 \\ \log S &= 2.49715\end{aligned}$$

$$S = 314.159 \text{ sq. in.}$$

$$\text{Area of segments}$$

$$\begin{aligned}&= 314.159 - 237.767 \\ &= 76.392 \text{ sq. in.}\end{aligned}$$

$\angle KAP$ or CAB is measured by one half arc BC .

$$\therefore KAP = 36^\circ.$$

$$PK = AP \tan 36^\circ.$$

$$\log AP = 0.76922$$

$$\log \tan 36^\circ = \underline{9.86126 - 10}$$

$$\log PK = 0.63048$$

Area of $\triangle AKB$

$$= \frac{1}{2} AB \times PK$$

$$= AP \times PK.$$

The area of the 5 $\triangle$ between points

$$= 5 \times AP \times PK.$$

$\log S$ of $\triangle$

$$= \log 5 + \log AP + \log PK.$$

$$\log 5 = 0.69897$$

$$\log AP = 0.76922$$

$$\log PK = 0.63048$$

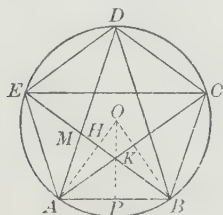
$$\log S = 2.09867$$

$$S = 125.508 \text{ sq. in.}$$

Area of segments + area of 'triangles' = 201.9 sq. in., area not covered by star.

$314.159 - 201.900 = 112.26 \text{ sq. in.}$, area of star.

6. In a circle of diameter 7.2 in. a regular five-pointed star is inscribed. The points are joined, thus forming a regular pentagon. There is also a regular pentagon formed in the center by the crossing of the lines of the star. The small pentagon is what fractional part of the large one?



$$\frac{S'}{S} = \frac{\overline{OK}^2}{\overline{OA}^2}.$$

$$OA = 3.6.$$

$$\frac{HK}{OK} = \sin 36^\circ.$$

$$OK = \frac{HK}{\sin 36^\circ}.$$

$$\frac{HK}{AK} = \sin 18^\circ.$$

$$HK = AK \sin 18^\circ.$$

$$\frac{AK}{AO} = \frac{\sin 36^\circ}{\sin 126^\circ}.$$

$$AK = \frac{3.6 \sin 36^\circ}{\sin 126^\circ}.$$

$$\therefore HK = \frac{3.6 \sin 36^\circ \sin 18^\circ}{\sin 126^\circ}.$$

and

$$OK = \frac{3.6 \sin 36^\circ \sin 18^\circ}{\sin 126^\circ \sin 36^\circ}$$

$$= \frac{3.6 \sin 18^\circ}{\sin 126^\circ}.$$

$$\log 3.6 = 0.55630$$

$$\log \sin 18^\circ = 9.48998 - 10$$

$$\text{colog } \sin 126^\circ = \underline{0.09204}$$

$$\log OK = 0.13832$$

$$\log \overline{OK}^2 = 0.27664$$

$$\text{colog } \overline{OA}^2 = \underline{8.88740 - 10}$$

$$\log \frac{\overline{OK}^2}{\overline{OA}^2} = 9.15404 - 10$$

$$\frac{\overline{OK}^2}{\overline{OA}^2} = 0.14279.$$

$\therefore \frac{S'}{S} = 0.14279$, ratio of inner pentagon to the inscribed pentagon

7. A circular hole is cut in a regular hexagonal plate of side 8 in. The radius of the circle is 4 in. What is the area of the rest of the plate?



Side of hexagon = 8 in.

Radius of hexagon = 8 in.

Apothem = $\sqrt{8^2 - 4^2} = 4\sqrt{3}$.

$$S = \frac{1}{2}(4\sqrt{3} \times 48)$$

$$= 2\sqrt{3} \times 48$$

$$= 96\sqrt{3}.$$

$$\log 96 = 1.98227$$

$$\log \sqrt{3} = 0.23856$$

$$\log S = 2.22083$$

Area of hexagon = 166.277 sq. in.

Area of circle = πr^2 .

$$\log \pi = 0.49715$$

$$\log r^2 = 1.20412$$

$$\log S = 1.70127$$

Area of circle = 50.265 sq. in.

$$166.277 - 50.265$$

= 116.012 sq. in., area of rest of plate.

8. A regular hexagon is formed by joining the mid-points of the sides of a regular hexagon. Find the ratio of the smaller hexagon to the larger.

Let a and $a' =$ one side of each of the two hexagons.

Draw OK .

OK makes $\angle$ of 60° with a .

$$\sin 60 = \frac{\frac{a'}{2}}{\frac{a}{2}} = \frac{a'}{a}.$$

$$a' = a \sin 60.$$

$$\frac{S'}{S} = \frac{a'^2 \sin^2 60^\circ}{a^2}.$$

$$\frac{S'}{S} = \sin^2 60^\circ$$

$$= \left(\frac{1}{2}\sqrt{3}\right)^2$$

$$= .75.$$

Therefore the ratio of the inner hexagon to the outer one is .75.

Exercise 64. Plane Sailing (Page 147)

1. A ship sails from latitude 40° N. on a course N.E. 26 mi. Find the difference of latitude and the departure.

Course = 45° .

Diff. of lat. = $26 \cos 45^\circ$.

Depart. = $26 \sin 45^\circ$.

$$\log 26 = 1.41497$$

$$\log \cos 45^\circ = 9.84949 - 10$$

$$\log \text{diff. of lat.} = 1.26446$$

$$\text{Diff. of lat.} = 18.385'$$

$$= 18' 23''.$$

$$\log 26 = 1.41497$$

$$\log \sin 45^\circ = 9.84949 - 10$$

$$\log \text{depart.} = 1.26446$$

$$\text{Depart.} = 18.385 \text{ mi.}$$

$$\text{Diff. of lat.} = 18' 23''.$$

2. A ship sails from latitude 35° N. on a course S.W. 53 mi. Find the difference of latitude and the departure.

$$\text{Course} = 45^{\circ}.$$

$$\text{Diff. of lat.} = 53 \cos 45^{\circ}.$$

$$\log 53 = 1.72428$$

$$\log \cos 45^{\circ} = 9.84949 - 10$$

$$\log \text{diff. of lat.} = 1.57377$$

$$\begin{aligned} \text{Diff. of lat.} &= 37.4775'. \\ &= 37' 28.65''. \end{aligned}$$

$$\text{Depart.} = 53 \sin 45^{\circ}.$$

$$\text{Depart.} = 37.4775 \text{ mi.}$$

$$\text{Diff. of lat.} = 37' 28.65''.$$

3. A ship sails from a point on the equator on a course N.E. by N. 62 mi. Find the difference of latitude and the departure.

$$\text{Course} = 33^{\circ} 45'.$$

$$\text{Diff. of lat.} = 62 \cos 33^{\circ} 45'.$$

$$\text{Depart.} = 62 \sin 33^{\circ} 45'.$$

$$\log 62 = 1.79239$$

$$\log \cos 33^{\circ} 45' = 9.91985 - 10$$

$$\log \text{diff. of lat.} = 1.71224$$

$$\text{Diff. of lat.} = 51.551'$$

$$= 51' 33''.$$

$$\log 62 = 1.79239$$

$$\log \sin 33^{\circ} 45' = 9.74474 - 10$$

$$\log \text{depart.} = 1.53713$$

$$\text{Depart.} = 34.445 \text{ mi.}$$

$$\text{Diff. of lat.} = 51' 33''.$$

4. A ship sails from latitude $43^{\circ} 45'$ S. on a course N. by E. 38 mi. Find the difference of latitude and the departure.

$$\text{Course} = 11^{\circ} 15'.$$

$$\text{Diff. of lat.} = 38 \cos 11^{\circ} 15'.$$

$$\text{Depart.} = 38 \sin 11^{\circ} 15'.$$

$$\log 38 = 1.57978$$

$$\log \cos 11^{\circ} 15' = 9.99157 - 10$$

$$\log \text{diff. of lat.} = 1.57135$$

$$\text{Diff. of lat.} = 37.269' = 37' 16''.$$

$$\log 38 = 1.57978$$

$$\log \sin 11^{\circ} 15' = 9.29024 - 10$$

$$\log \text{depart.} = 0.87002$$

$$\text{Depart.} = 7.4135 \text{ mi.}$$

$$\text{Diff. of lat.} = 37' 16''.$$

5. A ship sails from latitude $1^{\circ} 45'$ N. on a course S.E. by E. 25 mi. Find the difference of latitude and the departure.

$$\text{Course} = 56^{\circ} 15'.$$

$$\text{Diff. of lat.} = 25 \cos 56^{\circ} 15'.$$

$$\text{Depart.} = 25 \sin 56^{\circ} 15'.$$

$$\log 25 = 1.39794$$

$$\log \cos 56^{\circ} 15' = 9.74474 - 10$$

$$\log \text{diff. of lat.} = 1.14268$$

$$\text{Diff. of lat.} = 13.889' = 13' 53''.$$

$$\log 25 = 1.39794$$

$$\log \sin 56^{\circ} 15' = 9.91985 - 10$$

$$\log \text{depart.} = 1.31779$$

$$\text{Depart.} = 20.787 \text{ mi.}$$

$$\text{Diff. of lat.} = 13' 53''.$$

6. A ship sails from latitude $13^{\circ} 17'$ S. on a course N.E. by E. $\frac{3}{4}$ E., until the departure is 42 mi. Find the difference of latitude and the latitude reached.

$$\text{Course} = 64^{\circ} 41' 15''.$$

$$\text{Distance} = \frac{42}{\sin 64^{\circ} 41' 15''}.$$

$$\log 42 = 1.62325$$

$$\log \sin 64^{\circ} 41' 15'' = 0.04383$$

$$\log \text{distance} = 1.66708$$

$$\text{Distance} = 46.46 \text{ mi.}$$

$$\text{Diff. of lat.} = 46.46 \times \cos 64^{\circ} 41' 15''$$

$$\log 46.46 = 1.66708$$

$$\log \cos 64^{\circ} 41' 15'' = 9.63099 - 10$$

$$1.29807$$

$$\text{Diff. of lat.} = 19.864' = 19' 52''.$$

$$\text{Lat. reached} = 12^{\circ} 57' 8'' \text{ S.}$$

7. A ship sails from latitude $40^{\circ} 20'$ N. on a N.N.E. course for 92 mi. Find the departure

$$\text{Course} = 22^{\circ} 30'.$$

$$\text{Depart.} = 92 \sin 22^{\circ} 30'.$$

$$\log 92 = 1.96379$$

$$\log \sin 22^{\circ} 30' = \underline{9.58284 - 10}$$

$$\log \text{depart.} = 1.54663$$

$$\text{Depart.} = 35.207 \text{ mi.}$$

8. If a steamer sails S.W. by W. 20 mi., what is the departure and the difference of latitude?

$$\text{Course} = 56^{\circ} 15'.$$

$$\text{Depart.} = 20 \sin 56^{\circ} 15'.$$

$$\text{Diff. of lat.} = 20 \cos 56^{\circ} 15'$$

$$\log 20 = 1.30103$$

$$\log \sin 56^{\circ} 15' = \underline{9.91985 - 10}$$

$$\log \text{depart.} = 1.22088$$

$$\text{Depart.} = 16.6296 \text{ mi.}$$

$$\log 20 = 1.30103$$

$$\log \cos 56^{\circ} 15' = \underline{9.74474 - 10}$$

$$\log \text{diff. of lat.} = 1.04577$$

$$\text{Diff. of lat.} = 11.112'$$

$$= 11' 6.7''.$$

$$\text{Depart.} = 16.6296 \text{ mi.}$$

$$\text{Diff. of lat.} = 11' 6.7''.$$

9. If a sailboat sails N. 25° W. until the departure is 25 mi., what distance does it sail?

$$\text{Course} = 25^{\circ}.$$

$$\text{Depart.} = \text{distance}$$

$$\times \sin \text{course.}$$

$$\text{Distance} = \frac{25}{\sin 25^{\circ}}.$$

$$\log 25 = 1.39794$$

$$\log \sin 25^{\circ} = \underline{0.37405}$$

$$\log \text{distance} = 1.77199$$

$$\text{Distance sailed} = 59.155 \text{ mi.}$$

10. A ship sails from latitude $37^{\circ} 40'$ N. on a N.E. by E. course for 122 mi. Find the departure.

$$\text{Course} = 56^{\circ} 15'.$$

$$\text{Depart.} = 122 \sin 56^{\circ} 15'.$$

$$\log 122 = 2.08636$$

$$\log \sin 56^{\circ} 15' = \underline{9.91985 - 10}$$

$$\log \text{depart.} = 2.00621$$

$$\text{Depart.} = 101.44 \text{ mi.}$$

11. A yacht sails $6\frac{1}{2}$ points west of north, the distance being 12 mi. What is the departure?

$$\text{Course} = 73^{\circ} 7' 30''.$$

$$\text{Depart.} = 12 \sin 73^{\circ} 7' 30''.$$

$$\log 12 = 1.07918$$

$$\log \sin 73^{\circ} 7' 30'' = \underline{9.98088 - 10}$$

$$\log \text{depart.} = 1.06006$$

$$\text{Depart.} = 11.483 \text{ mi.}$$

12. A steamer sails S.W. by W. 28 mi. It then sails N.W. 30 mi. How far is it then to the west of its starting point?

$$\text{First course} = 56^{\circ} 15'.$$

$$\text{Second course} = 45^{\circ}.$$

$$\text{First depart.} = 28 \sin 56^{\circ} 15'$$

$$\text{Second depart.} = 30 \sin 45^{\circ}.$$

$$\log 28 = 1.44716$$

$$\log \sin 56^{\circ} 15' = \underline{9.91985 - 10}$$

$$\log \text{first depart.} = 1.36701$$

$$\text{First depart.} = 23.2816 \text{ mi.}$$

$$\log 30 = 1.47712$$

$$\log \sin 45^{\circ} = \underline{9.84949}$$

$$\log \text{second depart.} = 1.32661$$

$$\text{Second depart.} = 21.2133 \text{ mi.}$$

$$\text{Total depart.}$$

$$= 23.2816 + 21.2133$$

$$= 44.4949 \text{ mi.}$$

Ship was west of starting point 44.5 mi.

13. A ship sails on a course between S. and E. 24 mi., leaving latitude $2^{\circ} 52'$ S. and reaching latitude $2^{\circ} 58'$ S. Find the course and the departure.

$$\text{Diff. of lat.} = 6'$$

$$= 6 \text{ mi.}$$

$$6 = 24 \cos \text{course.}$$

$$\cos \text{course} = \frac{6}{24}$$

$$= .25.$$

$$\text{Course} = 75^{\circ} 31' 20''.$$

$$\text{Depart.} = 24 \sin 75^{\circ} 31' 20''.$$

$$\log 24 = 1.38021$$

$$\log \sin 75^{\circ} 31' 20'' = 9.98598 - 10$$

$$\log \text{depart.} = 1.36619$$

$$\text{Depart.} = 23.2374 \text{ mi.}$$

$$\text{Course} = \text{S. } 75^{\circ} 31' 20'' \text{ E.}$$

14. A ship sails from latitude $32^{\circ} 18' \text{ N.}$, on a course between N. and W., a distance of 34 mi. and a departure of 10 mi. Find the course and the latitude reached.

$$\frac{10}{34} = \sin \text{course.}$$

$$\log 10 = 1.00000$$

$$\text{colog } 34 = 8.46852 - 10$$

$$\log \sin \text{course} = 9.46852$$

$$\text{Course} = 17^{\circ} 6' 14''.$$

$$\text{Diff. of lat.} = 34 \cos 17^{\circ} 6' 14''.$$

$$\log 34 = 1.53148$$

$$\log \cos 17^{\circ} 6' 14'' = 9.98035 - 10$$

$$\log \text{diff. of lat.} = 1.51183$$

$$\text{Diff. of lat.} = 32.496'$$

$$= 32' 29.76''.$$

$$\text{Lat. reached} = 32^{\circ} 18' + 32' 30''$$

$$= 32^{\circ} 50' 30''.$$

$$\text{Course} = \text{N. } 17^{\circ} 6' 14'' \text{ W.}$$

$$\text{Latitude reached} = 32^{\circ} 50' 30'' \text{ N.}$$

15. A ship sails on a course between S. and E., making a difference of latitude 13 mi. and a departure of 20 mi. Find the distance and the course.

$$\frac{20}{13} = \tan \text{course.}$$

$$\log 20 = 1.30103$$

$$\text{colog } 13 = 8.88606 - 10$$

$$\log \tan \text{course} = 0.18709$$

$$\text{Course} = 56^{\circ} 58' 34''.$$

$$\text{Distance} = \frac{20}{\sin 56^{\circ} 58' 34''}$$

$$\log 20 = 1.30103$$

$$\text{colog } \sin 56^{\circ} 58' 34'' = 0.07653$$

$$\log \text{distance} = 1.37756$$

$$\text{Distance} = 23.854 \text{ mi.}$$

$$\text{Course} = \text{S. } 56^{\circ} 58' 34'' \text{ E.}$$

16. A ship sails on a course between N. and W., making a difference of latitude 17 mi. and a departure of 22 mi. Find the distance and the course.

$$\frac{22}{17} = \tan \text{course.}$$

$$\log 22 = 1.34242$$

$$\text{colog } 17 = 8.76955 - 10$$

$$\log \tan \text{course} = 10.11197$$

$$\text{Course} = 52^{\circ} 18' 21''.$$

$$\text{Distance} = \frac{22}{\sin 52^{\circ} 18' 21''}$$

$$\log 22 = 1.34242$$

$$\text{colog } \sin 52^{\circ} 18' 21'' = 0.10167$$

$$\log \text{distance} = 1.44409$$

$$\text{Distance sailed} = 27.803 \text{ mi.}$$

$$\text{Course} = \text{N. } 52^{\circ} 18' 21'' \text{ W.}$$

Exercise 65. Parallel Sailing (Page 148)

1. A ship in latitude $42^{\circ} 16' N.$, longitude $72^{\circ} 16' W.$, sails due east a distance of 149 mi. What is the position of the point reached?

$$\text{Diff. of long.} = 149 \sec 42^{\circ} 16'.$$

$$\log 149 = 2.17319$$

$$\log \sec 42^{\circ} 16' = 0.13076$$

$$\log \text{diff. of long.} = \underline{2.30395}$$

$$\text{Diff. of long.} = 201.35'$$

$$= 3^{\circ} 21' 21''.$$

The position of point reached is $42^{\circ} 16' N.$; $68^{\circ} 54' 39'' W.$

2. A ship in latitude $44^{\circ} 49' S.$, longitude $119^{\circ} 42' E.$, sails due west until it reaches longitude $117^{\circ} 16' E.$ Find the distance made.

$$\text{Diff. of long.} = 2^{\circ} 26' = 146'.$$

$$\text{Depart.} = 146 \cos 44^{\circ} 49'.$$

$$\log 146 = 2.16435$$

$$\log \cos 44^{\circ} 49' = 9.85087$$

$$\log \text{depart.} = \underline{2.01522}$$

$$\text{Depart.} = 103.57 \text{ mi.}$$

$$\text{Distance made} = 103.57 \text{ mi.}$$

3. A ship in latitude $60^{\circ} 15' N.$, longitude $60^{\circ} 15' W.$, sails due west a distance of 60 mi. What is the position of the point reached?

$$\text{Diff. of long.} = 60 \sec 60^{\circ} 15'.$$

$$\log 60 = 1.77815$$

$$\log \sec 60^{\circ} 15' = 0.30433$$

$$\log \text{diff. of long.} = 2.08248$$

$$\text{Diff. of long.} = 120.9139'$$

$$= 2^{\circ} 0' 54.8''.$$

$$60^{\circ} 15' + 2^{\circ} 0' 55'' = 62^{\circ} 15' 55'' W.$$

The position of point reached is $60^{\circ} 15' N.$; $62^{\circ} 15' 55'' W.$

Exercise 66. Middle Latitude Sailing (Page 149)

1. A ship leaves latitude $31^{\circ} 14' N.$, longitude $42^{\circ} 19' W.$, and sails E.N.E. 32 mi. Find the position reached.

$$\text{Course} = 67^{\circ} 30'.$$

$$\text{Diff. of lat.} = 32 \cos 67^{\circ} 30'.$$

$$\log 32 = 1.50515$$

$$\log \cos 67^{\circ} 30' = 9.58284$$

$$\log \text{diff. of lat.} = \underline{1.08799}$$

$$\text{Diff. of lat.} = 12.245'$$

$$= 12' 15''.$$

$$\text{Mid. lat.} = 31^{\circ} 20' 7''.$$

$$\text{Depart.} = 32 \sin 67^{\circ} 30'.$$

$$\text{Diff. of long.} = 32 \sin 67^{\circ} 30' \times \sec 31^{\circ} 20' 7''$$

$$\log 32 = 1.50515$$

$$\log \sin 67^{\circ} 30' = 9.96562$$

$$\log \sec 31^{\circ} 20' 7'' = \underline{0.06847}$$

$$\log \text{diff. long.} = \underline{1.53924}$$

$$\text{Diff. of long.} = 34.61'$$

$$= 34' 37''.$$

$$\text{Latitude reached, } 31^{\circ} 26' 15'' N.$$

$$\text{Longitude, } 41^{\circ} 44' 23'' W.,$$

2. Leaving latitude $49^{\circ} 57' N.$, longitude $15^{\circ} 16' W.$, a ship sails between S. and W. till the departure is 38 mi. and the latitude is $49^{\circ} 38' N.$ Find the course, distance, and longitude reached.

$$\text{Diff. of lat.} = 19' = 19 \text{ mi.}$$

$$\text{Mid. lat.} = 49^{\circ} 47' 30''.$$

$$\text{Depart.} = 38 \text{ mi.}$$

$$\text{Diff. of long.} = 38 \sec 49^{\circ} 47' 30''.$$

$$\log 38 = 1.57978$$

$$\log \sec 49^{\circ} 47' 30'' = 0.19005$$

$$\log \text{diff. of long.} = 1.76983$$

$$\text{Diff. of long.} = 58.86' = 58^{\circ} 52''.$$

$$\tan \text{course} = \frac{38}{19} = 2.$$

$$\text{Course} = 63^{\circ} 26'.$$

$$\begin{aligned} \text{Distance} &= \frac{\text{depart.}}{\sin 63^{\circ} 26'} \\ &= \frac{38}{\sin 63^{\circ} 26'}. \end{aligned}$$

$$\log 38 = 1.57978$$

$$\text{colog } \sin 63^{\circ} 26' = 0.04846$$

$$\log \text{distance} = 1.62824$$

$$\text{Distance} = 42.486 \text{ mi.}$$

$$\text{Course} = S. 63^{\circ} 26' W.$$

$$\text{Distance} = 42.486 \text{ mi.}$$

$$\text{Longitude reached, } 16^{\circ} 14' 52'' W.$$

3. Leaving latitude $42^{\circ} 30' N.$, longitude $58^{\circ} 51' W.$, a ship sails S.E. by S. 48 mi. Find the position reached.

$$\text{Course} = 33^{\circ} 45'.$$

$$\text{Diff. of lat.} = 48 \cos 33^{\circ} 45'.$$

$$\log 48 = 1.68124$$

$$\log \cos 33^{\circ} 45' = 9.91985$$

$$\log \text{diff. of lat.} = 1.60109$$

$$\text{Diff. of lat.} = 39.911'$$

$$= 39^{\circ} 55''.$$

$$\text{Mid. lat.} = 42^{\circ} 10' 3'' N.$$

$$\text{Depart.} = 48 \sin 33^{\circ} 45'.$$

$$\begin{aligned} \text{Diff. of long.} &= 48 \sin 33^{\circ} 45' \times \\ &\quad \sec 42^{\circ} 10' 3''. \end{aligned}$$

$$\log 48 = 1.68124$$

$$\log \sin 33^{\circ} 45' = 9.74474$$

$$\log \sec 42^{\circ} 10' 3'' = 0.13008$$

$$\log \text{diff. of long.} = 1.55606$$

$$\text{Diff. of long.} = 35.98'$$

$$= 35^{\circ} 59''.$$

$$\text{Latitude reached, } 41^{\circ} 50' 5'' N.$$

$$\text{Longitude, } 58^{\circ} 15' 1'' W.$$

4. Leaving latitude $49^{\circ} 57' N.$, longitude $30^{\circ} W.$, a ship sails S. $39^{\circ} W.$ and reaches latitude $49^{\circ} 44' N.$ Find the distance and the longitude reached.

$$\text{Diff. of lat.} = 13' = 13 \text{ mi.}$$

$$\cos 39^{\circ} = \frac{13}{\text{distance}}.$$

$$\text{Distance} = \frac{13}{\cos 39^{\circ}}.$$

$$\log 13 = 1.11394$$

$$\text{colog } \cos 39^{\circ} = 0.10950$$

$$\log \text{distance} = 1.22344$$

$$\text{Distance} = 16.727 \text{ mi.}$$

$$\text{Mid. lat.} = 49^{\circ} 50' 30''.$$

$$\text{Depart.} = 13 \tan 39^{\circ}.$$

$$\begin{aligned} \text{Diff. of long.} &= \text{depart.} \times \\ &\quad \sec 49^{\circ} 50' 30'' \\ &= 13 \tan 39^{\circ} \times \\ &\quad \sec 49^{\circ} 50' 30'' \end{aligned}$$

$$\log 13 = 1.11394$$

$$\log \tan 39^{\circ} = 9.90837$$

$$\log \sec 49^{\circ} 50' 30'' = 0.19050$$

$$\log \text{diff. of long.} = 1.21281$$

$$\text{Diff. of long.} = 16.319' = 16^{\circ} 19''$$

$$\text{Longitude reached, } 30^{\circ} 16' 19'' W$$

$$\text{Distance} = 16.727 \text{ mi.}$$

5. Leaving latitude 37° N., longitude $32^{\circ} 16'$ W., a ship sails between N. and W. 45 mi. and reaches latitude $37^{\circ} 10'$ N. Find the course and the longitude reached.

$$\text{Diff. of lat.} = 10'$$

$$= 10 \text{ mi.}$$

$$\text{Mid. lat.} = 37^{\circ} 5' \text{ N.}$$

$$\cos \text{course} = \frac{10}{45}.$$

$$\log 10 = 1.00000$$

$$\text{colog } 45 = 8.34679 - 10$$

$$\log \cos \text{course} = 9.84679$$

$$\text{Course} = 77^{\circ} 9' 38''.$$

$$\text{Depart.} = 45 \sin 77^{\circ} 9' 38''.$$

$$\text{Diff. of long.} = 45 \sin 77^{\circ} 9' 38'' \times \sec 37^{\circ} 5'.$$

$$\log 45 = 1.65321$$

$$\log \sin 77^{\circ} 9' 38'' = 9.98900 - 10$$

$$\log \sec 37^{\circ} 5' = 0.09813$$

$$\log \text{diff. of long.} = 1.74034$$

$$\text{Diff. of long.} = 55'.$$

$$\text{Longitude reached} = 33^{\circ} 11' \text{ W.}$$

$$\text{Course} = \text{N. } 77^{\circ} 9' 38'' \text{ W.}$$

6. A ship sails from latitude $40^{\circ} 28'$ N., longitude 74° W., on an E.S.E. course, 62 mi. Find the latitude and longitude reached.

$$\text{Course} = 67^{\circ} 30'.$$

$$\text{Diff. of lat.} = 62 \cos 67^{\circ} 30'.$$

$$\log 62 = 1.79239$$

$$\log \cos 67^{\circ} 30' = 9.58284$$

$$\log \text{diff. of lat.} = 1.37523$$

$$\text{Diff. of lat.} = 23.726'$$

$$= 23' 44''$$

$$\text{Mid. lat.} = 40^{\circ} 16' 8''.$$

$$\text{Depart.} = 62 \sin 67^{\circ} 30'.$$

$$\text{Diff. of long.} = 62 \sin 67^{\circ} 30' \times \sec 40^{\circ} 16' 8''.$$

$$\log 62 = 1.79239$$

$$\log \sin 67^{\circ} 30' = 9.96562 - 10$$

$$\log \sec 40^{\circ} 16' 8'' = 0.11746$$

$$\log \text{diff. of long.} = 1.87547$$

$$\text{Diff. of long.} = 75.07'$$

$$= 1^{\circ} 15' 4''.$$

$$\text{Latitude reached} = 40^{\circ} 4' 16'' \text{ N.}$$

$$\text{Longitude} = 72^{\circ} 44' 56'' \text{ W.}$$

7. A ship sails from latitude $42^{\circ} 20'$ N., longitude $71^{\circ} 4'$ W., on a N.N.E. course, 30 mi. Find the latitude and longitude reached.

$$\text{Course} = 22^{\circ} 30'.$$

$$\text{Diff. of lat.} = 30 \cos 22^{\circ} 30'.$$

$$\log 30 = 1.47712$$

$$\log \cos 22^{\circ} 30' = 9.96562 - 10$$

$$\log \text{diff. of lat.} = 1.44274$$

$$\text{Diff. of lat.} = 27.716'$$

$$= 27' 43''.$$

$$\text{Mid. lat.} = 42^{\circ} 33' 51'' \text{ N.}$$

$$\text{Depart.} = 30 \sin 22^{\circ} 30'.$$

$$\text{Diff. of long.} \times \sec \text{mid. lat.}$$

$$= 30 \sin 22^{\circ} 30' \sec 42^{\circ} 33' 51''.$$

$$\log 30 = 1.47712$$

$$\log \sin 22^{\circ} 30' = 9.58284 - 10$$

$$\log \sec 42^{\circ} 33' 51'' = 0.13281$$

$$\log \text{diff. of long.} = 1.19277$$

$$\text{Diff. of long.} = 15.587'$$

$$= 15' 35''.$$

$$\text{Latitude reached, } 42^{\circ} 47' 43'' \text{ N.}$$

$$\text{Longitude, } 70^{\circ} 48' 25'' \text{ W.}$$

Exercise 67. Traverse Sailing (Page 150)

1. Leaving latitude $37^{\circ} 16' S.$,
longitude $18^{\circ} 42' W.$, a ship sails
N.E. 104 mi., then N.N.W. 60 mi.,
then W. by S. 216 mi. Find the
position reached, and its bearing
and distance from the point left.

First course = 45° .

Diff. of lat. = $104 \cos 45^{\circ}$.

Depart. = $104 \sin 45^{\circ}$.

$\log 104 = 2.01703$

$\log \cos 45^{\circ} = \frac{9.84949 - 10}{}$

$\log \text{diff. of lat.} = 1.86652$

Diff. of lat. = 73.54 N.

Depart. = 73.54 E.

Second course = $22^{\circ} 30'$.

Diff. of lat. = $60 \cos 22^{\circ} 30'$.

Depart. = $60 \sin 22^{\circ} 30'$.

$\log 60 = 1.77815$

$\log \cos 22^{\circ} 30' = \frac{9.96562 - 10}{}$

$\log \text{diff. of lat.} = 1.74377$

Diff. of lat. = 55.434 N.

$\log 60 = 1.77815$

$\log \sin 22^{\circ} 30' = \frac{9.58284 - 10}{}$

$\log \text{depart.} = 1.36099$

Depart. = 22.961 W.

Third course = $78^{\circ} 45'$.

Diff. of lat. = $216 \cos 78^{\circ} 45'$.

Depart. = $216 \sin 78^{\circ} 45'$.

$\log 216 = 2.33445$

$\log \cos 78^{\circ} 45' = \frac{9.29024 - 10}{}$

$\log \text{diff. of lat.} = 1.62469$

Diff. of lat. = 42.14 S.

$\log 216 = 2.33445$

$\log \sin 78^{\circ} 45' = \frac{9.99157 - 10}{}$

$\log \text{depart.} = 2.32602$

Depart. = 211.85 W.

Total diff. of lat. = 86.834' N.

= $1^{\circ} 26' 50'' N.$

Lat. reached = $35^{\circ} 49' 10'' S.$

Mid. lat. = $36^{\circ} 32' 35''$.

Total depart. = 161.271 W.

Diff. of long. = $161.271 \sec 36^{\circ} 32' 35''$

$\log 161.271 = 2.20755$

$\log \sec 36^{\circ} 32' 35'' = \frac{0.09506}{}$

$\log \text{diff. of long.} = 2.30261$

Diff. of long. = 200.73'

= $3^{\circ} 20' 44''$.

Long. reached = $22^{\circ} 2' 44'' W.$

$\tan \text{course} = \frac{161.271}{86.834}$.

$\log 161.271 = 2.20755$

$\csc 86.834 = \frac{8.06131 - 10}{}$

$\log \tan \text{course} = 10.26886$

Course = $61^{\circ} 42'$.

Distance = $86.834 \sec 61^{\circ} 42'$.

$\log 86.834 = 1.93869$

$\log \sec 61^{\circ} 42' = 0.32414$

$\log \text{distance} = 2.26283$

Distance = 183.16 mi.

Course, N. $61^{\circ} 42' W.$; distance,
183.16 mi.; latitude reached, 35°
 $49' 10'' S.$; longitude, $22^{\circ} 2' 44'' W.$

2. A ship leaves Cape Cod ($42^{\circ} 2' \text{ N.}, 70^{\circ} 3' \text{ W.}$) and sails S.E. by S. 114 mi., then N. by E. 94 mi., then W.N.W. 42 mi. Find its position and the total distance.

$$\begin{aligned}\text{First course} &= 33^{\circ} 45'. \\ \text{Diff. of lat.} &= 114 \cos 33^{\circ} 45'. \\ \text{Depart.} &= 114 \sin 33^{\circ} 45'.\end{aligned}$$

$$\begin{aligned}\log 114 &= 2.05690 \\ \log \cos 33^{\circ} 45' &= \underline{9.91985 - 10} \\ \log \text{diff. of lat.} &= 1.97675\end{aligned}$$

$$\text{Diff. of lat.} = 94.787 \text{ S.}$$

$$\begin{aligned}\log 114 &= 2.05690 \\ \log \sin 33^{\circ} 45' &= \underline{9.74474 - 10} \\ \log \text{depart.} &= 1.80164\end{aligned}$$

$$\text{Depart.} = 63.334 \text{ E.}$$

$$\begin{aligned}\text{Second course} &= 11^{\circ} 15'. \\ \text{Diff. of lat.} &= 94 \cos 11^{\circ} 15'. \\ \text{Depart.} &= 94 \sin 11^{\circ} 15'.\end{aligned}$$

$$\begin{aligned}\log 94 &= 1.97313 \\ \log \cos 11^{\circ} 15' &= \underline{9.99157 - 10} \\ \log \text{diff. of lat.} &= 1.96470\end{aligned}$$

$$\text{Diff. of lat.} = 92.194 \text{ N.}$$

$$\begin{aligned}\log 94 &= 1.97313 \\ \log \sin 11^{\circ} 15' &= \underline{9.29024 - 10} \\ \log \text{depart.} &= 1.26337\end{aligned}$$

$$\text{Depart.} = 18.339 \text{ E.}$$

$$\begin{aligned}\text{Third course} &= 67^{\circ} 30'. \\ \text{Diff. of lat.} &= 42 \cos 67^{\circ} 30'. \\ \text{Depart.} &= 42 \sin 67^{\circ} 30'.\end{aligned}$$

$$\begin{aligned}\log 42 &= 1.62325 \\ \log \cos 67^{\circ} 30' &= \underline{9.58284 - 10} \\ \log \text{diff. of lat.} &= 1.20609\end{aligned}$$

$$\text{Diff. of lat.} = 16.073 \text{ N.}$$

$$\begin{aligned}\log 42 &= 1.62325 \\ \log \sin 67^{\circ} 30' &= \underline{9.96562 - 10} \\ \log \text{depart.} &= 1.58887\end{aligned}$$

$$\text{Depart.} = 38.804 \text{ W.}$$

$$\begin{aligned}\text{Total diff. of lat.} &= 13.48' \text{ N.} \\ &= 13^{\circ} 29' \text{ N.}\end{aligned}$$

$$\begin{aligned}\text{Lat. of Cape Cod} &= 42^{\circ} 2'. \\ \text{Lat. reached} &= 42^{\circ} 15' 29'' \text{ N.}\end{aligned}$$

$$\text{Mid. lat.} = 42^{\circ} 8' 44''.$$

$$\text{Total depart.} = 42.869 \text{ E.}$$

$$\text{Diff. of long.} = 42.369 \text{ sec } 42^{\circ} 8' 44''.$$

$$\begin{aligned}\log 42.869 &= 1.63214 \\ \log \sec 42^{\circ} 8' 44'' &= \underline{0.12992} \\ \log \text{diff. lo. g.} &= 1.76206\end{aligned}$$

$$\begin{aligned}\text{Diff. of long.} &= 57.817' \text{ E.} \\ &= 57^{\circ} 49'' \text{ E.}\end{aligned}$$

$$\begin{aligned}\text{Long. of Cape Cod} &= 70^{\circ} 3' \text{ W.} \\ \text{Long. reached} &= 69^{\circ} 5' 11'' \text{ W.}\end{aligned}$$

$$\tan \text{course} = \frac{42.869}{13.48}.$$

$$\begin{aligned}\log 42.869 &= 1.63214 \\ \log 13.48 &= \underline{8.87031 - 10} \\ \log \tan \text{course} &= 10.50245\end{aligned}$$

$$\text{Course} = 72^{\circ} 32' 40''.$$

$$\text{Distance} = 13.48 \text{ sec } 72^{\circ} 32' 40''.$$

$$\begin{aligned}\log 13.48 &= 1.12969 \\ \log \sec 72^{\circ} 32' 40'' &= \underline{0.52293} \\ \log \text{distance} &= 1.65262\end{aligned}$$

$$\text{Distance} = 44.939 \text{ mi.}$$

$$\begin{aligned}\text{Course N. } 72^{\circ} 32' 40'' \text{ E.; distance,} \\ 44.939 \text{ mi.; latitude reached, } 42^{\circ} \\ 15' 29'' \text{ N.; longitude, } 69^{\circ} 5' 11'' \text{ W.}\end{aligned}$$

3. A ship leaves Cape of Good Hope ($34^{\circ}22' \text{ S.}, 18^{\circ}30' \text{ E.}$) and sails N.W. 126 mi., then N. by E. 84 mi., then W.S.W. 217 mi. Find its position and the total distance.

First course = 45° .

Diff. of lat. = $126 \cos 45^{\circ}$.

Depart. = $126 \sin 45^{\circ}$.

$\log 126 = 2.10037$

$\log \cos 45^{\circ} = 9.84949 - 10$

$\log \text{diff. of lat.} = 1.94986$

Diff. of lat. = 89.096 N.

Depart. = 89.096 W.

Second course = $11^{\circ}15'$.

Diff. of lat. = $84 \cos 11^{\circ}15'$.

Depart. = $84 \sin 11^{\circ}15'$.

$\log 84 = 1.92428$

$\log \cos 11^{\circ}15' = 9.99157 - 10$

$\log \text{diff. of lat.} = 1.91585$

Diff. of lat. = 82.386 N.

$\log 84 = 1.92428$

$\log \sin 11^{\circ}15' = 9.29024 - 10$

$\log \text{depart.} = 1.21452$

Depart. = 16.388 E.

Third course = $67^{\circ}30'$.

Diff. of lat. = $217 \cos 67^{\circ}30'$.

Depart. = $217 \sin 67^{\circ}30'$.

$\log 217 = 2.33646$

$\log \cos 67^{\circ}30' = 9.58284 - 10$

$\log \text{diff. of lat.} = 1.91930$

Diff. of lat. = 83.042 S.

$\log 217 = 2.33646$

$\log \sin 67^{\circ}30' = 9.96562 - 10$

$\log \text{depart.} = 2.30208$

Depart. = 200.49 W.

Total diff. of lat. = $88.440' \text{ N.}$

= $1^{\circ}28'26'' \text{ N.}$

Lat. reached = $32^{\circ}53'34'' \text{ S.}$

Mid. lat. = $33^{\circ}37'47''$.

Total depart. = 273.198 W.

Diff. long. = $273.198 \sec 33^{\circ}37'47''$

$\log 273.198 = 2.43648$

$\log \sec 33^{\circ}37'47'' = 0.07954$

$\log \text{diff. of long.} = 2.51602$

Diff. of long. = $328.11'$

= $5^{\circ}28'7''$.

Long. reached = $13^{\circ}1'53'' \text{ E.}$

$\tan \text{course} = \frac{273.198}{88.44}$.

$\log 273.198 = 2.43648$

$\text{colog } 88.44 = 8.05335 - 10$

$\log \tan \text{course} = 10.48983$

Course = $72^{\circ}3'43''$.

Distance = $88.44 \sec 72^{\circ}3'43''$.

$\log 88.44 = 1.94665$

$\log \sec 72^{\circ}3'43'' = 0.51147$

$\log \text{distance} = 2.45812$

Distance = 287.16 mi.

Course N. $72^{\circ}3'43'' \text{ W.}$; distance, 287.16 mi. ; latitude reached, $32^{\circ}53'34'' \text{ S.}$, longitude, $13^{\circ}1'53'' \text{ E.}$

4. A ship in latitude 40° N. and longitude $67^{\circ}4' \text{ W.}$ sails N.W. 60 mi., then N. by W. 52 mi., and then W.S.W. 83 mi. Find its position

First course = 45° .

Diff. of lat. = $60 \cos 45^\circ$.

Depart. = $60 \sin 45^\circ$.

$\log 60 = 1.77815$

$\log \cos 45^\circ = \frac{9.84949 - 10}{}$

$\log \text{diff. of lat.} = 1.62764$

Diff. of lat. = 42.427 N.

Depart. = 42.427 W.

Second course = $11^\circ 15'$.

Diff. of lat. = $52 \cos 11^\circ 15'$.

Depart. = $52 \sin 11^\circ 15'$.

$\log 52 = 1.71600$

$\log \cos 11^\circ 15' = \frac{9.99157 - 10}{}$

$\log \text{diff. of lat.} = 1.70757$

Diff. of lat. = 51 N.

$\log 52 = 1.71600$

$\log \sin 11^\circ 15' = \frac{9.29024 - 10}{}$

$\log \text{depart.} = 1.00624$

Depart. = 10.145 W.

Third course = $67^\circ 30'$.

Diff. of lat. = $83 \cos 67^\circ 30'$.

Depart. = $83 \sin 67^\circ 30'$.

$\log 83 = 1.91908$

$\log \cos 67^\circ 30' = \frac{9.58284 - 10}{}$

$\log \text{diff. of lat.} = 1.50192$

Diff. of lat. = 31.763 S.

$\log 83 = 1.91908$

$\log \sin 67^\circ 30' = \frac{9.96562 - 10}{}$

$\log \text{depart.} = 1.88470$

Depart. = 76.683 W.

Total diff. of lat. = $61.66' \text{ N.}$

= $1^\circ 1' 40'' \text{ N.}$

Lat. reached = $41^\circ 1' 40'' \text{ N.}$

Mid. lat. = $40^\circ 30' 50''$.

Total depart. = 129.255 W.

Diff. of long. = $129.255 \times$

$\sec 40^\circ 30' 50''$.

$\log 129.255 = 2.11145$

$\log \sec 40^\circ 30' 50'' = 0.11904$

$\log \text{diff. of long.} = 2.23049$

Diff. of long. = $170.02' \text{ W.}$

= $2^\circ 50' 1'' \text{ W.}$

Long. reached = $69^\circ 54' 1'' \text{ W.}$

The position reached is latitude $41^\circ 1' 40'' \text{ N.}$, longitude $69^\circ 54' 1'' \text{ W.}$

5. A ship sailed S.S.W. 48 mi., then S.W. by S. 36 mi., and then N.E. 24 mi. Find the difference in latitude and the departure.

First course = $22^\circ 30'$.

Diff. of lat. = $48 \cos 22^\circ 30'$.

Depart. = $48 \sin 22^\circ 30'$.

$\log 48 = 1.68124$

$\log \cos 22^\circ 30' = \frac{9.96562 - 10}{}$

$\log \text{diff. of lat.} = 1.64686$

Diff. of lat. = $44.347' \text{ S.}$

$\log 48 = 1.68124$

$\log \sin 22^\circ 30' = \frac{9.58284 - 10}{}$

$\log \text{depart.} = 1.26408$

Depart. = 18.369 mi. W.

Second course = $33^\circ 45'$.

Diff. of lat. = $36 \cos 33^\circ 45'$.

Depart. = $36 \sin 33^\circ 45'$.

$\log 36 = 1.55630$

$\log \cos 33^\circ 45' = 9.91985$

$\log \text{diff. of lat.} = 1.47615$

Diff. of lat. = $29.933' \text{ S.}$

$\log 36 = 1.55630$

$\log \sin 33^\circ 45' = \frac{9.74474}{}$

$\log \text{depart.} = 1.30104$

Depart. = 20 mi. W.

Third course = 45° .Diff. of lat. = $24 \cos 45^\circ$.Depart. = $24 \sin 45^\circ$. $\log 24 = 1.38021$ $\log \cos 45^\circ = 9.84949$ $\log \text{diff. of lat.} = 1.22970$ Diff. of lat. = $16.97' \text{ N.}$ Depart. = 16.97 mi. E. Total diff. of lat. = $57' 19'' \text{ S.}$ Total depart. = 21.40 mi. W. Second course = $39^\circ 22' 30''$.Diff. of lat. = $37 \cos 39^\circ 22' 30''$.Depart. = $37 \sin 39^\circ 22' 30''$. $\log 37 = 1.56820$ $\log \cos 39^\circ 22' 30'' = 9.88819$ $\log \text{diff. of lat.} = 1.45639$ Diff. of lat. = $28.601' \text{ S.}$ $\log 37 = 1.56820$ $\log \sin 39^\circ 22' 30'' = 9.80236$ $\log \text{depart.} = 1.37056$ Depart. = 23.473 mi. W.

6. A ship sailed S. $\frac{1}{2}$ E. 18 mi., S.W. $\frac{1}{2}$ S. 37 mi., and then S.S.W. $\frac{1}{4}$ W. 56 mi. Find the difference in latitude and the departure.

First course = $5^\circ 37' 30''$.Diff. of lat. = $18 \cos 5^\circ 37' 30''$.Depart. = $18 \sin 5^\circ 37' 30''$. $\log 18 = 1.25527$ $\log \cos 5^\circ 37' 30'' = 9.99791$ $\log \text{diff. of lat.} = 1.25318$ Diff. of lat. = $17.916' \text{ S.}$ $\log 18 = 1.25527$ $\log \sin 5^\circ 37' 30'' = 8.99130$ $\log \text{depart.} = 0.24657$ Depart. = 1.764 mi. E. Third course = $25^\circ 18' 45''$.Diff. of lat. = $56 \cos 25^\circ 18' 45''$.Depart. = $56 \sin 25^\circ 18' 45''$. $\log 56 = 1.74819$ $\cos 25^\circ 18' 45'' = 9.95617$ $\log \text{diff. of lat.} = 1.70436$ Diff. of lat. = $50.624' \text{ S.}$ $\log 56 = 1.74819$ $\sin 25^\circ 18' 45'' = 9.63099$ $\log \text{depart.} = 1.37918$ Depart. = 23.943 mi. W. Total diff. of lat. = $1^\circ 37' 8''$.Total depart. = 45.652 mi. W. Total diff. of lat. = $1^\circ 37' 8'' \text{ S.}$ Total depart. = 45.652 mi. W.

Exercise 68. Radians (Page 152)

Express the following in radians:

1. $270^\circ = \frac{270}{180} \pi = \frac{3}{2} \pi$.

2. $11.25^\circ = \frac{11.25}{180} \pi = \frac{1}{16} \pi$.

3. $56.25^\circ = \frac{56\frac{1}{4}}{180} \pi = \frac{5}{16} \pi$.

4. $7.5^\circ = \frac{7.5}{180} \pi = \frac{1}{24} \pi$.

5. $196.5^\circ = \frac{196.5}{180} \pi = \frac{131}{120} \pi$

6. $1440^\circ = \frac{1440}{180} \pi = 8 \pi$

7. $200^\circ = \frac{200}{180} \pi = \frac{10}{9} \pi$.

8. $3000^\circ = \frac{3000}{180} \pi = \frac{50}{3} \pi$.

Express the following in degree measure :

9. $1\frac{1}{2}\pi = 270^\circ$. 11. $1\frac{1}{6}\pi = 210^\circ$. 13. $\frac{1}{2}\pi = 90^\circ$. 15. $6\pi = 1080^\circ$.
 10. $1\frac{1}{3}\pi = 240^\circ$. 12. $1\frac{1}{4}\pi = 225^\circ$. 14. $3\pi = 540^\circ$. 16. $10\pi = 1800^\circ$.

State the quadrant in which the following angles lie :

17. $\frac{2}{3}\pi = 120^\circ$, quadrant II. 21. $2.5\pi = 450^\circ$, quadrant II.
 18. $\frac{4}{5}\pi = 144^\circ$, quadrant II. 22. $-3.4\pi = -612^\circ$, quadrant II.
 19. $1\frac{3}{8}\pi = 247^\circ 30'$, quadrant III. 23. $1 = 57^\circ 17' 45''$, quadrant I.
 20. $1\frac{4}{5}\pi = 324^\circ$, quadrant IV. 24. $-2 = -114^\circ 35' 30''$, quadrant III.

Express the following in degrees and also in radians :

25. $\frac{3}{5}$ of four right angles. $\frac{3}{5} \times 360^\circ = 216^\circ$; $0.017453 \times 216 = 3.769848$ radians.

26. $\frac{5}{6}$ of four right angles. $\frac{5}{6}$ of $360^\circ = 300^\circ$; $0.017453 \times 300 = 5.2359$ radians.

27. $\frac{2}{3}$ of two right angles. $\frac{2}{3}$ of $180^\circ = 120^\circ$; $0.017453 \times 120 = 2.09436$ radians.

28. $\frac{3}{8}$ of one right angle. $\frac{3}{8} \times 90^\circ = 33^\circ 45'$; $0.017453 \times 33\frac{3}{4} = 0.589039$ radians.

29. What decimal part of a radian is 1° ? $1'$?

$$(a) 1^\circ = \frac{1}{180}\pi = 0.017453 \text{ radian.}$$

$$(b) 1' = \frac{1}{180}\pi \times \frac{1}{60} = \frac{1}{10800}\pi = 0.0002909 \text{ radian.}$$

30. How many minutes in a radian? How many seconds?

$$1 \text{ radian} = 57^\circ 17' 45'' = 3437.75' = 206,265''.$$

31. Express in radians the angle of an equilateral triangle.

$$\frac{1}{3} \text{ of } 180^\circ = 60^\circ; 0.017453 \times 60 = 1.04718 \text{ radians.}$$

32. Over what part of a radian does the minute hand of a clock move in 15 min.?

$$15 \text{ minute spaces} = \frac{1}{4} \text{ of } 360^\circ = 90^\circ; 0.017453 \times 90 = 1.57077 \text{ radians.}$$

Exercise 69. Sines and Small Angles (Page 154)

1. Find four angles whose sine is 0.2756.

By § 138,

$$\sin x = \sin [n\pi + (-1)^n x].$$

$$n = 0, 1, 2, 3.$$

$$\sin x = \sin x, \sin (\pi - x),$$

$$\sin (2\pi + x),$$

$$\sin (3\pi - x).$$

$$\sin x = 0.2756.$$

$$\therefore x = 16^\circ, 164^\circ, 376^\circ, 524^\circ.$$

2. Find six angles whose sine is 0.5000.

$$\sin x = \sin [n\pi + (-1)^n x].$$

$$n = 0, 1, 2, 3, 4, 5.$$

$$\sin x = \sin x, \sin (\pi - x),$$

$$\sin (2\pi + x), \sin (3\pi - x),$$

$$\sin (4\pi + x), \sin (5\pi - x).$$

$$\sin x = 0.5000.$$

$$x = 30^\circ, 150^\circ, 390^\circ, 510^\circ, 750^\circ,$$

$$870^\circ.$$

3. Find eight angles having the same sine as $\frac{1}{6}\pi$.

$$\frac{1}{6}\pi = 30^\circ.$$

From Ex. 2, six angles having the same sine as $\frac{1}{6}\pi$ are 30° , 150° , 390° , 510° , 750° , 870° ;

and $\sin 30^\circ = \sin(6\pi + 30^\circ)$,
 $\sin(7\pi - 30^\circ)$.

Hence the required angles are 30° , 150° , 390° , 510° , 750° , 870° , 1110° , and 1230° .

4. Find four angles having the same cosecant as $\frac{3}{8}\pi$.

$$\csc x = \frac{1}{\sin x}.$$

From Ex. 1, $\sin x = \sin x$, $\sin(\pi - x)$,
 $\sin(2\pi + x)$, $\sin(3\pi - x)$.

$$\begin{aligned}\csc x &= \csc x, \csc(\pi - x), \\ &\csc(2\pi + x), \csc(3\pi - x). \\ \frac{3}{8}\pi &= 67^\circ 30'.$$

Hence the required angles are $67^\circ 30'$, $112^\circ 30'$, $427^\circ 30'$, and $472^\circ 30'$.

5. Find four angles having the same cosecant as 0.1π .

From Ex. 4, $\csc x = \csc x$, $\csc(\pi - x)$, $\csc(2\pi + x)$, $\csc(3\pi - x)$.

$$0.1\pi = 18^\circ.$$

Hence the required angles are 18° , 162° , 378° , and 522° .

Given $\pi = 3.141592653589$, compute to eleven decimal places:

6. $\cos 1'$.

The circular measure of $1'$ is

$$\frac{\pi}{10800} = \frac{3.141592653589}{10800} = 0.0002908882086 +, \text{ the next figure being } 6.$$

Taking the value of $\sin 1'$ as computed in the textbook, $0.00029088 +$, we have

$$\begin{aligned}\cos 1' &> \sqrt{1 - (0.00029088)^2} \\ &> \sqrt{1 - 0.00000064617} \\ &> \sqrt{0.99999915383} \\ &> 0.99999957691.\end{aligned}$$

Also

$$\begin{aligned}\cos 1' &< \sqrt{1 - (0.00029088)^2} \\ &< \sqrt{1 - 0.00000064611} \\ &< \sqrt{0.99999915389} \\ &< 0.99999957694.\end{aligned}$$

Hence $\cos 1' = 0.9999995769$, correct to eleven decimal places.

7. $\sin 1'$.

By § 137, $\sin x > x \cos x$.

By Ex. 6, $x = 0.0002908882086$,

and $\cos x = 0.9999995769$.

$$\begin{aligned}\therefore \sin 1' &> 0.0002908882086 \times 0.9999995769 \\ &> 0.0002908882086 (1 - 0.0000004231) \\ &> 0.0002908882086 - 0.00000000012 \\ &> 0.000290888196.\end{aligned}$$

Therefore $\sin 1'$ lies between 0.000290888196 and 0.0002908882087.
Hence, to eleven places,

$$\sin 1' = 0.00029088820.$$

8. $\tan 1'$.

From the values of $\sin 1'$ and $\cos 1'$ we have

$$\begin{aligned}\tan 1' &= \frac{\sin 1'}{\cos 1'} \\ &= \frac{0.00029088820}{0.99999995769} \\ &= 0.000290888212.\end{aligned}$$

9. $\sin 2'$.

The circular measure of $2'$ is $\frac{\pi}{5400} = \frac{3.141592653589}{5400}$
 $= 0.000581776417 + .$

Hence $\sin 2'$ lies between 0 and 0.000581776418.

$$\begin{aligned}\therefore \cos 2' &> \sqrt{1 - 0.000581776418^2} \\ &> \sqrt{1.000581776418 \times 0.999418223582} \\ &> \sqrt{0.9999999661536199459089276} \\ &> 0.999999830768.\end{aligned}$$

But $\sin x > x \cos x$.

$$\begin{aligned}\therefore \sin 2' &> 0.000581776418 \times 0.999999830768 \\ &> 0.000581776418 \times (10.000000169232) \\ &> 0.000581776418 - 0.000000000098 \\ &> 0.000581776320.\end{aligned}$$

Hence $\sin 2' = 0.00058177632$, correct to eleven decimal places.

10. From the results of Exs. 6 and 7, and by the aid of the formula $\sin 2x = 2 \sin x \cos x$, compute $\sin 2'$, carrying the multiplication to six decimal places. Compare the result with that of Ex. 9.

$$\begin{aligned}\sin 2' &= \sin (1' + 1') \\ &= 2 \sin 1' \cos 1' .\end{aligned}$$

By Ex. 6,

$$\sin 1' = 0.0002908 + .$$

By Ex. 7,

$$\cos 1' = 0.9999999 + .$$

$$2 \sin 1' \cos 1' = 2 (0.0002908 \times 0.9999999).$$

$$\sin 2' = 0.0005790.$$

By Ex. 9,

$$\sin 2' = 0.0005817.$$

The method of Ex. 9 gives a result larger by 0.0000018 than that of Ex. 10.

11. Compute $\sin 1^\circ$ to four decimal places.

The circular measure of 1° is

$$\frac{\pi}{180} = \frac{3.141592653589}{180} = 0.01745329.$$

Hence

$$\begin{aligned}\cos 1^\circ &> \sqrt{1 - (0.01746)^2} \\ &> \sqrt{0.99969515} \\ &> 0.999847.\end{aligned}$$

$$\sin x > x \cos x.$$

$$\begin{aligned}\therefore \sin 1^\circ &> 0.017453 \times 0.999847 \\ &> 0.017453 (1 - 0.000153) \\ &> 0.017453 - 0.0000027 \\ &> 0.0174503.\end{aligned}$$

Hence, to four decimal places,
 $\sin 1^\circ = 0.0175$.

12. From the formula

$$\cos x = 1 - 2 \sin^2 \frac{x}{2},$$

show that $\cos x > 1 - \frac{x^2}{2}$.

$$\sin x < x.$$

$$\therefore \sin \frac{x}{2} < \frac{x}{2}.$$

$$\sin^2 \frac{x}{2} < \frac{x^2}{4}.$$

$$1 - 2 \sin^2 \frac{x}{2} > 1 - \frac{x^2}{2}.$$

$$\therefore \cos x > 1 - \frac{x^2}{2}.$$

Exercise 70. Angles having the Same Functions (Page 155)

1. Find two positive angles that have $\frac{1}{2}$ as their cosine.

$$\cos x = \cos (2n\pi \pm x).$$

$$n = 0, 1.$$

$$\cos x = \cos \pm x, \cos (2\pi \pm x).$$

$$\frac{1}{2} = \cos 60^\circ.$$

Hence the required angles are 60° and 300° .

2. Find two negative angles that have $\frac{1}{2}$ as their cosine.

From the formula in Ex. 1,

$$n = 0, -1.$$

$$\cos x = \cos (-x) \text{ and } \cos (-2\pi \pm x).$$

Hence the required angles are -60° and -300° .

3. Find four angles whose cosine is the same as the cosine of 25°

From Ex. 1, $\cos x = \cos (\pm x)$, $\cos (2\pi \pm x)$, and $\cos (4\pi \pm x)$.

The other angles are $2\pi - 25^\circ$, $2\pi + 25^\circ$, $4\pi - 25^\circ$.

Hence the required angles are -25° , 335° , 385° , and 695° .

4. Find four angles that have 2 as their secant.

$$\sec x = \frac{1}{\cos x}.$$

$$\sec x = \sec (\pm x), \sec (2\pi \pm x), \sec (4\pi \pm x).$$

$$\sec 60^\circ = 2.$$

Hence the required angles are 60° , 300° , 420° , and 660° .

5. Find two positive angles that have 1 as their tangent.

By § 140,

$$\tan x = \tan (n\pi + x).$$

$$n = 0, 1.$$

$$\tan x = \tan x, \tan (\pi + x).$$

$$\tan 45^\circ = 1.$$

Hence the required angles are 45° and 225° .

6. Find two negative angles that have 1 as their tangent.

$$\tan x = \tan (n\pi + x).$$

$$n = -1 \text{ or } -2.$$

$$\tan x = \tan (-\pi + x), \tan (-2\pi + x)$$

$$\tan 45^\circ = 1.$$

Hence the required angles are -135° and -315° .

7. Find four angles that have $\sqrt{3}$ as their tangent.

$$\tan x = \tan (n\pi + x).$$

$$n = 0, 1, 2, 3.$$

$$\tan x = \tan x, \tan (\pi + x), \tan (2\pi + x), \tan (3\pi + x).$$

$$\tan 60^\circ = \sqrt{3}.$$

Hence the required angles are 60° , 240° , 420° , and 600° .

8. Find four angles that have $\sqrt{3}$ as their cotangent.

By § 140,

$$\tan x = \tan (n\pi + x).$$

Since

$$\cot x = \frac{1}{\tan x},$$

$$\cot x = \cot (n\pi + x).$$

$$n = 0, 1, 2, \text{ and } 3.$$

$$\cot x = \cot x, \cot (\pi + x), \cot (2\pi + x), \cot (3\pi + x).$$

$$\cot 30^\circ = \sqrt{3}.$$

Hence the required angles are 30° , 210° , 390° , and 570° .

9. Find four angles that have 0.5000 as their tangent.

$$\tan x = \tan (n\pi + x).$$

$$n = 0, 1, 2, 3.$$

$$\tan x = \tan x, \tan (\pi + x), \tan (2\pi + x), \tan (3\pi + x).$$

$$0.5000 = \tan 26^\circ 34'.$$

Hence the required angles are $26^\circ 34'$, $206^\circ 34'$, $386^\circ 34'$, and $566^\circ 34'$.

10. Find four negative angles whose cotangent is 0.5000.

$$\cot x = \cot (n\pi + x).$$

$$n = -1, -2, -3, -4.$$

$$\cot x = \cot (-\pi + x), \cot (-2\pi + x), \cot (-3\pi + x), \cot (-4\pi + x).$$

$$0.5000 = \cot 63^\circ 26'.$$

Hence the required angles are $-116^\circ 34'$, $-296^\circ 34'$, $-476^\circ 34'$, and $-656^\circ 34'$.

Exercise 71. Inverse Functions (Page 156)*Prove the following formulas :*

1. $\sin^{-1} x + \cos^{-1} x = \frac{1}{2} \pi.$

2. $\tan^{-1} x + \cot^{-1} x = \frac{1}{2} \pi.$

$$\begin{array}{ll} \text{Let} & y = \sin^{-1} x; \\ \text{then} & 90^\circ - y = \cos^{-1} x. \end{array}$$

$$\begin{array}{ll} \text{Let} & y = \tan^{-1} x; \\ \text{then} & 90^\circ - y = \cot^{-1} x \end{array}$$

$$\begin{array}{l} \text{Add,} \\ \sin^{-1} x + \cos^{-1} x = 90^\circ. \end{array}$$

$$\begin{array}{l} \text{Add,} \\ \tan^{-1} x + \cot^{-1} x = 90^\circ \end{array}$$

$\therefore \sin^{-1} x + \cos^{-1} x = \frac{1}{2} \pi$

$= \frac{1}{2} \pi.$

3. $\sec^{-1} x + \csc^{-1} x = \frac{1}{2} \pi.$

$$\begin{array}{ll} \text{Let} & y = \sec^{-1} x; \\ \text{then} & 90^\circ - y = \csc^{-1} x. \end{array}$$

$$\text{Add,} \quad \sec^{-1} x + \csc^{-1} x = 90^\circ = \frac{1}{2} \pi.$$

4. $\sin^{-1}(-x) = -\sin^{-1} x.$

$$\begin{aligned} \sin^{-1}(-x) &= \text{the angle whose sine is } -x \\ &= -\text{the angle whose sine is } x \\ &= -\sin^{-1} x. \end{aligned}$$

Find two values of each of the following :

5. $\sin^{-1} \frac{1}{2} \sqrt{3}.$

9. $\sec^{-1} 2.$

$\sin x = \sin(\pi - x).$

$\cos x = \cos(2\pi \pm x).$

$\sin^{-1} \frac{1}{2} \sqrt{3} = 60^\circ \text{ and } (\pi - 60^\circ).$

$\therefore \sec x = \sec(2\pi \pm x).$

$\therefore \angle$ are 60° and $120^\circ.$

$\sec^{-1} 2 = 60^\circ \text{ and } (2\pi - 60^\circ).$

$\therefore \angle$ are 60° and $300^\circ.$

6. $\csc^{-1} \sqrt{2}.$

10. $\cos^{-1}(-\frac{1}{2} \sqrt{2}).$

$\csc x = \csc(\pi - x).$

The $\angle$ must be in second and third quadrants.

$\csc^{-1} \sqrt{2} = 45^\circ \text{ and } (\pi - 45^\circ).$

$\therefore \angle$ are 45° and $135^\circ.$

$\cos x = \cos(2\pi \pm x)$

Since $\cos^{-1} \frac{1}{2} \sqrt{2} = 45^\circ,$

$\cos^{-1}(-\frac{1}{2} \sqrt{2}) = 135$

and $(2\pi - 135^\circ).$

$\therefore \angle$ are 135° and $225^\circ.$

7. $\tan^{-1} \frac{1}{3} \sqrt{3}.$

$\tan x = \tan(\pi + x).$

$\tan^{-1} \frac{1}{3} \sqrt{3} = 30^\circ \text{ and } (\pi + 30^\circ).$

$\therefore \angle$ are 30° and $210^\circ.$

11. Find the value of the sine of the angle whose cosine is $\frac{1}{2}$; that is, the value of $\sin(\cos^{-1} \frac{1}{2}).$

8. $\tan^{-1} \infty.$

$\cos^{-1} \frac{1}{2} = 60^\circ.$

$\tan^{-1} \infty = 90^\circ \text{ and } (\pi + 90^\circ).$

$\sin 60^\circ = \frac{1}{2} \sqrt{3}.$

$\therefore \angle$ are 90° and $270^\circ.$

$\therefore \sin(\cos^{-1} \frac{1}{2}) = \frac{1}{2} \sqrt{3}.$

Find the values of the following :

12. $\sin(\cos^{-1}\frac{1}{2}\sqrt{3})$.

$$\cos^{-1}\frac{1}{2}\sqrt{3} = 30^\circ.$$

$$\sin 30^\circ = \frac{1}{2}.$$

$$\therefore \sin(\cos^{-1}\frac{1}{2}\sqrt{3}) = \frac{1}{2}.$$

14. $\cos(\cot^{-1}1)$.

$$\cot^{-1}1 = 45^\circ.$$

$$\cos 45^\circ = \frac{1}{2}\sqrt{2}.$$

$$\therefore \cos(\cot^{-1}1) = \frac{1}{2}\sqrt{2}.$$

13. $\sin(\tan^{-1}1)$.

$$\tan^{-1}1 = 45^\circ.$$

$$\sin 45^\circ = \frac{1}{2}\sqrt{2}.$$

$$\therefore \sin(\tan^{-1}1) = \frac{1}{2}\sqrt{2}.$$

Prove the following formulas :

15. $\tan(\tan^{-1}x + \tan^{-1}y)$

$$= \frac{x+y}{1-xy}.$$

By [26], $\tan(\tan^{-1}x + \tan^{-1}y)$

$$= \frac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\tan(\tan^{-1}y)}$$

$$= \frac{x+y}{1-xy}.$$

$$= \frac{x+y}{1-xy}.$$

16. $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1}x.$

Let $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = y;$

then $\frac{x}{\sqrt{1-x^2}} = \tan y.$

By [20], $\sec^2 y = 1 + \tan^2 y$

$$= 1 + \left(\frac{x}{\sqrt{1-x^2}}\right)^2$$

$$= \frac{1}{1-x^2}.$$

$$\therefore \cos^2 y = \frac{1}{\sec^2 y} = 1 - x^2.$$

By [16], $\sin^2 y = 1 - \cos^2 y = x^2.$

$$\therefore \sin y = x.$$

$$\therefore y = \sin^{-1}x.$$

$$\therefore \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1}x.$$

17. $\tan(2\tan^{-1}x) = \frac{2x}{1-x^2}.$

Let $\tan^{-1}x = y.$

then $x = \tan y.$

$$\tan(2\tan^{-1}x) = \tan 2y;$$

by [31], $= \frac{2\tan y}{1-\tan^2 y}$

$$= \frac{2x}{1-x^2}.$$

18. $\sin(2\tan^{-1}x) = \frac{2x}{1+x^2}.$

Let $\tan^{-1}x = y;$

then $x = \tan y.$

$$\sin(2\tan^{-1}x) = \sin 2y$$

$$= 2\sin y \cos y$$

$$= 2\frac{\sin y}{\cos y} \cos^2 y$$

$$= \frac{2\tan y}{\sec^2 y}$$

$$= \frac{2\tan y}{1+\tan^2 y}$$

$$= \frac{2x}{1+x^2}.$$

Find four values of each of the following :

19. $\cot^{-1}0.5774.$

$$\cot^{-1}0.5774 = 60^\circ.$$

$\therefore$ The four values are $60^\circ, \pi + 60^\circ, 2\pi + 60^\circ, 3\pi + 60^\circ$, or $60^\circ, 240^\circ, 420^\circ$, and 600° .

20. $\cot^{-1}0.6249$.

$$\cot^{-1}0.6249 = 58^\circ.$$

$\therefore$ The four values are $58^\circ, \pi + 58^\circ, 2\pi + 58^\circ$, and $3\pi + 58^\circ$, or $58^\circ, 238^\circ, 418^\circ$, and 598° .

21. $\sin^{-1}0.9613$.

$$\sin^{-1}0.9613 = 74^\circ.$$

$\therefore$ The four values are $74^\circ, \pi - 74^\circ, 2\pi + 74^\circ$, and $3\pi - 74^\circ$, or $74^\circ, 106^\circ, 434^\circ$, and 466° .

22. $\sin^{-1}0.3256$.

$$\sin^{-1}0.3256 = 19^\circ.$$

$\therefore$ The four values are $19^\circ, \pi - 19^\circ, 2\pi + 19^\circ$, and $3\pi - 19^\circ$, or $19^\circ, 161^\circ, 379^\circ$, and 521° .

23. $\tan^{-1}0.2756$.

$$\tan^{-1}0.2756 = 15^\circ 24' 30''.$$

$\therefore$ The four values are $15^\circ 24' 30'', \pi + 15^\circ 24' 30'', 2\pi + 15^\circ 24' 30'',$ and $3\pi + 15^\circ 24' 30'',$ or $15^\circ 24' 30'', 195^\circ 24' 30'', 375^\circ 24' 30'',$ and $555^\circ 24' 30''$.

24. $\cos^{-1}0.9455$.

$$\cos^{-1}0.9455 = 19^\circ.$$

$\therefore$ The four values are $19^\circ, 2\pi - x, 2\pi + x$, and $4\pi - x$, or $19^\circ, 341^\circ, 379^\circ$, and 701° .

27. If $\sin^{-1}x = 2 \cos^{-1}x$, find the value of x .

$$\sin^{-1}x = 90^\circ - \cos^{-1}x.$$

$$\therefore 90^\circ - \cos^{-1}x = 2 \cos^{-1}x.$$

$$3 \cos^{-1}x = 90^\circ.$$

$$\cos^{-1}x = 30^\circ, 150^\circ, \text{ or } 270^\circ.$$

$$x = \pm \frac{1}{2}\sqrt{3}, \text{ or } 0.$$

Prove the following formulas:

28. $\cos(\sin^{-1}x) = \sqrt{1-x^2}$.

Let
then

$$\sin^{-1}x = y;$$

$$x = \sin y.$$

$$\sqrt{1-x^2} = \cos y = \cos(\sin^{-1}x).$$

25. Solve the equation $y = \sin^{-1}\frac{1}{3}$.

$$\sin y = \frac{1}{3} = 0.3333 +.$$

By § 138,

$$\sin y = \sin[n\pi + (-1)^n y].$$

$$y = 19^\circ 28' \text{ or } [n\pi + (-1)^n 19^\circ 28'].$$

26. Find the value of

$$\sin(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}).$$

Let $\tan^{-1}\frac{1}{2} = x,$

$$\tan^{-1}\frac{1}{3} = y;$$

then $\tan x = \frac{1}{2},$

and $\tan y = \frac{1}{3}.$

By [20], $\sec^2 x = 1 + \tan^2 x = \frac{5}{4}.$

By [8], $\cos^2 x = \frac{4}{5}.$

$$\cos x = \frac{\pm 2}{\sqrt{5}}.$$

By [16], $\sin x = \frac{\pm 1}{\sqrt{5}}.$

$$\cos y = \frac{\pm 3}{\sqrt{10}}.$$

$$\sin y = \frac{\pm 1}{\sqrt{10}}.$$

$$\sin(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3})$$

$$= \sin(x + y)$$

$$= \sin x \cos y + \cos x \sin y$$

$$= \frac{1}{\sqrt{5}} \times \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{10}}$$

$$= \frac{\pm 5}{\sqrt{50}} = \pm \frac{1}{2}\sqrt{2}.$$

$$29. \cos(2 \sin^{-1} x) = 1 - 2x^2.$$

Let
then

$$\sin^{-1} x = y;$$

$$x = \sin y.$$

$$\begin{aligned} \cos(2 \sin^{-1} x) &= \cos 2y = \cos^2 y - \sin^2 y = 1 - \sin^2 y - \sin^2 y \\ &= 1 - 2 \sin^2 y = 1 - 2x^2. \end{aligned}$$

$$30. \sin(\sin^{-1} x) = x. \quad \sin^{-1} x \text{ is an angle whose sine is } x.$$

$$\therefore \sin(\sin^{-1} x) = x.$$

$$31. \sin(\sin^{-1} x + \sin^{-1} y) = x \sqrt{1 - y^2} + y \sqrt{1 - x^2}.$$

$$\begin{aligned} \text{By [22], } \sin(\sin^{-1} x + \sin^{-1} y) &= \sin(\sin^{-1} x) \cos(\sin^{-1} y) + \\ &\quad \cos(\sin^{-1} x) \sin(\sin^{-1} y) \\ &= x \cos(\sin^{-1} y) + y \cos(\sin^{-1} x) \\ &= x \sqrt{1 - y^2} + y \sqrt{1 - x^2}. \end{aligned}$$

By [16],

$$32. \tan^{-1} 2 + \tan^{-1} \frac{1}{2} = \frac{1}{2} \pi.$$

$$\begin{aligned} \text{By [26], } \tan(\tan^{-1} 2 + \tan^{-1} \frac{1}{2}) &= \frac{\tan(\tan^{-1} 2) + \tan(\tan^{-1} \frac{1}{2})}{1 - \tan(\tan^{-1} 2) \tan(\tan^{-1} \frac{1}{2})} \\ &= \frac{2 + \frac{1}{2}}{1 - 2 \cdot \frac{1}{2}} = \frac{2\frac{1}{2}}{0} = \infty. \end{aligned}$$

$$\tan^{-1} \infty = 90^\circ = \frac{1}{2} \pi.$$

$$\therefore \tan^{-1} 2 + \tan^{-1} \frac{1}{2} = \frac{1}{2} \pi.$$

$$33. 2 \tan^{-1} x = \tan^{-1} [2x : (1 - x^2)].$$

$$\text{By [31], } \tan(2 \tan^{-1} x) = \frac{2 \tan(\tan^{-1} x)}{1 - \tan^2(\tan^{-1} x)} = \frac{2x}{1 - x^2}.$$

$$\therefore 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1 - x^2} = \tan^{-1} [2x : (1 - x^2)].$$

$$34. 2 \sin^{-1} x = \sin^{-1} (2x \sqrt{1 - x^2}).$$

$$\begin{aligned} \text{By [30], } \sin(2 \sin^{-1} x) &= 2 \sin(\sin^{-1} x) \cos(\sin^{-1} x) = 2x \sqrt{1 - x^2} \\ \therefore 2 \sin^{-1} x &= \sin^{-1} (2x \sqrt{1 - x^2}). \end{aligned}$$

$$35. 2 \cos^{-1} x = \cos^{-1} (2x^2 - 1).$$

$$\text{By [32] and [16], } \cos(2 \cos^{-1} x) = 2 \cos^2(\cos^{-1} x) - 1 = 2x^2 - 1.$$

$$\therefore 2 \cos^{-1} x = \cos^{-1} (2x^2 - 1).$$

$$36. 3 \tan^{-1} x = \tan^{-1} [(3x - x^3) : (1 - 3x^2)].$$

$$\tan(3 \tan^{-1} x) = \tan(\tan^{-1} x + 2 \tan^{-1} x)$$

$$\begin{aligned} \text{By [26], } &= \frac{\tan(\tan^{-1} x) + \tan(2 \tan^{-1} x)}{1 - \tan(\tan^{-1} x) \tan(2 \tan^{-1} x)} \\ &= \frac{x + \tan(2 \tan^{-1} x)}{1 - x \tan(2 \tan^{-1} x)} \end{aligned}$$

$$\begin{aligned} &= \frac{x + \frac{2x}{1 - x^2}}{1 - x \frac{2x}{1 - x^2}} = \frac{3x - x^3}{1 - 3x^2}. \end{aligned}$$

$$\therefore 3 \tan^{-1} x = \tan^{-1} [(3x - x^3) : (1 - 3x^2)].$$

$$\begin{aligned} 37. \sin^{-1} \sqrt{x:y} \\ = \tan^{-1} \sqrt{x:(y-x)}. \end{aligned}$$

$$\text{Let } \sin^{-1} \sqrt{\frac{x}{y}} = n.$$

$$\text{Then } \sqrt{\frac{x}{y}} = \sin n.$$

$$\sqrt{\frac{y-x}{y}} = \cos n.$$

$$\begin{aligned} \sqrt{\frac{x}{y-x}} &= \tan n. \\ \therefore n &= \tan^{-1} \sqrt{\frac{x}{y-x}}. \end{aligned}$$

$$\therefore \sin^{-1} \sqrt{x:y} = \tan^{-1} \sqrt{x:(y-x)}.$$

$$\begin{aligned} 38. \sin^{-1} \sqrt{(x-y):(x-z)} \\ = \tan^{-1} \sqrt{(x-y):(y-z)}. \end{aligned}$$

$$\text{Let } \sin^{-1} \sqrt{\frac{x-y}{x-z}} = n.$$

$$\text{Then } \sqrt{\frac{x-y}{x-z}} = \sin n.$$

$$\sqrt{\frac{y-z}{x-z}} = \cos n.$$

$$\begin{aligned} \sqrt{\frac{x-y}{y-z}} &= \tan n. \\ n &= \tan^{-1} \sqrt{\frac{x-y}{y-z}}. \end{aligned}$$

$$\begin{aligned} \therefore \sin^{-1} \sqrt{(x-y):(x-z)} \\ = \tan^{-1} \sqrt{(x-y):(y-z)}. \end{aligned}$$

$$41. \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{1}{4} \pi.$$

$$\text{By [26], } \tan(\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = 1.$$

$$\therefore \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} 1 = \frac{1}{4} \pi.$$

$$42. \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{4}{7}.$$

$$\text{By [26], } \tan(\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5}) = \frac{\frac{1}{3} + \frac{1}{5}}{1 - \frac{1}{3} \times \frac{1}{5}} = \frac{4}{7}.$$

$$\therefore \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{4}{7}.$$

$$39. \sin^{-1} x = \sec^{-1} (1 : \sqrt{1-x^2}).$$

$$\text{Let } \sin^{-1} x = n.$$

$$\text{Then } \frac{x}{\sqrt{1-x^2}} = \sec n.$$

$$\frac{1}{\sqrt{1-x^2}} = \sec n.$$

$$n = \sec^{-1} \frac{1}{\sqrt{1-x^2}}.$$

$$\therefore \sin^{-1} x = \sec^{-1} (1 : \sqrt{1-x^2})$$

$$\begin{aligned} 40. 2 \sec^{-1} x \\ = \tan^{-1} [2\sqrt{x^2-1} : (2-x^2)]. \end{aligned}$$

$$\text{Let } 2 \sec^{-1} x = n.$$

$$\text{Then } x = \sec \frac{1}{2} n.$$

$$\frac{1}{x} = \cos \frac{1}{2} n.$$

$$\text{By [32] and [16],}$$

$$2 \left(\frac{1}{x} \right)^2 - 1 = \cos n.$$

$$\frac{2-x^2}{x^2} = \cos n.$$

$$\frac{x^2}{2-x^2} = \sec n.$$

$$\text{By [20],}$$

$$\left(\frac{x^2}{2-x^2} \right)^2 - 1 = \tan^2 n.$$

$$\frac{4x^2-4}{(2-x^2)^2} = \tan^2 n.$$

$$\tan n = \frac{2\sqrt{x^2-1}}{2-x^2}.$$

$$n = \tan^{-1} \frac{2\sqrt{x^2-1}}{2-x^2}$$

$$\therefore 2 \sec^{-1} x = \tan^{-1} [2\sqrt{x^2-1} : (2-x^2)].$$

$$43. \sin^{-1} \frac{2}{3} + \sin^{-1} \frac{1}{3} = \sin^{-1} \frac{6}{5}.$$

$$\begin{aligned} \text{By [22],} \quad \sin(\sin^{-1} \frac{2}{3} + \sin^{-1} \frac{1}{3}) &= \sin(\sin^{-1} \frac{2}{3}) \cos(\sin^{-1} \frac{1}{3}) + \cos(\sin^{-1} \frac{2}{3}) \sin(\sin^{-1} \frac{1}{3}) \\ &= \frac{2}{3} \times \frac{4}{5} + \frac{4}{5} \times \frac{1}{3} = \frac{6}{5}. \\ \therefore \sin^{-1} \frac{2}{3} + \sin^{-1} \frac{1}{3} &= \sin^{-1} \frac{6}{5}. \end{aligned}$$

$$44. \sin^{-1} \frac{1}{8} \sqrt{82} + \sin^{-1} \frac{4}{41} \sqrt{41} = \frac{1}{4} \pi.$$

$$\begin{aligned} \text{By [22],} \quad \sin\left(\sin^{-1} \frac{1}{\sqrt{82}} + \sin^{-1} \frac{4}{\sqrt{41}}\right) &= \frac{1}{\sqrt{82}} \times \frac{5}{\sqrt{41}} + \frac{9}{\sqrt{82}} \times \frac{4}{\sqrt{41}} \\ &= \frac{5}{41\sqrt{2}} + \frac{36}{41\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2} \sqrt{2}. \\ \therefore \sin^{-1} \frac{1}{8} \sqrt{82} + \sin^{-1} \frac{4}{41} \sqrt{41} &= \sin^{-1} \frac{1}{2} \sqrt{2} = \frac{1}{4} \pi. \end{aligned}$$

$$45. \sec^{-1} \frac{5}{3} + \sec^{-1} \frac{1}{2} = 75^\circ 45'.$$

$$\begin{aligned} \sec^{-1} \frac{5}{3} + \sec^{-1} \frac{1}{2} &= \cos^{-1} \frac{3}{5} + \cos^{-1} \frac{1}{2}. \\ \text{By [24],} \quad \cos(\cos^{-1} \frac{3}{5} + \cos^{-1} \frac{1}{2}) &= \frac{3}{5} \times \frac{1}{2} - \frac{4}{5} \times \frac{\sqrt{3}}{2} = \frac{1}{6}. \\ \therefore \cos^{-1} \frac{3}{5} + \cos^{-1} \frac{1}{2} &= \cos^{-1} \frac{1}{6}. \\ \sec^{-1} \frac{5}{3} + \sec^{-1} \frac{1}{2} &= \sec^{-1} \frac{6}{5} = 75^\circ 45'. \end{aligned}$$

$$46. \tan^{-1}(2 + \sqrt{3}) - \tan^{-1}(2 - \sqrt{3}) = \sec^{-1} 2.$$

$$\text{Let } \tan^{-1}(2 + \sqrt{3}) - \tan^{-1}(2 - \sqrt{3}) = n.$$

$$\begin{aligned} \text{By [27],} \quad \tan n &= \frac{(2 + \sqrt{3}) - (2 - \sqrt{3})}{1 + (2 + \sqrt{3})(2 - \sqrt{3})} \\ &= \frac{2\sqrt{3}}{2} = \sqrt{3}. \\ \therefore n &= 60^\circ. \end{aligned}$$

$$\sec n = 2.$$

$$\therefore \tan^{-1}(2 + \sqrt{3}) - \tan^{-1}(2 - \sqrt{3}) = \sec^{-1} 2.$$

$$47. \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{1}{4} \pi.$$

$$\begin{aligned} \text{Let } \quad \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} &= n, \\ \text{and} \quad \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} &= v. \end{aligned}$$

$$\begin{aligned} \text{By [26],} \quad \tan n &= \frac{\frac{1}{3} + \frac{1}{5}}{1 - \frac{1}{3} \times \frac{1}{5}} = \frac{4}{7}. \\ \tan v &= \frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \frac{1}{8}} = \frac{3}{11}. \\ \tan(n + v) &= \frac{\frac{4}{7} + \frac{3}{11}}{1 - \frac{4}{7} \times \frac{3}{11}} = 1. \\ n + v &= \tan^{-1} 1 = \frac{1}{4} \pi. \end{aligned}$$

$$\therefore \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{1}{4} \pi.$$

$$48. \sin^{-1}x + \sin^{-1}\sqrt{1-x^2} = \frac{1}{2}\pi.$$

By [22],

$$\begin{aligned} \sin(\sin^{-1}x + \sin^{-1}\sqrt{1-x^2}) &= \sin(\sin^{-1}x) \cos(\sin^{-1}\sqrt{1-x^2}) \\ &\quad + \cos(\sin^{-1}x) \sin(\sin^{-1}\sqrt{1-x^2}) \\ &= x \cos(\sin^{-1}\sqrt{1-x^2}) + \sqrt{1-x^2} \cos(\sin^{-1}x); \\ \text{by [16],} \quad &= x \cos(\cos^{-1}\sqrt{1-(1-x^2)}) \\ &\quad + \sqrt{1-x^2} \cos(\cos^{-1}\sqrt{1-x^2}) \\ &= x^2 + 1 - x^2 \\ &= 1. \end{aligned}$$

$$\sin^{-1}1 = 90^\circ.$$

$$\therefore \sin^{-1}x + \sin^{-1}\sqrt{1-x^2} = 90^\circ = \frac{1}{2}\pi.$$

$$49. \sin^{-1}0.5 + \sin^{-1}\frac{1}{2}\sqrt{3} = \sin^{-1}1.$$

$$\sin^{-1}0.5 = 30^\circ,$$

and

$$\sin^{-1}\frac{1}{2}\sqrt{3} = 60^\circ.$$

$$\therefore \sin^{-1}0.5 + \sin^{-1}\frac{1}{2}\sqrt{3} = 90^\circ.$$

$$\sin 90^\circ = 1.$$

$$\therefore \sin^{-1}0.5 + \sin^{-1}\frac{1}{2}\sqrt{3} = \sin^{-1}1.$$

$$50. \tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}.$$

$$\text{By [26],} \quad \tan(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}) = \frac{\frac{1}{4} + \frac{2}{9}}{1 - \frac{1}{4} \times \frac{2}{9}} = \frac{1}{2}.$$

$$\therefore \tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}.$$

$$51. \tan^{-1}0.5 + \tan^{-1}0.2 + \tan^{-1}0.125 = \frac{1}{4}\pi.$$

Let

$$\tan^{-1}0.5 + \tan^{-1}0.2 = n$$

and

$$\tan^{-1}0.125 = v.$$

By [26],

$$\tan n = \frac{0.5 + 0.2}{1 - 0.5 \times 0.2} = \frac{7}{9}.$$

$$\tan v = \frac{1}{8}.$$

$$\tan(n+v) = \frac{\frac{7}{9} + \frac{1}{8}}{1 - \frac{7}{9} \times \frac{1}{8}} = 1.$$

$$n+v = \tan^{-1}1 = 45^\circ = \frac{1}{4}\pi.$$

$$\therefore \tan^{-1}0.5 + \tan^{-1}0.2 + \tan^{-1}0.125 = \frac{1}{4}\pi.$$

$$52. \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi.$$

Let

$$\tan^{-1}1 + \tan^{-1}2 = n$$

and

$$\tan^{-1}3 = v.$$

By [26],

$$\tan n = \frac{1+2}{1-(1 \times 2)} = -3.$$

$$\tan v = 3.$$

By [26],

$$\tan(n+v) = \frac{-3+3}{1-(-3 \times 3)} = 0.$$

$$n+v = \tan^{-1}0 = 180^\circ.$$

$$\therefore \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = 180^\circ = \pi.$$

$$53. \tan^{-1} \frac{2}{3} + \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{1} = \frac{1}{2} \pi.$$

Let $\tan^{-1} \frac{2}{3} + \tan^{-1} \frac{1}{4} = n,$
and $\tan^{-1} \frac{1}{1} = v.$

By [26], $\tan n = \frac{\frac{2}{3} + \frac{1}{4}}{1 - \frac{2}{3} \times \frac{1}{4}} = \frac{11}{10}.$

$$\tan v = \frac{1}{1}.$$

By [26], $\tan(n + v) = \frac{\frac{11}{10} + \frac{1}{1}}{1 - \frac{11}{10} \times \frac{1}{1}} = \infty.$

$$n + v = \tan^{-1} \infty = 90^\circ = \frac{1}{2} \pi.$$

$$\therefore \tan^{-1} \frac{2}{3} + \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{1} = \frac{1}{2} \pi.$$

$$54. \cos^{-1} \frac{3}{10} \sqrt{10} + \sin^{-1} \frac{1}{5} \sqrt{5} = \frac{1}{4} \pi.$$

Let $\cos^{-1} \frac{3}{10} \sqrt{10} = x,$
and $\sin^{-1} \frac{1}{5} \sqrt{5} = y.$

$$\cos x = \frac{3}{10} \sqrt{10}.$$

$$\sin x = \sqrt{1 - \left(\frac{3\sqrt{10}}{10}\right)^2} = \frac{1}{10} \sqrt{10}.$$

$$\sin y = \frac{1}{5} \sqrt{5}.$$

$$\cos y = \sqrt{1 - \left(\frac{1}{5} \sqrt{5}\right)^2} = \frac{2}{5} \sqrt{5}.$$

$$\begin{aligned} \sin(x + y) &= \sin x \cos y + \cos x \sin y \\ &= \frac{1}{10} \sqrt{10} \times \frac{2}{5} \sqrt{5} + \frac{3}{10} \sqrt{10} \times \frac{1}{5} \sqrt{5} \\ &= \frac{1}{2} \sqrt{2}. \end{aligned}$$

$$x + y = \sin^{-1} \frac{1}{2} \sqrt{2} = 45^\circ = \frac{1}{4} \pi.$$

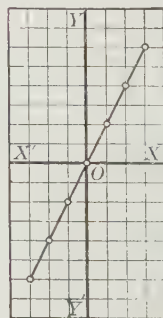
$$\therefore \cos^{-1} \frac{3}{10} \sqrt{10} + \sin^{-1} \frac{1}{5} \sqrt{5} = \frac{1}{4} \pi.$$

Exercise 72. Graphs (Page 158)

1. Plot the graph of the function $2x$.

Let $y = 2x.$

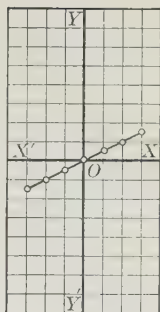
x	y	x	y
0	0		
1	2	-1	-2
2	4	-2	-4
3	6	-3	-6



2. Plot the graph of the function $\frac{1}{2}x$.

Let $y = \frac{1}{2}x$.

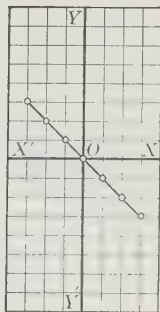
x	y	x	y
0	0		
1	$\frac{1}{2}$	-1	$-\frac{1}{2}$
2	1	-2	-1
3	$1\frac{1}{2}$	-3	$-1\frac{1}{2}$



3. Plot the graph of the function $-x$.

Let $y = -x$.

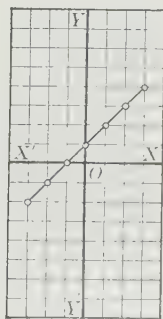
x	y	x	y
0	0		
1	-1	-1	1
2	-2	-2	2
3	-3	-3	3



4. Plot the graph of the function $x + 1$.

Let $y = x + 1$.

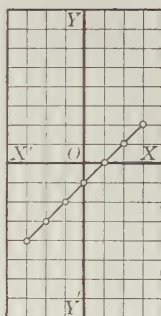
x	y	x	y
0	1		
1	2	-1	0
2	3	-2	-1
3	4	-3	-2



5. Plot the graph of the function $x - 1$.

Let $y = x - 1$.

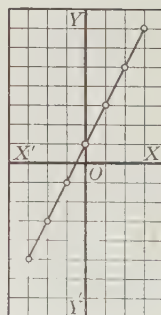
x	y	x	y
0	-1		
1	0	-1	-2
2	1	-2	-3
3	2	-3	-4



6. Plot the graph of the function $2x + 1$.

Let $y = 2x + 1$.

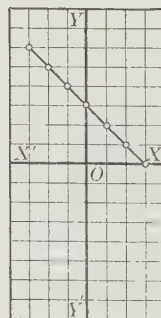
x	y	x	y
0	1		
1	3	-1	-1
2	5	-2	-3
3	7	-3	-5



7. Plot the graph of the function $3 - x$.

Let $y = 3 - x$.

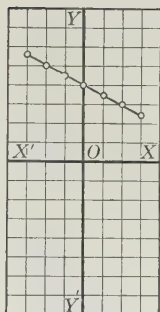
x	y	x	y
0	3		
1	2	-1	4
2	1	-2	5
3	0	-3	6



6. Plot the graph of the function $4 - \frac{1}{2}x$.

Let $y = 4 - \frac{1}{2}x$.

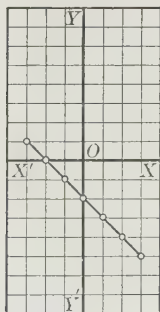
x	y	x	y
0	4		
1	$3\frac{1}{2}$	-1	$4\frac{1}{2}$
2	3	-2	5
3	$2\frac{1}{2}$	-3	$5\frac{1}{2}$



9. Plot the graph of the function $-2 - x$.

Let $y = -2 - x$.

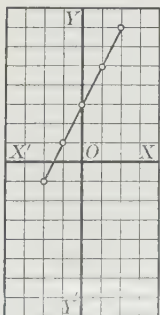
x	y	x	y
0	-2		
1	-3	-1	-1
2	-4	-2	0
3	-5	-3	1



10. Plot the graph of the function $2x + 3$.

Let $y = 2x + 3$.

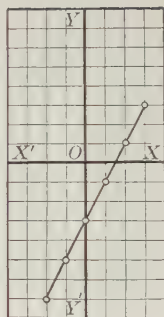
x	y	x	y
0	3		
1	5	-1	1
2	7	-2	-1



11. Plot the graph of the function $2x - 3$.

Let $y = 2x - 3$.

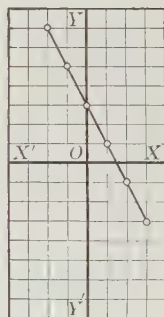
x	y	x	y
0	-3		
1	-1	-1	-5
2	1	-2	-7
3	3		



12. Plot the graph of the function $3 - 2x$.

Let $y = 3 - 2x$.

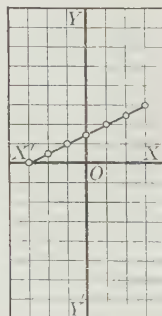
x	y	x	y
0	3		
1	1	-1	5
2	-1	-2	7
3	-3		



13. Plot the graph of the function $0.5x + 1.5$.

Let $y = 0.5x + 1.5$.

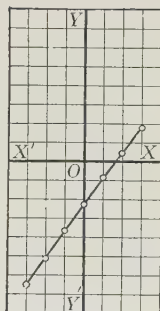
x	y	x	y
0	1.5		
1	2	-1	1
2	2.5	-2	0.5
3	3	-3	0



14. Plot the graph of the function $1.4x - 2.3$.

Let $y = 1.4x - 2.3$.

x	y	x	y
0	-2.3		
1	-0.9	-1	-3.7
2	0.5	-2	-5.1
3	1.9	-3	-6.5

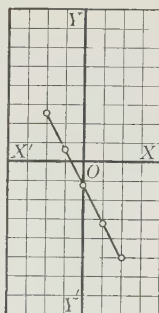


15. Plot the graph of the function $-\frac{1.5}{4}x - 2\frac{1}{2}$.

Let $y = -\frac{1.5}{4}x - 2\frac{1}{2}$,

and let 1 space on the Y-axis = 2 units.

x	y	x	y
0	$-2\frac{1}{2}$		
1	$-6\frac{1}{4}$	-1	$1\frac{1}{4}$
2	-10	-2	5

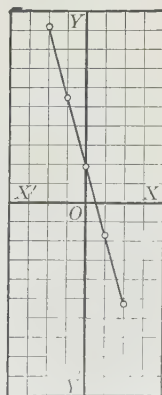


16. Plot the graph of the function $-\frac{2.9}{4}x + 3\frac{3}{4}$.

Let $y = -\frac{2.9}{4}x + 3\frac{3}{4}$,

and let 1 space on the Y-axis = 2 units.

x	y	x	y
0	$3\frac{3}{4}$		
1	$-3\frac{1}{2}$	-1	11
2	$-10\frac{3}{4}$	-2	$18\frac{1}{4}$

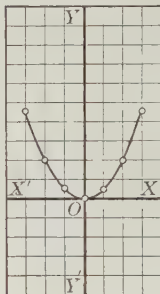


Exercise 73. Graphs of Quadratic Functions (Page 159)

1. Plot the graph of the function x^2 .

Let $y = x^2$, and let 1 space on the Y -axis = 2 units.

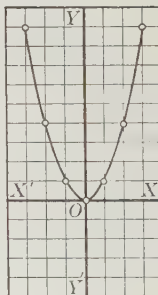
x	y	x	y
0	0		
1	1	- 1	1
2	4	- 2	4
3	9	- 3	9



2. Plot the graph of the function $2x^2$.

Let $y = 2x^2$, and let 1 space on the Y -axis = 2 units.

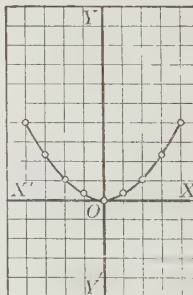
x	y	x	y
0	0		
1	2	- 1	2
2	8	- 2	8
3	18	- 3	18



3. Plot the graph of the function $\frac{1}{2}x^2$.

Let $y = \frac{1}{2}x^2$, and let 1 space on the Y -axis = 2 units.

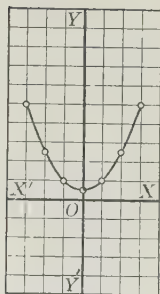
x	y	x	y
0	0		
1	$\frac{1}{2}$	- 1	$\frac{1}{2}$
2	2	- 2	2
3	$4\frac{1}{2}$	- 3	$4\frac{1}{2}$
4	8	- 4	8



4. Plot the graph of the function $x^2 + 1$.

Let $y = x^2 + 1$, and let 1 space on the Y -axis = 2 units.

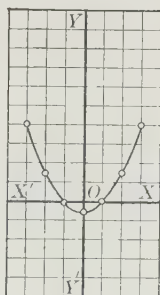
x	y	x	y
0	1		
1	2	-1	2
2	5	-2	5
3	10	-3	10



5. Plot the graph of the function $x^2 - 1$.

Let $y = x^2 - 1$, and let 1 space on the Y -axis = 2 units.

x	y	x	y
0	-1		
1	0	-1	0
2	3	-2	3
3	8	-3	8

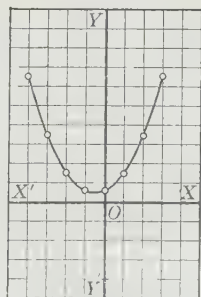


6. Plot the graph of the function $x^2 + x + 1$.

Let $y = x^2 + x + 1$,

and let 1 space on the Y -axis = 2 units.

x	y	x	y
0	1	-1	1
1	3	-2	3
2	7	-3	7
3	13	-4	13

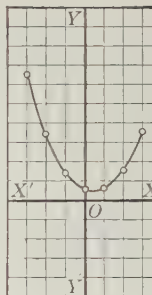


7. Plot the graph of the function $x^2 - x + 1$.

Let $y = x^2 - x + 1$,

and let 1 space on the Y -axis = 2 units.

x	y	x	y
0	1		
1	1	-1	3
2	3	-2	7
3	7	-3	13

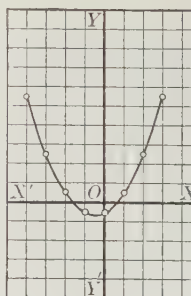


8. Plot the graph of the function $x^2 + x - 1$.

Let $y = x^2 + x - 1$,

and let 1 space on the Y -axis = 2 units.

x	y	x	y
0	-1	-1	-1
1	1	-2	1
2	5	-3	5
3	11	-4	11

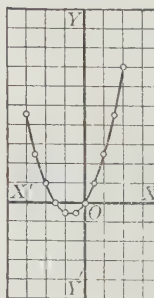


9. Plot the graph of the function $2x^2 + 3x$.

Let $y = 2x^2 + 3x$,

and let 1 space on the Y -axis = 2 units.

x	y	x	y
0	0	$-\frac{1}{2}$	-1
$\frac{1}{2}$	2	-1	-1
1	5	$-1\frac{1}{2}$	0
$1\frac{1}{2}$	9	-2	2
2	14	$-2\frac{1}{2}$	5
		-3	9

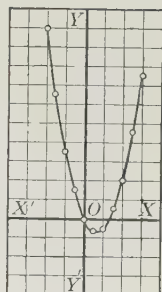


10. Plot the graph of the function $3x^2 - 4x$.

Let $y = 3x^2 - 4x$,

and let 1 space on the Y -axis = 2 units.

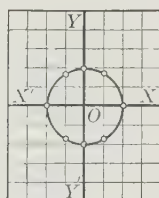
x	y	x	y
0	0		
$\frac{1}{2}$	$-1\frac{1}{4}$	$-\frac{1}{2}$	$2\frac{3}{4}$
1	-1	-1	7
$1\frac{1}{2}$	$\frac{3}{4}$	$-1\frac{1}{2}$	$12\frac{3}{4}$
2	4	-2	20
$2\frac{1}{2}$	$8\frac{3}{4}$		
3	15		



11. Plot the graph of the function $\pm \sqrt{4 - x^2}$.

Let $y = \pm \sqrt{4 - x^2}$.

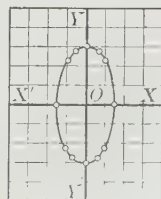
x	y	x	y
0	± 2		
1	± 1.73	-1	± 1.73
2	0	-2	0
3	$\pm \sqrt{-5}$	-3	$\pm \sqrt{-5}$



12. Plot the graph of the function $\pm \sqrt{9 - 4x^2}$.

Let $y = \pm \sqrt{9 - 4x^2}$.

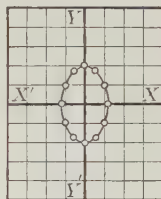
x	y	x	y
0	± 3		
$\frac{1}{2}$	± 2.8	$-\frac{1}{2}$	± 2.8
1	± 2.24	-1	± 2.24
$1\frac{1}{2}$	0	$-1\frac{1}{2}$	0
2	$\pm \sqrt{-7}$	-2	$\pm \sqrt{-7}$



13. Plot the graph of the function $\pm\sqrt{4-3x^2}$.

Let $y = \pm\sqrt{4-3x^2}$.

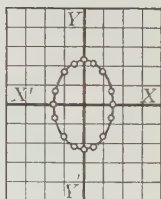
x	y	x	y
0	± 2		
$\frac{1}{2}$	± 1.8	$-\frac{1}{2}$	± 1.8
1	± 1	-1	± 1
1.15	0	-1.15	0



14. Plot the graph of the function $\pm\sqrt{5-2x^2}$.

Let $y = \pm\sqrt{5-2x^2}$.

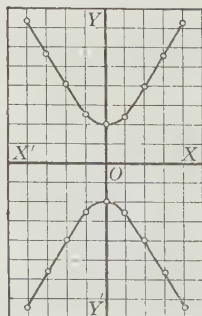
x	y	x	y
0	± 2.24		
$\frac{1}{2}$	± 2.12	$-\frac{1}{2}$	± 2.12
1	± 1.73	-1	± 1.73
$1\frac{1}{2}$	± 0.7	$-1\frac{1}{2}$	± 0.7
1.6	0	-1.6	0



15. Plot the graph of the function $\pm\sqrt{4+3x^2}$.

Let $y = \pm\sqrt{4+3x^2}$.

x	y	x	y
0	± 2		
1	± 2.65	-1	± 2.65
2	± 4	-2	± 4
3	± 5.57	-3	± 5.57
4	± 7.21	-4	± 7.21



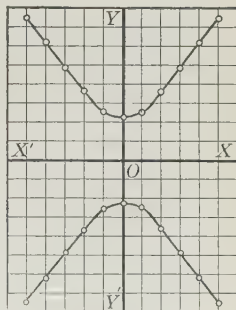
16. Plot the graph of the function

$$\pm\sqrt{5+2x^2}.$$

Let

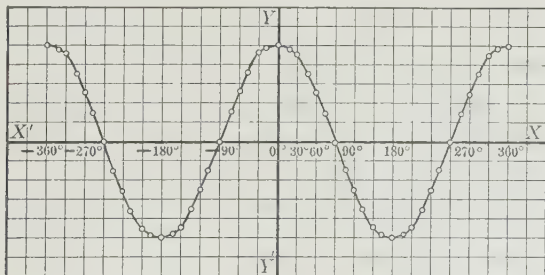
$$y = \pm\sqrt{5+2x^2}.$$

x	y	x	y
0	± 2.2		
1	± 2.65	-1	± 2.65
2	± 3.6	-2	± 3.6
3	± 4.8	-3	± 4.8
4	± 6.1	-4	± 6.1
5	± 7.42	-5	± 7.42



Exercise 74. Graphs of Trigonometric Functions (Page 161)

1. Verify the following plot of the graph of $\cos x$:

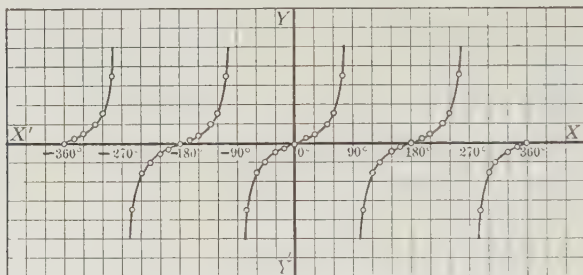


Let 1 space on the y -axis = $\frac{1}{5}$, and 1 space on the x -axis = 30° .

$x =$	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
$y =$	1	.97	.87	.7	.5	.26	0	-.26	-.5	-.7	-.87	-.97	-1
$x =$	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°	
$y =$	-.97	-.87	-.7	-.5	-.26	0	.26	.5	.7	.87	.97	1	
$x =$	0°	-15°	-30°	-45°	-60°	-75°	-90°	-105°	-120°	-135°	-150°	-165°	-180° ...
$y =$	1	.97	.87	.7	.5	.26	0	-.26	-.5	-.7	-.87	-.97	-1...

2. What is the period of $\cos x$?

Problem 1 shows that the values of the cosine repeat themselves after 360° . The period of the cosine is therefore 360° , or 2π .

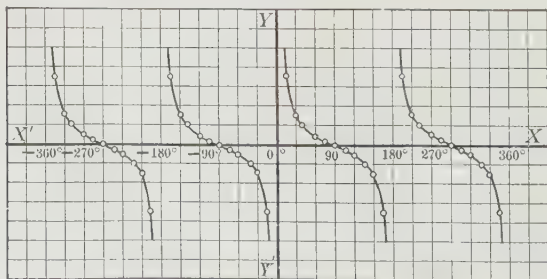
3. Verify the following plot of the graph of $\tan x$:

Let 1 space on the y -axis = 1, and 1 space on the x -axis = 30° .

$x =$	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
$y =$	0	.27	.58	1	1.7	3.7	$\pm \infty$	-3.7	-1.7	-1	-.58	-.27	0
$x =$	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°	
$y =$	-.27	-.58	-1	-1.7	-3.7	$\pm \infty$	-3.7	-1.7	-1	-.58	-.27	0	
$x =$	0°	-15°	-30°	-45°	-60°	-75°	-90°	-105°	-120°	-135°	-150°	-165°	$-180^\circ \dots$
$y =$	0	-.27	-.58	-1	-1.7	-3.7	$\pm \infty$	3.7	1.7	1	.58	.27	0 ...

4. What is the period of $\tan x$?

Problem 3 shows that the values of the tangent repeat themselves after 180° . The period of the tangent is therefore 180° , or π .

5. Verify the following plot of the graph of $\cot x$:

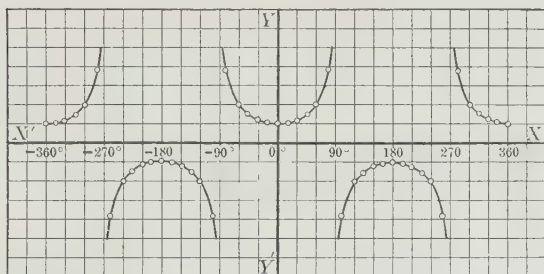
Let 1 space on the y -axis = 1, and 1 space on the x -axis = 30° .

$x =$	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
$y =$	∞	3.7	1.7	1	.58	.27	0	-.27	-.58	-1	-1.7	-3.7	$\mp\infty$
$x =$	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°	
$y =$	3.7	1.7	1	.58	.27	0	-.27	-.58	-1	-1.7	-3.7	$\mp\infty$	
$x =$	0°	-15°	-30°	-45°	-60°	-75°	-90°	-105°	-120°	-135°	-150°	-165°	$-180^\circ \dots$
$y =$	∞	3.7	1.7	1	.58	.27	0	-.27	-.58	-1	-1.7	-3.7	$\mp\infty \dots$

6. What is the period of $\cot x$?

Problem 5 shows that the values of the cotangent repeat after 180° . The period of the cotangent is therefore 180° , or π .

7. Verify the following plot of the graph of $\sec x$:



Let 1 space on the y -axis = 1, and 1 space on the x -axis = 30° .

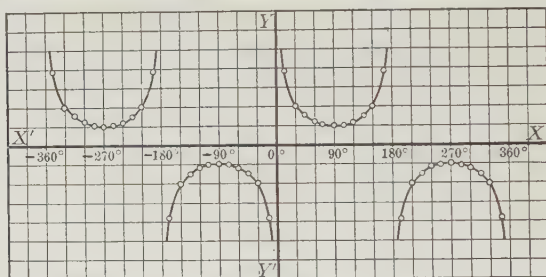
$x =$	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
$y =$	1	1.03	1.1	1.4	2	3.9	$\pm\infty$	-3.9	-2	-1.4	-1.1	-1.03	-1
$x =$	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°	
$y =$	-1.03	-1.1	-1.4	-2	-3.9	$\mp\infty$	3.9	2	1.4	1.1	1.03	1	
$x =$	0°	-15°	-30°	-45°	-60°	-75°	-90°	-105°	-120°	-135°	-150°	-165°	-180°
$y =$	1	1.03	1.1	1.4	2	3.9	$\pm\infty$	-3.9	-2	-1.4	-1.1	-1.03	-1

8. What is the period of $\sec x$?

Problem 7 shows that the values of the secant repeat after 360° . The period of the secant is therefore 360° , or 2π .

9. Plot the graph of $\csc x$, and state the period. Also state at what values of x the sign of $\csc x$ changes.

Let 1 space on the y -axis = 1, and 1 space on the x -axis = 30° .

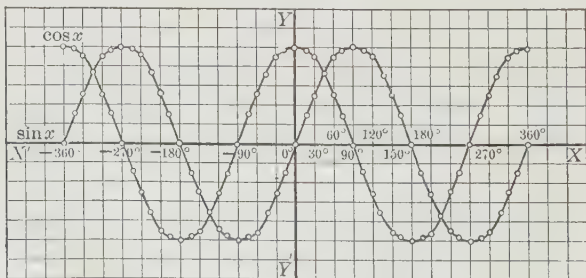


$x =$	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
$y =$	∞	3.9	2	1.4	1.1	1.03	1	1.03	1.1	1.4	2	3.9	$\pm \infty$
$x =$	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°	
$y =$	-3.9	-2	-1.4	-1.1	-1.03	-1	-1.03	-1.1	-1.4	-2	-3.9	$\mp \infty$	
$x =$	0°	-15°	-30°	-45°	-60°	-75°	-90°	-105°	-120°	-135°	-150°	-165°	-180°
$y =$	∞	-3.9	-2	-1.4	-1.1	-1.03	-1	-1.03	-1.1	-1.4	-2	-3.9	$\mp \infty$

The period of the cosecant is 360° , or 2π . The sign of the cosecant changes at 0° , 180° , 360° , -180° , and -360° .

10. Plot the graphs of $\sin x$ and $\cos x$ on the same paper. What does the figure tell as to the mutual relation of these functions?

Let 1 space on the y -axis = $\frac{1}{5}$, and 1 space on the x -axis = 30° .



The figure shows that $\sin x$ and $\cos x$ have the same period and that the curves are similar. For equal values of x the curves are 90 degrees apart, and therefore these functions of x are complementary

Exercise 75. Miscellaneous Exercise (Page 162)

Find the areas of the triangles in which:

1. $a = 25, b = 25, c = 25.$

$$\begin{aligned} S &= \frac{1}{2} ab \sin C \\ &= \frac{1}{2} (25 \times 25 \times \frac{1}{2} \sqrt{3}) \\ &= \frac{625 \times \sqrt{3}}{4}. \end{aligned}$$

$$\log 625 = 2.79588$$

$$\log \sqrt{3} = 0.23856$$

$$\text{colog } 4 = 9.39794$$

$$\log S = 2.43238$$

$$S = 270.63.$$

2. $a = 25, b = 33\frac{1}{3}, c = 41\frac{2}{3}.$

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$s = \frac{1}{2} (33\frac{1}{3} + 41\frac{2}{3} + 25) = 50.$$

$$s - a = 25.$$

$$s - b = 16\frac{2}{3}.$$

$$s - c = 8\frac{1}{3}.$$

$$\log 50 = 1.69897$$

$$\log 25 = 1.39794$$

$$\log 16\frac{2}{3} = 1.22182$$

$$\log 8\frac{1}{3} = 0.92082$$

$$2 \overline{) 5.23955}$$

$$\log S = 2.61977$$

$$S = 416.65.$$

3. $a = 74, b = 75, c = 92.$

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$s = \frac{1}{2} (74 + 75 + 92) = 120.5.$$

$$s - a = 46.5.$$

$$s - b = 45.5.$$

$$s - c = 28.5.$$

$$\log 120.5 = 2.08099$$

$$\log 46.5 = 1.66745$$

$$\log 45.5 = 1.65801$$

$$\log 28.5 = 1.45484$$

$$2 \overline{) 6.86129}$$

$$\log S = 3.430645$$

$$S = 2695.8.$$

4. $a = 2\frac{1}{2}, b = 3\frac{1}{3}, c = 4\frac{1}{4}.$

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$s = \frac{1}{2} (2.5 + 3.333 + 4.25) = 5.0416.$$

$$s - a = 2.5416.$$

$$s - b = 1.7086.$$

$$s - c = 0.7916.$$

$$\log 5.0416 = 0.70257$$

$$\log 2.5416 = 0.40510$$

$$\log 1.7086 = 0.23258$$

$$\log 0.7916 = 9.89851 - 10$$

$$2 \overline{) 1.23876}$$

$$\log S = 0.61938$$

$$S = 4.163.$$

5. Consider the area of a triangle with sides 17.2, 26.4, 43.6.

$$17.2 + 26.4 = 43.6.$$

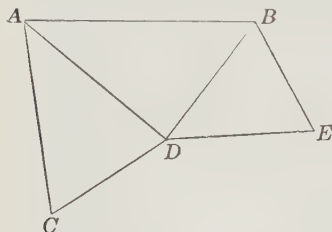
$\therefore$ the figure is not a triangle.

6. Consider the area of a triangle with sides 26.3, 42.4, 73.9.

$$73.9 > (26.3 + 42.4)$$

$\therefore$ the triangle is impossible.

7. Two inaccessible points A and B are visible from D , but no other point can be found from which both points are visible. Take some point C from which both A and D can be seen and measure CD , 200 ft.; angle ADC , 89° ; and angle ACD , $50^\circ 30'$. Then take some point E from which both D and B are visible, and measure DE , 200 ft.; angle BDE , $54^\circ 30'$; and angle BED , $88^\circ 30'$. At D measure angle ADB , $72^\circ 30'$. Compute the distance AB .



$$AD = CD \frac{\sin ACD}{\sin CAD} \\ = 200 \frac{\sin 50^\circ 30'}{\sin 40^\circ 30'}$$

$$\begin{aligned} \log 200 &= 2.30103 \\ \log \sin 50^\circ 30' &= 9.88741 \\ \text{colog} \sin 40^\circ 30' &= 0.18746 \\ \log AD &= 2.37590 \\ AD &= 237.63. \end{aligned}$$

$$BD = DE \frac{\sin BED}{\sin DBE} \\ = 200 \frac{\sin 88^\circ 30'}{\sin 37^\circ}$$

$$\begin{aligned} \log 200 &= 2.30103 \\ \log \sin 88^\circ 30' &= 9.99985 \\ \text{colog} \sin 37^\circ &= 0.22054 \\ \log BD &= 2.52142 \\ BD &= 332.22. \end{aligned}$$

$$\begin{aligned} \tan \frac{1}{2}(DAB - DBA) \\ &= \frac{BD - AD}{BD + AD} \tan \frac{1}{2}(DAB + DBA) \\ &= \frac{94.59}{569.85} \tan 53^\circ 45'. \end{aligned}$$

$$\begin{aligned} \log 94.59 &= 1.97585 \\ \text{colog} 569.85 &= 7.24424 \\ \log \tan 53^\circ 45' &= 10.13476 \\ \log \tan \frac{1}{2}(DAB - DBA) &= 9.35485 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}(DAB - DBA) &= 12^\circ 45' 21'' \\ \frac{1}{2}(DAB + DBA) &= 53^\circ 45' \\ DAB &= 66^\circ 30' 21'' \end{aligned}$$

$$\begin{aligned} AB &= BD \frac{\sin ADB}{\sin DAB} \\ &= 332.22 \frac{\sin 72^\circ 30'}{\sin 66^\circ 30' 21''} \end{aligned}$$

$$\begin{aligned} \log 332.22 &= 2.52142 \\ \log \sin 72^\circ 30' &= 9.97942 \\ \text{colog} \sin 66^\circ 30' 21'' &= 0.03758 \\ \log AB &= 2.53842 \\ AB &= 345.48. \end{aligned}$$

Distance AB , 345.48 ft.

8. Show by aid of the table of natural sines that $\sin x$ and x agree to four places of decimals for all angles less than $4^\circ 40'$.

The circular measure of $4^\circ 40'$, or $280'$, is

$$\frac{280\pi}{10800} = \frac{7\pi}{270} = \frac{7 \times 3.141592653589}{270} = 0.0814487.$$

The circular measure of $4^\circ 41'$ is

$$0.0814487 + 0.0002909 = 0.0817396.$$

From the table,

$$\begin{aligned} \sin 4^\circ 40' &= 0.0814, \\ \sin 4^\circ 41' &= 0.0816. \end{aligned}$$

Hence $\sin x$ and the circular measure of x agree for $4^\circ 40'$, and therefore for all smaller angles, to four decimal places; but they differ for larger angles.

9. If the values of $\log x$ and $\log \sin x$ agree to five decimal places, find from the table the greatest value x can have.

Let x be expressed in seconds. Then its circular measure is $\frac{\pi x}{648000}$, and its logarithm is

$$\begin{aligned}\log x'' + (\log \pi - \log 648,000) &= \log x'' + (0.49715 - 5.81158) \\ &= \log x'' - 5.31443 \\ &= \log x'' + 4.68557 - 10.\end{aligned}$$

But from the explanation preceding Table VII, if we remember that log sines are given in the table increased by 10, we have

$$\begin{aligned}\log \sin x + 10 &= \log x'' + S, \\ \therefore \log \sin x &= \log x'' + S - 10.\end{aligned}$$

Hence, if for five decimal places $\log \sin x = \log x$, we have

$$\begin{aligned}\log x'' + 4.68557 - 10 &= \log x'' + S - 10, \\ \therefore S &= 4.68557.\end{aligned}$$

But the greatest angle for which this value of S can be used is given in the table as $2409''$.

Hence the greatest angle for which $\log x$ and $\log \sin x$ agree to five decimal places is

$$2409'' = 40' 9''.$$

10. Find four angles whose cosine is the same as the cosine of 175° .

$$\cos x = \cos(2n\pi \pm x).$$

If $n = 0, 1, 2$,

$$\begin{aligned}\cos x &= \cos(\pm x), \cos(2\pi \pm x), \cos(4\pi \pm x), \\ \cos 175^\circ &= \cos \pm 175^\circ, \cos(360^\circ \pm 175^\circ), \cos(720^\circ \pm 175^\circ).\end{aligned}$$

Hence the angles are $-175^\circ, 185^\circ, 535^\circ, 545^\circ$.

11. Find four angles whose cosine is the same as the cosine of 200° .

$$\cos x = \cos(2n\pi \pm x).$$

If $n = 0, 1, 2$,

$$\begin{aligned}\cos x &= \cos(\pm x), \cos(2\pi \pm x), \cos(4\pi \pm x), \\ \cos 200^\circ &= \cos \pm 200^\circ, \cos(360^\circ \pm 200^\circ), \cos(720^\circ \pm 200^\circ).\end{aligned}$$

Hence the angles are $-200^\circ, 160^\circ, 560^\circ, 520^\circ$.

12. How many radians in the angle subtended by an arc 7.2 in. long, the radius being 3.6 in.? How many degrees?

$$\frac{7.2}{3.6} = 2 \text{ radians} = 114^\circ 35' 30''.$$

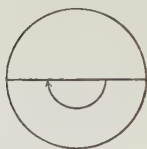
13. How many radians in the angle subtended by an arc 1.62 in. long, the radius being 4.86 in.? How many degrees?

$$\frac{1.62}{4.86} = \frac{1}{3} \text{ radian} = 19^\circ 5' 55''.$$

Draw the following angles :

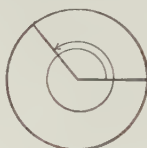
14. $-\pi$.

$$-\pi = -180^\circ.$$



18. 2.7π .

$$2.7\pi = 486^\circ.$$



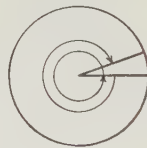
15. -2 .

$$-2 \text{ radians} = -(114^\circ 35' 30'').$$



19. $2\pi - 6$.

$$\begin{aligned} 2\pi - 6 &= 360^\circ - 343^\circ 46' 30'' \\ &= 16^\circ 13' 30''. \end{aligned}$$



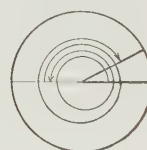
16. $-\frac{1}{2}\pi$.

$$-\frac{1}{2}\pi = -90^\circ.$$



20. $3\pi - 9$.

$$\begin{aligned} 3\pi - 9 &= 540^\circ - 515^\circ 39' 45'' \\ &= 24^\circ 20' 15''. \end{aligned}$$



17. $-\frac{1}{2}$.

$$\frac{1}{2} \text{ of } 57^\circ 17' 45'' = (-28^\circ 38' 52.5'').$$



21. $4 - \pi$.

$$4 - \pi = 229^\circ 11' - 180^\circ = 49^\circ 11'.$$



22. Find four angles whose tangent is $\frac{1}{\sqrt{3}}$.

$$\tan x = \tan(\pi + x).$$

If $n = 0, 1, 2, 3$,

$$\tan x = \tan x, \tan(\pi + x), \tan(2\pi + x), \tan(3\pi + x).$$

$$\tan^{-1} \frac{1}{\sqrt{3}} = 30^\circ.$$

$$\tan x = \tan 30^\circ, \tan(180^\circ + 30^\circ), \tan(360^\circ + 30^\circ), \tan(540^\circ + 30^\circ).$$

Hence the angles are $30^\circ, 210^\circ, 390^\circ, 570^\circ$.

23. Find four angles whose cotangent is $\frac{1}{\sqrt{3}}$.

$$\cot^{-1} \frac{1}{\sqrt{3}} = 60^\circ.$$

From Problem 22 it is evident that

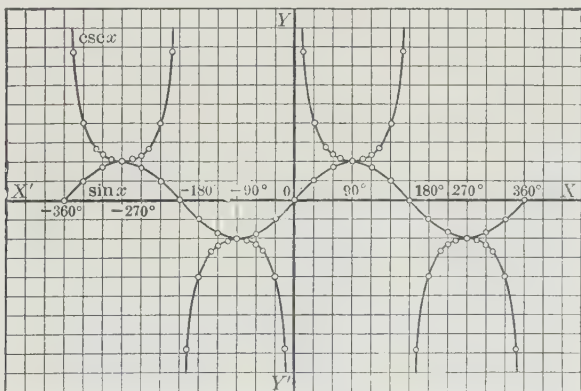
$$\cot x = \cot x, \cot(\pi + x), \cot(2\pi + x), \cot(3\pi + x).$$

$$\cot x = \cot 60^\circ, \cot(180^\circ + 60^\circ), \cot(360^\circ + 60^\circ), \cot(540^\circ + 60^\circ).$$

Hence the angles are $60^\circ, 240^\circ, 420^\circ, 600^\circ$.

24. Plot the graphs of $\sin x$ and $\csc x$ on the same paper. What does the figure tell as to the mutual relation of these functions?

Let 1 space on the y -axis = $\frac{1}{2}$, and 1 space on the x -axis = 30° .

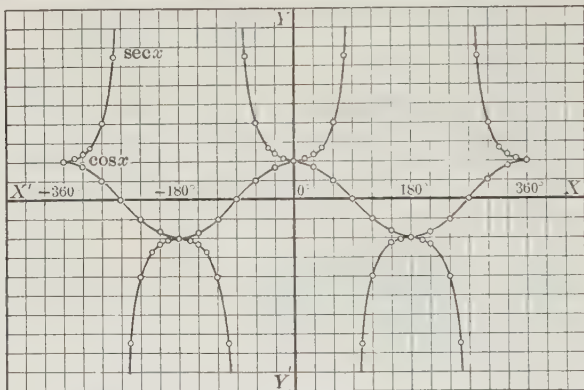


As x increases from 0° to 90° , $\sin x$ increases from 0 to 1 and $\csc x$ decreases from ∞ to 1. From 90° to 180° , $\sin x$ decreases from 1 to 0 and $\csc x$ increases from 1 to ∞ . In each quadrant $\csc x$ decreases as $\sin x$ increases and vice versa. The sine and cosecant are mutually reciprocal,

$$\text{is, for any given value of } x, \sin x = \frac{1}{\csc x}.$$

25. Plot the graphs of $\cos x$ and $\sec x$ on the same paper. What does the figure tell as to the mutual relation of these functions?

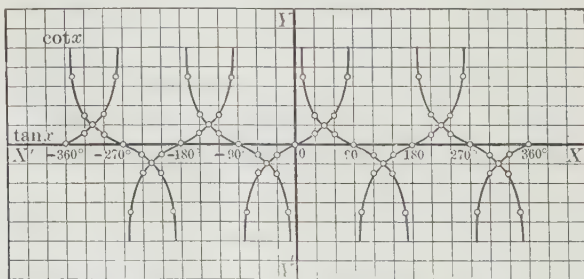
Let 1 space on the y -axis = $\frac{1}{2}$, and 1 space on the x -axis = 30° .



As x increases from 0° to 90° , $\cos x$ decreases from 1 to 0 and $\sec x$ increases from 1 to ∞ . From 90° to 180° , $\cos x$ is negative, increasing in absolute value from 0 to 1, while $\sec x$ is also negative and decreases in absolute value from ∞ to 1. In each quadrant $\sec x$ decreases as $\cos x$ increases, and for any given value of x the cosine and the secant are reciprocal.

26. Plot the graphs of $\tan x$ and $\cot x$ on the same paper. What does the figure tell as to the mutual relation of these functions?

Let 1 space on the y -axis = 1, and 1 space on the x -axis = 30° .



As x increases from 0° to 90° , $\tan x$ increases from 0 to ∞ and $\cot x$ decreases from ∞ to 0. In each quadrant this same relation obtains, cotangent decreasing with tangent increasing, and for any given value of x the tangent and the cotangent are reciprocals,

Exercise 76. Identities (Page 164)

Prove the following identities :

$$1. \tan x = \frac{2 \tan \frac{1}{2} x}{1 - \tan^2 \frac{1}{2} x}.$$

Let $x = 2y.$

By [31], $\tan 2y = \frac{2 \tan y}{1 - \tan^2 y}.$

Hence $\tan x = \frac{2 \tan \frac{1}{2} x}{1 - \tan^2 \frac{1}{2} x}.$

$$2. \sin x = \frac{2 \tan \frac{1}{2} x}{1 + \tan^2 \frac{1}{2} x}.$$

By [36],

$$\begin{aligned} \frac{2 \tan \frac{1}{2} x}{1 + \tan^2 \frac{1}{2} x} &= \frac{2 \sqrt{\frac{1 - \cos x}{1 + \cos x}}}{1 + \frac{1 - \cos x}{1 + \cos x}} = \frac{2 \sqrt{\frac{1 - \cos x}{1 + \cos x}}}{\frac{1 + \cos x}{1 + \cos x}} = (1 + \cos x) \sqrt{\frac{1 - \cos x}{1 + \cos x}} \\ &= \sqrt{(1 + \cos x)(1 - \cos x)} = \sqrt{1 - \cos^2 x}. \end{aligned}$$

By [16], $\sin x = \sqrt{1 - \cos^2 x}.$

$$\therefore \sin x = \frac{2 \tan \frac{1}{2} x}{1 + \tan^2 \frac{1}{2} x}.$$

$$3. \sin 2x = \frac{2 \tan x}{1 + \tan^2 x}.$$

$$\frac{2 \tan x}{1 + \tan^2 x} = \frac{2 \tan x}{\sec^2 x} = 2 \tan x \cos^2 x = 2 \sin x \cos x = \sin 2x.$$

$$4. 2 \sin x + \sin 2x = \frac{2 \sin^3 x}{1 - \cos x}.$$

$$\begin{aligned} 2 \sin x + \sin 2x &= 2 \sin x + 2 \sin x \cos x \\ &= 2 \sin x (1 + \cos x). \end{aligned}$$

By [16], $1 - \cos^2 x = \sin^2 x.$

$$\therefore 1 + \cos x = \frac{\sin^2 x}{1 - \cos x}.$$

$$\therefore 2 \sin x + \sin 2x = 2 \sin x \frac{\sin^2 x}{1 - \cos x} = \frac{2 \sin^3 x}{1 - \cos x}.$$

$$5. \sin 3x = \frac{\sin^2 2x - \sin^2 x}{\sin x}.$$

By [42], $\sin 2x + \sin x = 2 \sin \frac{3}{2} x \cos \frac{1}{2} x.$

By [43], $\sin 2x - \sin x = 2 \cos \frac{3}{2} x \sin \frac{1}{2} x.$

$$\therefore \sin^2 2x - \sin^2 x = 2 \sin \frac{3}{2} x \cos \frac{3}{2} x \times 2 \sin \frac{1}{2} x \cos \frac{1}{2} x;$$

by [30], $= \sin 3x \sin x.$

$$\therefore \sin 3x = \frac{\sin^2 2x - \sin^2 x}{\sin x}.$$

$$6. \tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$$

$$\tan 3x = \tan(2x + x);$$

by [26],

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x};$$

by [31],

$$\begin{aligned} &= \frac{\frac{2 \tan x}{1 - \tan^2 x} + \tan x}{1 - \frac{2 \tan x}{1 - \tan^2 x} \tan x} \\ &= \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}. \end{aligned}$$

$$7. \frac{\tan 2x + \tan x}{\tan 2x - \tan x} = \frac{\sin 3x}{\sin x}$$

By [46],

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}.$$

By composition and division,

$$\frac{\sin A}{\sin B} = \frac{\tan \frac{1}{2}(A + B) + \tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B) - \tan \frac{1}{2}(A - B)}$$

$$\text{Put } 3x \text{ for } A, x \text{ for } B, \frac{\sin 3x}{\sin x} = \frac{\tan 2x + \tan x}{\tan 2x - \tan x}.$$

$$8. \frac{3 \cos x + \cos 3x}{3 \sin x - \sin 3x} = \cot^3 x.$$

by [44],

$$\begin{aligned} 3 \cos x + \cos 3x &= 2 \cos x + (\cos x + \cos 3x); \\ &= 2 \cos x + 2 \cos 2x \cos x \\ &= 2 \cos x (1 + \cos 2x); \\ &= 4 \cos^3 x. \end{aligned}$$

by [35],

by [43],

$$\begin{aligned} 3 \sin x - \sin 3x &= 2 \sin x - (\sin 3x - \sin x); \\ &= 2 \sin x - 2 \cos 2x \sin x \\ &= 2 \sin x (1 - \cos 2x); \\ &= 4 \sin^3 x. \end{aligned}$$

by [34],

$$\begin{aligned} \therefore \frac{3 \cos x + \cos 3x}{3 \sin x - \sin 3x} &= \frac{4 \cos^3 x}{4 \sin^3 x} \\ &= \cot^3 x. \end{aligned}$$

$$9. \frac{\sin 3x + \sin 5x}{\cos 3x - \cos 5x} = \cot x.$$

By [42],

$$\sin 3x + \sin 5x = 2 \sin 4x \cos x.$$

By [45],

$$\cos 3x - \cos 5x = 2 \sin 4x \sin x.$$

$$\therefore \frac{\sin 3x + \sin 5x}{\cos 3x - \cos 5x} = \frac{\cos x}{\sin x} = \cot x.$$

$$10. \frac{\sin 3x + \sin 5x}{\sin x + \sin 3x} = 2 \cos 2x.$$

$$\text{By [42],} \quad \sin 3x + \sin 5x = 2 \sin 4x \cos x.$$

$$\text{By [42],} \quad \sin x + \sin 3x = 2 \sin 2x \cos x.$$

$$\begin{aligned} \therefore \frac{\sin 3x + \sin 5x}{\sin x + \sin 3x} &= \frac{\sin 4x}{\sin 2x}; \\ &= \frac{2 \sin 2x \cos 2x}{\sin 2x} \\ &= 2 \cos 2x. \end{aligned}$$

by [30],

$$11. \sin x + \sin 3x + \sin 5x = \frac{\sin^2 3x}{\sin x}.$$

$$\text{By [42],} \quad \sin x + \sin 5x = 2 \sin 3x \cos 2x.$$

$$\begin{aligned} \therefore \sin x + \sin 3x + \sin 5x &= \sin 3x + 2 \sin 3x \cos 2x \\ &= \sin 3x (1 + 2 \cos 2x). \end{aligned}$$

$$\text{By [43],} \quad \sin 3x - \sin x = 2 \cos 2x \sin x.$$

$$\frac{\sin 3x}{\sin x} - 1 = 2 \cos 2x.$$

$$1 + 2 \cos 2x = \frac{\sin 3x}{\sin x}.$$

$$\begin{aligned} \sin x + \sin 3x + \sin 5x &= \sin 3x \frac{\sin 3x}{\sin x} \\ &= \frac{\sin^2 3x}{\sin x}. \end{aligned}$$

$$12. \tan 2x + \sec 2x = \frac{\cos x + \sin x}{\cos x - \sin x}.$$

$$\begin{aligned} \tan 2x + \sec 2x &= \frac{\sin 2x}{\cos 2x} + \frac{1}{\cos 2x} \\ &= \frac{\sin 2x + 1}{\cos 2x} \end{aligned}$$

$$= \frac{1 - \cos(2x + 90^\circ)}{\sin(2x + 90^\circ)};$$

$$= \frac{2 \sin^2(x + 45^\circ)}{2 \sin(x + 45^\circ) \cos(x + 45^\circ)}$$

$$= \frac{\sin(x + 45^\circ)}{\cos(x + 45^\circ)};$$

$$= \frac{\sin x \cos 45^\circ + \cos x \sin 45^\circ}{\cos x \cos 45^\circ - \sin x \sin 45^\circ}$$

$$= \frac{\frac{1}{2}\sqrt{2} \sin x + \frac{1}{2}\sqrt{2} \cos x}{\frac{1}{2}\sqrt{2} \cos x - \frac{1}{2}\sqrt{2} \sin x}$$

$$= \frac{\cos x + \sin x}{\cos x - \sin x}.$$

by [34] and [30],

by [22] and [24],

$$13. \tan x + \tan y = \frac{\sin(x+y)}{\cos x \cos y}.$$

$$\begin{aligned} \tan x + \tan y &= \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} \\ &= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}; \end{aligned}$$

$$\text{by [22],} \quad = \frac{\sin(x+y)}{\cos x \cos y}.$$

$$14. \tan(x+y) = \frac{\sin 2x + \sin 2y}{\cos 2x + \cos 2y}.$$

$$\text{By [42],} \quad \sin 2x + \sin 2y = 2 \sin(x+y) \cos(x-y).$$

$$\text{By [44],} \quad \cos 2x + \cos 2y = 2 \cos(x+y) \cos(x-y).$$

$$\begin{aligned} \therefore \frac{\sin 2x + \sin 2y}{\cos 2x + \cos 2y} &= \frac{2 \sin(x+y) \cos(x-y)}{2 \cos(x+y) \cos(x-y)} \\ &= \tan(x+y). \end{aligned}$$

$$15. \frac{\sin x + \cos y}{\sin x - \cos y} = \frac{\tan[\frac{1}{2}(x+y) + 45^\circ]}{\tan[\frac{1}{2}(x-y) - 45^\circ]}.$$

$$\begin{aligned} \text{by [42],} \quad \sin x + \cos y &= \sin x + \sin(y + 90^\circ); \\ &= 2 \sin \frac{1}{2}(x+y+90^\circ) \cos \frac{1}{2}(x-y-90^\circ) \end{aligned}$$

$$\text{By [43],} \quad \sin x - \cos y = 2 \cos \frac{1}{2}(x+y+90^\circ) \sin \frac{1}{2}(x-y-90^\circ)$$

$$\begin{aligned} \therefore \frac{\sin x + \cos y}{\sin x - \cos y} &= \frac{\tan \frac{1}{2}(x+y+90^\circ)}{\tan \frac{1}{2}(x-y-90^\circ)} \\ &= \frac{\tan[\frac{1}{2}(x+y) + 45^\circ]}{\tan[\frac{1}{2}(x-y) - 45^\circ]}. \end{aligned}$$

$$16. \sin 2x + \sin 4x = 2 \sin 3x \cos x.$$

$$\text{By [42],} \quad \sin 2x + \sin 4x = 2 \sin 3x \cos x.$$

$$17. \sin 4x = 4 \sin x \cos x - 8 \sin^3 x \cos x.$$

$$\begin{aligned} \text{By [30],} \quad \sin 4x &= 2 \sin 2x \cos 2x; \\ \text{by [16], [30], and [32],} \quad &= 4 \sin x \cos x (1 - 2 \sin^2 x) \\ &= 4 \sin x \cos x - 8 \sin^3 x \cos x. \end{aligned}$$

$$18. \sin 4x = 8 \cos^3 x \sin x - 4 \cos x \sin x.$$

$$\begin{aligned} \text{By [30],} \quad \sin 4x &= 2 \sin 2x \cos 2x; \\ \text{by [32] and [16],} \quad &= 4 \sin x \cos x (2 \cos^2 x - 1) \\ &= 8 \cos^3 x \sin x - 4 \sin x \cos x. \end{aligned}$$

$$19. \cos 4x = 1 - 8 \cos^2 x + 8 \cos^4 x = 1 - 8 \sin^2 x + 8 \sin^4 x.$$

$$\begin{aligned} \text{By [16] and [32], } \cos 4x &= 2 \cos^2 2x - 1; \\ \text{by [16] and [32],} \quad &= 2(2 \cos^2 x - 1)^2 - 1 \\ &= 8 \cos^4 x - 8 \cos^2 x + 2 - 1 \\ &= 1 - 8 \cos^2 x + 8 \cos^4 x. \end{aligned}$$

$$\begin{aligned} \text{Again,} \quad \cos 4x &= 1 - 2 \sin^2 2x; \\ \text{by [30],} \quad &= 1 - 2(4 \sin^2 x \cos^2 x) \\ &= 1 - 8 \sin^2 x (1 - \sin^2 x) \\ &= 1 - 8 \sin^2 x + 8 \sin^4 x. \end{aligned}$$

$$20. \cos 2x + \cos 4x = 2 \cos 3x \cos x.$$

$$\text{By [44],} \quad \cos 2x + \cos 4x = 2 \cos 3x \cos x.$$

$$21. \sin 3x - \sin x = 2 \cos 2x \sin x.$$

$$\text{By [43],} \quad \sin 3x - \sin x = 2 \cos 2x \sin x.$$

$$22. \sin^3 x \sin 3x + \cos^3 x \cos 3x = \cos^3 2x.$$

$$\begin{aligned} \text{by [16],} \quad \sin^3 x \sin 3x &= \sin x \sin^2 x \sin 3x; \\ &= \sin x (1 - \cos^2 x) \sin 3x \\ &= \sin x \sin 3x - \sin x \cos^2 x \sin 3x. \\ \cos^3 x \cos 3x &= \cos x \cos^2 x \cos 3x \\ &= \cos x (1 - \sin^2 x) \cos 3x \\ &= \cos x \cos 3x - \cos x \sin^2 x \cos 3x. \end{aligned}$$

$$\begin{aligned} \therefore \sin^3 x \sin 3x + \cos^3 x \cos 3x &= \sin x \sin 3x + \cos x \cos 3x \\ &\quad - \sin x \cos^2 x \sin 3x - \cos x \sin^2 x \cos 3x; \\ \text{by [25],} \quad &= \cos(3x - x) \\ &\quad - \sin x \cos x (\cos x \sin 3x + \sin x \cos 3x); \\ \text{by [22],} \quad &= \cos 2x - \sin x \cos x \sin(3x + x); \\ \text{by [30],} \quad &= \cos 2x - \frac{1}{2} \sin 2x \sin 4x; \\ \text{by [30],} \quad &= \cos 2x - \sin^2 2x \cos 2x; \\ \text{by [16],} \quad &= \cos 2x (1 - \sin^2 2x); \\ \text{by [32],} \quad &= \cos^3 2x. \end{aligned}$$

$$23. \cos^4 x - \sin^4 x = \cos 2x.$$

$$\begin{aligned} \cos^4 x - \sin^4 x &= (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x); \\ \text{by [16] and [32],} \quad &= 1 \times \cos 2x \\ &= \cos 2x. \end{aligned}$$

$$24. \cos^4 x + \sin^4 x = 1 - \frac{1}{2} \sin^2 2x.$$

$$\begin{aligned} \cos^4 x + \sin^4 x &= (\cos^2 x + \sin^2 x)^2 - 2 \sin^2 x \cos^2 x; \\ \text{by [16],} \quad &= 1 - 2 \sin^2 x \cos^2 x; \\ \text{by [30],} \quad &= 1 - \frac{1}{2} \sin^2 2x. \end{aligned}$$

$$25. \cos^6 x - \sin^6 x = (1 - \sin^2 x \cos^2 x) \cos 2x.$$

$$\cos^6 x - \sin^6 x = (\cos^2 x - \sin^2 x)(\cos^4 x + \cos^2 x \sin^2 x + \sin^4 x);$$

$$\text{by [32],} \quad = \cos 2x [(\cos^2 x + \sin^2 x)^2 - \cos^2 x \sin^2 x];$$

$$\text{by [16],} \quad = \cos 2x (1 - \cos^2 x \sin^2 x).$$

$$26. \cos^6 x + \sin^6 x = 1 - 3 \sin^2 x \cos^2 x.$$

$$\cos^6 x + \sin^6 x = (\cos^2 x + \sin^2 x)(\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x);$$

$$\text{by [16],} \quad = \cos^4 x - \cos^2 x \sin^2 x + \sin^4 x$$

$$= (\cos^2 x + \sin^2 x)^2 - 3 \cos^2 x \sin^2 x;$$

$$\text{by [16],} \quad = 1 - 3 \cos^2 x \sin^2 x.$$

$$27. \csc x - 2 \cot 2x \cos x = 2 \sin x.$$

$$\csc x - 2 \cot 2x \cos x = \csc x - 2 \frac{\cos 2x}{\sin 2x} \cos x;$$

$$\text{by [30],} \quad = \csc x - \frac{2 \cos 2x \cos x}{2 \sin x \cos x}$$

$$= \frac{1}{\sin x} - \frac{\cos 2x}{\sin x}$$

$$= \frac{1 - \cos 2x}{\sin x};$$

$$\text{by [16] and [32],} \quad = \frac{2 \sin^2 x}{\sin x}$$

$$= 2 \sin x.$$

$$28. (\sin 2x - \sin 2y) \tan(x + y) = 2(\sin^2 x - \sin^2 y).$$

$$\text{By [43],} \quad \sin 2x - \sin 2y = 2 \cos(x + y) \sin(x - y).$$

$$\therefore (\sin 2x - \sin 2y) \tan(x + y) = 2 \sin(x + y) \sin(x - y).$$

$$\text{By [42],} \quad \sin x + \sin y = 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y).$$

$$\text{By [43],} \quad \sin x - \sin y = 2 \cos \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y).$$

$$\therefore \sin^2 x - \sin^2 y = 4 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y) \cos \frac{1}{2}(x - y);$$

$$\text{by [30],} \quad = \sin(x + y) \sin(x - y).$$

$$\therefore 2(\sin^2 x - \sin^2 y) = 2 \sin(x + y) \sin(x - y) \\ = (\sin 2x - \sin 2y) \tan(x + y).$$

$$29. \sin 3x = 4 \sin x \sin(60^\circ + x) \sin(60^\circ - x).$$

$$\text{By [22],} \quad \sin(60^\circ + x) = \frac{1}{2} \sqrt{3} \cos x + \frac{1}{2} \sin x.$$

$$\text{By [23],} \quad \sin(60^\circ - x) = \frac{1}{2} \sqrt{3} \cos x - \frac{1}{2} \sin x.$$

$$\therefore \sin(60^\circ + x) \sin(60^\circ - x) = \frac{3}{4} \cos^2 x - \frac{1}{4} \sin^2 x;$$

$$\text{by [16],} \quad = \frac{3(1 - \sin^2 x) - \sin^2 x}{4}$$

$$= \frac{3 - 4 \sin^2 x}{4}.$$

$$4 \sin x \sin(60^\circ + x) \sin(60^\circ - x) = \sin x (3 - 4 \sin^2 x)$$

$$= 3 \sin x - 4 \sin^3 x;$$

$$\text{by Ex. 15, Exercise 44,} \quad = \sin 3x.$$

$$30. \sin 4x = 2 \sin x \cos 3x + \sin 2x.$$

$$\begin{aligned} \text{By [43],} \quad \sin 4x - \sin 2x &= 2 \cos 3x \sin x. \\ \therefore \sin 4x &= 2 \cos 3x \sin x + \sin 2x. \end{aligned}$$

$$31. \sin x + \sin(x - \frac{2}{3}\pi) + \sin(\frac{1}{3}\pi - x) = 0.$$

$$\begin{aligned} \text{By [42],} \quad \sin(x - \frac{2}{3}\pi) + \sin(\frac{1}{3}\pi - x) \\ &= 2 \sin(-\frac{1}{6}\pi) \cos(x - \frac{1}{2}\pi) \\ &= -2 \sin \frac{1}{6}\pi \sin x \\ &= -\sin x. \\ \therefore \sin x + \sin(x - \frac{2}{3}\pi) + \sin(\frac{1}{3}\pi - x) &= 0. \end{aligned}$$

$$32. \cos x \sin(y - z) + \cos y \sin(z - x) + \cos z \sin(x - y) = 0.$$

$$\begin{aligned} \text{By [23],} \quad \cos x \sin(y - z) &= \cos x \sin y \cos z - \cos x \cos y \sin z, \\ \cos y \sin(z - x) &= \cos y \sin z \cos x - \cos y \cos z \sin x, \\ \cos z \sin(x - y) &= \cos z \sin x \cos y - \cos z \cos x \sin y. \\ \therefore \cos x \sin(y - z) + \cos y \sin(z - x) + \cos z \sin(x - y) &= 0. \end{aligned}$$

$$\begin{aligned} 33. \cos(x + y) \sin y - \cos(x + z) \sin z \\ &= \sin(x + y) \cos y - \sin(x + z) \cos z. \end{aligned}$$

$$\begin{aligned} \text{By [22] and [24],} \\ \sin(x + y) \cos y - \cos(x + y) \sin y &= \sin x (\cos^2 y + \sin^2 y) = \sin x, \\ \sin(x + z) \cos z - \cos(x + z) \sin z &= \sin x (\cos^2 z + \sin^2 z) = \sin x. \\ \therefore \sin(x + y) \cos y - \cos(x + y) \sin y &= \sin(x + z) \cos z - \cos(x + z) \sin z. \\ \therefore \cos(x + y) \sin y - \cos(x + z) \sin z &= \sin(x + y) \cos y - \sin(x + z) \cos z. \end{aligned}$$

$$\begin{aligned} 34. \cos(x + y + z) + \cos(x + y - z) + \cos(x - y + z) + \cos(y + z - x) \\ &= 4 \cos x \cos y \cos z. \end{aligned}$$

$$\begin{aligned} \text{By [44],} \quad \cos[(x + y) + z] + \cos[(x + y) - z] &= 2 \cos(x + y) \cos z, \\ \cos[z + (x - y)] + \cos[z - (x - y)] &= 2 \cos z \cos(x - y). \\ \therefore \cos(x + y + z) + \cos(x + y - z) + \cos(x - y + z) + \cos(y + z - x) \\ &= 2 \cos(x + y) \cos z + 2 \cos(x - y) \cos z \\ &= 2 \cos z [\cos(x + y) + \cos(x - y)] \\ &= 2 \cos z (2 \cos x \cos y) \\ &= 4 \cos x \cos y \cos z. \end{aligned}$$

$$\begin{aligned} 35. \sin(x + y) \cos(x - y) + \sin(y + z) \cos(y - z) \\ + \sin(z + x) \cos(z - x) &= \sin 2x + \sin 2y + \sin 2z. \end{aligned}$$

$$\begin{aligned} \text{By [42],} \quad \sin(x + y) \cos(x - y) &= \frac{1}{2}(\sin 2x + \sin 2y), \\ \sin(y + z) \cos(y - z) &= \frac{1}{2}(\sin 2y + \sin 2z), \\ \sin(z + x) \cos(z - x) &= \frac{1}{2}(\sin 2z + \sin 2x). \\ \therefore \sin(x + y) \cos(x - y) + \sin(y + z) \cos(y - z) + \sin(z + x) \cos(z - x) \\ &= \sin 2x + \sin 2y + \sin 2z. \end{aligned}$$

$$36. \sin(x+y) + \cos(x-y) = 2 \sin(x + \tfrac{1}{4}\pi) \sin(y + \tfrac{1}{4}\pi).$$

$$\text{By [22],} \quad \sin(x+y) = \sin x \cos y + \cos x \sin y.$$

$$\text{By [25],} \quad \cos(x-y) = \cos x \cos y + \sin x \sin y.$$

$$\therefore \sin(x+y) + \cos(x-y) = (\sin x + \cos x) \cos y + (\cos x + \sin x) \sin y \\ = (\sin x + \cos x) (\sin y + \cos y).$$

$$\text{But} \quad \sin x + \cos x = \sqrt{2} (\tfrac{1}{2} \sqrt{2} \sin x + \tfrac{1}{2} \sqrt{2} \cos x); \\ \text{by [22],} \quad = \sqrt{2} \sin(x + \tfrac{1}{4}\pi).$$

$$\text{Similarly,} \quad \sin y + \cos y = \sqrt{2} \sin(y + \tfrac{1}{4}\pi).$$

$$\therefore \sin(x+y) + \cos(x-y) = 2 \sin(x + \tfrac{1}{4}\pi) \sin(y + \tfrac{1}{4}\pi).$$

$$37. \sin(x+y) - \cos(x-y) = -2 \sin(x - \tfrac{1}{4}\pi) \sin(y - \tfrac{1}{4}\pi).$$

$$\text{By [22],} \quad \sin(x+y) = \sin x \cos y + \cos x \sin y.$$

$$\text{By [25],} \quad \cos(x-y) = \cos x \cos y + \sin x \sin y.$$

$$\therefore \sin(x+y) - \cos(x-y) = (\sin x - \cos x) \cos y + (\cos x - \sin x) \sin y \\ = (\sin x - \cos x) (\cos y - \sin y) \\ = -2 \sin(x - \tfrac{1}{4}\pi) \sin(y - \tfrac{1}{4}\pi).$$

$$38. \cos(x+y) \cos(x-y) = \cos^2 x - \sin^2 y.$$

$$\text{By [24] and [25],} \quad \cos(x+y) \cos(x-y) \\ = (\cos x \cos y - \sin x \sin y) (\cos x \cos y + \sin x \sin y) \\ = \cos^2 x \cos^2 y - \sin^2 x \sin^2 y;$$

$$\text{by [16],} \quad = (1 - \sin^2 x) (1 - \sin^2 y) - \sin^2 x \sin^2 y \\ = 1 - \sin^2 x - \sin^2 y + \sin^2 x \sin^2 y - \sin^2 x \sin^2 y \\ = 1 - \sin^2 x - \sin^2 y;$$

$$\text{by [16],} \quad = \cos^2 x - \sin^2 y.$$

$$39. \sin(x+y) \sin(x-y) = \sin^2 x - \sin^2 y.$$

$$\text{By [22] and [23],} \quad \sin(x+y) \sin(x-y) \\ = (\sin x \cos y + \cos x \sin y) (\sin x \cos y - \cos x \sin y) \\ = \sin^2 x \cos^2 y - \cos^2 x \sin^2 y;$$

$$\text{by [16],} \quad = \sin^2 x (1 - \sin^2 y) - (1 - \sin^2 x) \sin^2 y \\ = \sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y \\ = \sin^2 x - \sin^2 y.$$

$$40. \sin x + 2 \sin 3x + \sin 5x = 4 \cos^2 x \sin 3x.$$

$$\text{By [42],} \quad \sin x + \sin 5x = 2 \sin 3x \cos 2x.$$

$$\therefore \sin x + 2 \sin 3x + \sin 5x = 2 \sin 3x + 2 \sin 3x \cos 2x \\ = 2 \sin 3x (1 + \cos 2x);$$

$$\text{by [32],} \quad = 2 \sin 3x (\cos^2 x - \sin^2 x + 1);$$

$$\text{by [16],} \quad = 2 \sin 3x (\cos^2 x - 1 + \cos^2 x + 1) \\ = 4 \cos^2 x \sin 3x.$$

If A, B, C are the angles of a triangle, prove that :

$$41. \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

$$A + B + C = 180^\circ.$$

$$\begin{aligned} \text{By [42],} \quad \sin 2A + \sin 2B &= 2 \sin (A + B) \cos (A - B) \\ &= 2 \sin C \cos (A - B). \end{aligned}$$

$$\text{By [30],} \quad \sin 2C = 2 \sin C \cos C.$$

$$\begin{aligned} \therefore \sin 2A + \sin 2B + \sin 2C &= 2 \sin C \cos (A - B) + 2 \sin C \cos C \\ &= 2 \sin C [\cos (A - B) + \cos (A + B)]; \end{aligned}$$

$$\begin{aligned} \text{by [45],} \quad &= 2 \sin C [-2 \sin A \sin (-B)] \\ &= 4 \sin C \sin A \sin B. \end{aligned}$$

$$42. \cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$$

$$\begin{aligned} \text{By [44],} \quad \cos 2A + \cos 2B &= 2 \cos (A + B) \cos (A - B) \\ &= -2 \cos C \cos (A - B). \end{aligned}$$

$$\begin{aligned} \therefore \cos 2A + \cos 2B + \cos 2C &= -2 \cos C \cos (A - B) + \cos 2C; \\ \text{by [32],} \quad &= -2 \cos C \cos (A - B) + 2 \cos^2 C - 1 \end{aligned}$$

$$\begin{aligned} &= 2 \cos C [\cos C - \cos (A - B)] - 1 \\ &= 2 \cos C [-\cos (A + B) - \cos (A - B)] - 1. \end{aligned}$$

$$\begin{aligned} \text{by [44],} \quad &= 2 \cos C (-2 \cos A \cos B) - 1 \\ &= -4 \cos A \cos B \cos C - 1. \end{aligned}$$

$$43. \sin 3A + \sin 3B + \sin 3C = -4 \cos \frac{3}{2}A \cos \frac{3}{2}B \cos \frac{3}{2}C.$$

$$\begin{aligned} \text{By [42],} \quad \sin 3A + \sin 3B &= 2 \sin \frac{3}{2}(A + B) \cos \frac{3}{2}(A - B) \\ &= 2 \sin \frac{3}{2}(180^\circ - C) \cos \frac{3}{2}(A - B) \\ &= -2 \cos \frac{3}{2}C \cos \frac{3}{2}(A - B). \end{aligned}$$

$$\begin{aligned} \text{By [30],} \quad \sin 3C &= 2 \sin \frac{3}{2}C \cos \frac{3}{2}C \\ &= 2 \sin \frac{3}{2}(180^\circ - A - B) \cos \frac{3}{2}C \\ &= -2 \cos \frac{3}{2}(A + B) \cos \frac{3}{2}C. \end{aligned}$$

$$\begin{aligned} \therefore \sin 3A + \sin 3B + \sin 3C &= -2 \cos \frac{3}{2}C [\cos \frac{3}{2}(A - B) + \cos \frac{3}{2}(A + B)]; \\ \text{by [24] and [25],} \quad &= -2 \cos \frac{3}{2}C (2 \cos \frac{3}{2}A \cos \frac{3}{2}B) \\ &= -4 \cos \frac{3}{2}A \cos \frac{3}{2}B \cos \frac{3}{2}C. \end{aligned}$$

$$44. \cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$$

By [35],

$$\cos^2 A = \frac{1 + \cos 2A}{2}, \quad \cos^2 B = \frac{1 + \cos 2B}{2}, \quad \cos^2 C = \frac{1 + \cos 2C}{2}.$$

$$\cos^2 A + \cos^2 B + \cos^2 C = \frac{3 + \cos 2A + \cos 2B + \cos 2C}{2}.$$

By Ex. 42, Exercise 76,

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$$

$$\therefore \cos^2 A + \cos^2 B + \cos^2 C = \frac{3 - 1 - 4 \cos A \cos B \cos C}{2}$$

$$= 1 - 2 \cos A \cos B \cos C.$$

If $A + B + C = 90^\circ$, prove that :

$$45. \tan A \tan B + \tan B \tan C + \tan C \tan A = 1.$$

$$\tan A \tan B + \tan B \tan C + \tan C \tan A$$

$$= \tan A \tan B + (\tan A + \tan B) \tan C$$

$$= \tan A \tan B + \frac{\tan A + \tan B}{\tan(A + B)};$$

by [26],

$$= \tan A \tan B + \frac{\tan A + \tan B}{\frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$= \tan A \tan B + 1 - \tan A \tan B = 1.$$

$$46. \sin^2 A + \sin^2 B + \sin^2 C = 1 - 2 \sin A \sin B \sin C.$$

$$\sin C = \cos(A + B);$$

by [24],

$$= \cos A \cos B - \sin A \sin B.$$

$$\sin C + \sin A \sin B = \cos A \cos B,$$

$$\sin^2 C + 2 \sin A \sin B \sin C + \sin^2 A \sin^2 B$$

$$= \cos^2 A \cos^2 B,$$

$$\sin^2 C + 2 \sin A \sin B \sin C = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B;$$

by [16],

$$= (1 - \sin^2 A)(1 - \sin^2 B) - \sin^2 A \sin^2 B$$

$$= 1 - \sin^2 A - \sin^2 B.$$

$$\therefore \sin^2 A + \sin^2 B + \sin^2 C = 1 - 2 \sin A \sin B \sin C.$$

$$47. \sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C.$$

$$\text{By [42], } \sin 2A + \sin 2B = 2 \sin(A + B) \cos(A - B)$$

$$= 2 \cos C \cos(A - B).$$

By [30],

$$\sin 2C = 2 \sin C \cos C.$$

$$\therefore \sin 2A + \sin 2B + \sin 2C = 2 \cos C \cos(A - B) + 2 \sin C \cos C$$

$$= 2 \cos C [\cos(A - B) + \sin C]$$

$$= 2 \cos C [\cos(A - B) + \cos(A + B)];$$

by [44],

$$= 4 \cos A \cos B \cos C.$$

$$48. \text{ Prove that } \cot^{-1} 3 + \csc^{-1} \sqrt{5} = \frac{1}{4} \pi.$$

$$\cot^{-1} 3 = 18^\circ 26' 7''$$

$$\csc^{-1} \sqrt{5} = 26^\circ 33' 53''$$

$$\cot^{-1} 3 + \csc^{-1} \sqrt{5} = \overline{45^\circ}$$

$$= \frac{1}{4} \pi.$$

49. Prove that $x + \tan^{-1}(\cot 2x) = \tan^{-1}(\cot x)$.

$$\begin{aligned}\text{Let} \quad \tan^{-1}(\cot 2x) &= y. \\ \tan y &= \cot 2x.\end{aligned}$$

$$\begin{aligned}\text{By [26],} \quad \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ &= \frac{\tan x + \cot 2x}{1 - \tan x \cot 2x};\end{aligned}$$

$$\begin{aligned}\text{by [12],} \quad &= \frac{\tan x + \frac{1}{\tan 2x}}{1 - \frac{\tan x}{\tan 2x}} \\ &= \frac{\tan x \tan 2x + 1}{\tan 2x - \tan x};\end{aligned}$$

$$\begin{aligned}\text{by [31],} \quad &= \frac{\tan x \frac{2 \tan x}{1 - \tan^2 x} + 1}{\frac{2 \tan x}{1 - \tan^2 x} - \tan x} \\ &= \frac{1 + \tan^2 x}{\tan x(1 + \tan^2 x)} \\ &= \frac{1}{\tan x};\end{aligned}$$

$$\text{by [12],} \quad = \cot x.$$

$$\therefore x + \tan^{-1}(\cot 2x) = \tan^{-1}(\cot x).$$

Prove the following statements:

$$50. \quad \frac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \tan 60^\circ.$$

$$\begin{aligned}\text{By [46],} \quad \frac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} &= \frac{\tan 45^\circ}{\tan 30^\circ} \\ &= \tan 60^\circ.\end{aligned}$$

$$51. \quad \sin 60^\circ + \sin 120^\circ = 2 \sin 90^\circ \cos 30^\circ.$$

$$\begin{aligned}\text{By [42],} \quad \sin 120^\circ + \sin 60^\circ &= 2 \sin \frac{1}{2}(120^\circ + 60^\circ) \cos \frac{1}{2}(120^\circ - 60^\circ) \\ &= 2 \sin 90^\circ \cos 30^\circ.\end{aligned}$$

$$52. \quad \cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0.$$

$$\begin{aligned}\text{By [44],} \quad \cos 20^\circ + \cos 100^\circ &= 2 \cos 60^\circ \cos 40^\circ \\ &= \cos 40^\circ.\end{aligned}$$

$$\begin{aligned}\text{Also,} \quad \cos 140^\circ &= \cos(180^\circ - 40^\circ) \\ &= -\cos 40^\circ.\end{aligned}$$

$$\therefore \cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0.$$

$$53. \cos 36^\circ + \sin 36^\circ = \sqrt{2} \cos 9^\circ.$$

$$\begin{aligned} \text{By [25],} \quad \cos(x - \tfrac{1}{4}\pi) &= \cos x \cos \tfrac{1}{4}\pi + \sin x \sin \tfrac{1}{4}\pi \\ &= \tfrac{1}{2}\sqrt{2} \cos x + \tfrac{1}{2}\sqrt{2} \sin x \\ &= \tfrac{1}{2}\sqrt{2} (\cos x + \sin x). \\ \therefore \sin x + \cos x &= \sqrt{2} \cos(x - \tfrac{1}{4}\pi). \\ \cos 36^\circ + \sin 36^\circ &= \sqrt{2} \cos(36^\circ - \tfrac{1}{4}\pi) \\ &= \sqrt{2} \cos(-9^\circ) \\ &= \sqrt{2} \cos 9^\circ. \end{aligned}$$

$$54. \tan 11^\circ 15' + 2 \tan 22^\circ 30' + 4 \tan 45^\circ = \cot 11^\circ 15'.$$

$$\begin{aligned} \cot x - \tan x &= \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{2 \cos 2x}{\sin 2x} = 2 \cot 2x. \\ \cot 11^\circ 15' - \tan 11^\circ 15' &= 2 \cot 22^\circ 30'. \\ 2 \cot 22^\circ 30' - 2 \tan 22^\circ 30' &= 4 \cot 45^\circ. \\ \therefore \cot 11^\circ 15' - \tan 11^\circ 15' - 2 \tan 22^\circ 30' &= 4 \cot 45^\circ = 4 \tan 45^\circ. \\ \therefore \tan 11^\circ 15' + 2 \tan 22^\circ 30' + 4 \tan 45^\circ &= \cot 11^\circ 15'. \end{aligned}$$

Exercise 77. Trigonometric Equations (Page 166)

Solve the following equations:

$$1. \sin x = 2 \sin(\tfrac{1}{3}\pi + x).$$

$$\begin{aligned} \text{by [22],} \quad \sin x &= 2 \sin(\tfrac{1}{3}\pi + x); \\ &= 2 \sin \tfrac{1}{3}\pi \cos x + 2 \cos \tfrac{1}{3}\pi \sin x \\ &= \sqrt{3} \cos x + \sin x. \\ \sqrt{3} \cos x &= 0. \\ \cos x &= 0. \\ \therefore x &= \tfrac{1}{2}\pi \text{ or } \tfrac{3}{2}\pi. \end{aligned}$$

$$2. \sin 2x = 2 \cos x.$$

$$\begin{aligned} \sin 2x &= 2 \cos x. \\ 2 \sin x \cos x &= 2 \cos x. \\ \text{By [30],} \quad 2 \cos x (\sin x - 1) &= 0. \end{aligned}$$

$$\begin{aligned} \text{(i)} \quad 2 \cos x &= 0. \\ \cos x &= 0. \\ \therefore x &= 90^\circ \text{ or } 270^\circ. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \sin x - 1 &= 0. \\ \sin x &= 1. \\ \therefore x &= 90^\circ. \\ \therefore x &= 90^\circ \text{ or } 270^\circ. \end{aligned}$$

$$3. \cos 2x = 2 \sin x.$$

$$\cos 2x = 2 \sin x.$$

$$1 - 2 \sin^2 x = 2 \sin x.$$

$$\text{By [16] and [32], } 2 \sin^2 x + 2 \sin x = 1.$$

$$4 \sin^2 x + 4 \sin x + 1 = 3.$$

$$2 \sin x + 1 = \pm \sqrt{3}.$$

$$2 \sin x = -1 \pm \sqrt{3}.$$

$$\sin x = \frac{-1 \pm \sqrt{3}}{2}$$

$$= \frac{-1 \pm 1.7320}{2}$$

$$= 0.3660 \text{ or } -1.3660.$$

$$\therefore x = 21^\circ 28' \text{ or } 158^\circ 32'.$$

$$4. \sin x + \cos x = 1.$$

$$\sin x + \cos x = 1.$$

$$\sin^2 x + 2 \sin x \cos x + \cos^2 x = 1.$$

$$2 \sin x \cos x = 0.$$

$$(i) \quad \sin x = 0.$$

$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

$$(ii) \quad \cos x = 0.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

$$\therefore x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.$$

But the values $x = 180^\circ$ and $x = 270^\circ$ do not satisfy the given equation

$$\therefore x = 0^\circ \text{ or } 90^\circ.$$

$$5. \sin x + \cos 2x = 4 \sin^2 x.$$

$$\sin x + \cos 2x = 4 \sin^2 x.$$

By [16] and [32],

$$\sin x + 1 - 2 \sin^2 x = 4 \sin^2 x.$$

$$6 \sin^2 x - \sin x - 1 = 0.$$

$$(2 \sin x - 1)(3 \sin x + 1) = 0.$$

$$(i) \quad 2 \sin x - 1 = 0.$$

$$2 \sin x = 1.$$

$$\sin x = \frac{1}{2}.$$

$$\therefore x = 30^\circ \text{ or } 150^\circ.$$

$$(ii) \quad 3 \sin x + 1 = 0.$$

$$3 \sin x = -1.$$

$$\sin x = -\frac{1}{3}.$$

$$\therefore x = 199^\circ 28' \text{ or } 340^\circ 32'.$$

$$\therefore x = 30^\circ, 150^\circ, 199^\circ 28', \text{ or } 340^\circ 32'.$$

$$6. 4 \cos 2x + 3 \cos x = 1.$$

$$4 \cos 2x + 3 \cos x = 1.$$

By [16] and [32],

$$8 \cos^2 x - 4 + 3 \cos x = 1.$$

$$8 \cos^2 x + 3 \cos x - 5 = 0.$$

$$(\cos x + 1)(8 \cos x - 5) = 0.$$

$$(i) \quad \cos x + 1 = 0.$$

$$\cos x = -1.$$

$$\therefore x = 180^\circ.$$

$$(ii) \quad 8 \cos x - 5 = 0.$$

$$8 \cos x = 5.$$

$$\cos x = \frac{5}{8} = 0.6250.$$

$$\therefore x = 51^\circ 19' \text{ or } 308^\circ 41'.$$

$$\therefore x = 51^\circ 19', 180^\circ, \text{ or } 308^\circ 41'.$$

$$7. \sin x = \cos 2x.$$

$$\sin x = \cos 2x.$$

By [16] and [32],

$$\sin x = 1 - 2 \sin^2 x.$$

$$2 \sin^2 x + \sin x - 1 = 0.$$

$$(\sin x + 1)(2 \sin x - 1) = 0.$$

$$(i) \quad \sin x + 1 = 0.$$

$$\sin x = -1.$$

$$\therefore x = 270^\circ.$$

$$(ii) \quad 2 \sin x - 1 = 0.$$

$$2 \sin x = 1.$$

$$\sin x = \frac{1}{2}.$$

$$\therefore x = 30^\circ \text{ or } 150^\circ.$$

$$\therefore x = 30^\circ, 150^\circ, \text{ or } 270^\circ.$$

$$8. \tan x \tan 2x = 2.$$

$$\tan x \tan 2x = 2.$$

$$\text{By [31],} \quad \tan x \frac{2 \tan x}{1 - \tan^2 x} = 2.$$

$$2 \tan^2 x = 2 - 2 \tan^2 x.$$

$$4 \tan^2 x = 2.$$

$$\tan^2 x = \frac{1}{2}.$$

$$\tan x = \pm \sqrt{\frac{1}{2}} = \pm \frac{1}{2} \sqrt{2} = \pm 0.7071.$$

$$\therefore x = 35^\circ 16', 144^\circ 44', 215^\circ 16', \text{ or } 324^\circ 44'.$$

$$9. \sec x = 4 \csc x.$$

$$\sec x = 4 \csc x.$$

$$\frac{1}{\cos x} = \frac{4}{\sin x}.$$

$$\sin x = 4 \cos x.$$

$$\sin^2 x = 16 \cos^2 x.$$

$$1 - \cos^2 x = 16 \cos^2 x.$$

$$17 \cos^2 x = 1.$$

$$\cos x = \pm \frac{1}{\sqrt{17}} \sqrt{17} = \pm 0.2425.$$

$$\therefore x = 75^\circ 58', 104^\circ 2', 255^\circ 58', \text{ or } 284^\circ 2'.$$

But the values $x = 104^\circ 2'$ and $x = 284^\circ 2'$ do not satisfy the given equation.

$$10. \cos \theta + \cos 2\theta = 0. \quad \therefore x = 75^\circ 58' \text{ or } 255^\circ 58'.$$

$$\cos \theta + \cos 2\theta = 0.$$

$$\text{By [32], } \cos \theta + \cos^2 \theta - \sin^2 \theta = 0.$$

$$\cos \theta + \cos^2 \theta - 1 + \cos^2 \theta = 0.$$

$$2 \cos^2 \theta + \cos \theta - 1 = 0.$$

$$(\cos \theta + 1)(2 \cos \theta - 1) = 0.$$

$$(i) \quad \cos \theta + 1 = 0.$$

$$\cos \theta = -1.$$

$$\therefore \theta = 180^\circ.$$

$$(ii) \quad 2 \cos \theta - 1 = 0.$$

$$2 \cos \theta = 1.$$

$$\cos \theta = \frac{1}{2}.$$

$$\therefore \theta = 60^\circ \text{ or } 300^\circ.$$

$$\therefore \theta = 60^\circ, 180^\circ, \text{ or } 300^\circ.$$

$$11. \cot \frac{1}{2} \theta + \csc \theta = 2.$$

$$\cot \frac{1}{2} \theta + \csc \theta = 2.$$

$$\text{By [37], } \pm \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}} + \frac{1}{\sin \theta} = 2.$$

$$\pm \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}} = 2 - \frac{1}{\sin \theta}.$$

$$\frac{1 + \cos \theta}{1 - \cos \theta} = 4 - \frac{4}{\sin \theta} + \frac{1}{\sin^2 \theta}.$$

$$\frac{1 + 2 \cos \theta + \cos^2 \theta}{1 - \cos^2 \theta} = 4 - \frac{4}{\sin \theta} - \frac{1}{\sin^2 \theta}.$$

$$\frac{1 + 2 \cos \theta + \cos^2 \theta}{\sin^2 \theta} = 4 - \frac{4}{\sin \theta} - \frac{1}{\sin^2 \theta}.$$

$$\text{Extract square root, } \frac{1 + \cos \theta}{\sin \theta} = 2 - \frac{1}{\sin \theta}.$$

$$1 + \cos \theta = 2 \sin \theta - 1.$$

$$2 + \cos \theta = 2 \sin \theta.$$

$$\text{By [16], } 2 + \cos \theta = 2 \sqrt{1 - \cos^2 \theta}.$$

$$4 + 4 \cos \theta + \cos^2 \theta = 4 - 4 \cos^2 \theta.$$

$$5 \cos^2 \theta + 4 \cos \theta = 0.$$

$$\cos \theta (5 \cos \theta + 4) = 0.$$

$$\therefore \cos \theta = 0 \text{ or } -0.8.$$

$$\therefore \theta = 90^\circ \text{ or } 143^\circ 8'.$$

$$12. \cot x \tan 2x = 3.$$

$$\cot x \tan 2x = 3.$$

$$\text{By [31],} \quad \frac{1}{\tan x} \times \frac{2 \tan x}{1 - \tan^2 x} = 3.$$

$$\frac{2}{1 - \tan^2 x} = 3.$$

$$2 = 3 - 3 \tan^2 x.$$

$$3 \tan^2 x = 1.$$

$$\tan^2 x = \frac{1}{3}.$$

$$\tan x = \pm \sqrt{\frac{1}{3}} = \pm \frac{1}{3} \sqrt{3}.$$

$$\therefore x = 30^\circ, 150^\circ, 210^\circ, \text{ or } 330^\circ.$$

$$13. \sin x + \sin 2x = \sin 3x.$$

$$\sin x + \sin 2x = \sin 3x.$$

$$\text{By [30],} \quad \sin x + \sin 2x = \sin x + 2 \sin x \cos x.$$

$$\sin 3x = \sin (2x + x);$$

$$\text{by [22],} \quad = \sin 2x \cos x + \cos 2x \sin x;$$

$$\text{by [30] and [32],} \quad = 2 \sin x \cos^2 x + (\cos^2 x - \sin^2 x) \sin x$$

$$= 3 \sin x \cos^2 x - \sin^3 x;$$

$$\text{by [16],} \quad = 3 \sin x (1 - \sin^2 x) - \sin^3 x$$

$$= 3 \sin x - 4 \sin^3 x.$$

$$\therefore \sin x + 2 \sin x \cos x = 3 \sin x - 4 \sin^3 x.$$

$$4 \sin^3 x - 2 \sin x + 2 \sin x \cos x = 0.$$

$$\sin x (2 \sin^2 x - 1 + \cos x) = 0.$$

$$\sin x (1 - 2 \cos^2 x + \cos x) = 0.$$

$$\sin x (1 - \cos x) (1 + 2 \cos x) = 0.$$

$$(i) \quad \sin x = 0.$$

$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

$$(ii) \quad 1 - \cos x = 0.$$

$$\cos x = 1.$$

$$\therefore x = 0^\circ.$$

$$(iii) \quad 1 + 2 \cos x = 0.$$

$$2 \cos x = -1.$$

$$\cos x = -\frac{1}{2}.$$

$$\therefore x = 120^\circ \text{ or } 240^\circ.$$

$$\therefore x = 0^\circ, 120^\circ, 180^\circ, \text{ or } 240^\circ.$$

$$14. \sin 2x = 3 \sin^2 x - \cos^2 x.$$

$$\sin 2x = 3 \sin^2 x - \cos^2 x.$$

$$\text{By [30],} \quad 2 \sin x \cos x = 3 \sin^2 x - \cos^2 x.$$

$$3 \sin^2 x - 2 \sin x \cos x - \cos^2 x = 0.$$

$$(3 \sin x + \cos x)(\sin x - \cos x) = 0.$$

$$(i) \quad 3 \sin x + \cos x = 0.$$

$$3 \tan x + 1 = 0.$$

$$\tan x = -\frac{1}{3}.$$

$$\therefore x = 161^\circ 34' \text{ or } 341^\circ 34'.$$

$$(ii) \quad \sin x - \cos x = 0.$$

$$\sin x = \cos x.$$

$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

$$\therefore x = 45^\circ, 161^\circ 34', 225^\circ, \text{ or } 341^\circ 34'.$$

$$15. \cot \theta = \frac{1}{3} \tan \theta.$$

$$\cot \theta = \frac{1}{3} \tan \theta.$$

$$\frac{1}{\tan \theta} = \frac{\tan \theta}{3}.$$

$$\tan^2 \theta = 3.$$

$$\tan \theta = \pm \sqrt{3}.$$

$$\therefore \theta = 60^\circ, 120^\circ, 240^\circ, \text{ or } 300^\circ.$$

$$16. 2 \sin \theta = \cos \theta.$$

$$2 \sin \theta = \cos \theta.$$

$$4 \sin^2 \theta = \cos^2 \theta.$$

$$4 \sin^2 \theta = 1 - \sin^2 \theta.$$

$$5 \sin^2 \theta = 1.$$

$$\sin^2 \theta = \frac{1}{5}.$$

$$\sin \theta = \pm \frac{1}{\sqrt{5}} \sqrt{5} = \pm 0.4472.$$

$$\therefore \theta = 26^\circ 34', 153^\circ 26', 206^\circ 34', \text{ or } 333^\circ 26'.$$

But the values $\theta = 153^\circ 26'$ and $\theta = 333^\circ 26'$ do not satisfy the given equation.

$$\therefore \theta = 26^\circ 34' \text{ or } 206^\circ 34'.$$

$$17. 2 \sin^2 x + 5 \sin x = 3.$$

$$2 \sin^2 x + 5 \sin x = 3.$$

$$2 \sin^2 x + 5 \sin x - 3 = 0.$$

$$(\sin x + 3)(2 \sin x - 1) = 0.$$

$$(i) \quad \sin x + 3 = 0.$$

$$\sin x = -3.$$

$\therefore x$ is impossible.

$$(ii) \quad 2 \sin x - 1 = 0.$$

$$2 \sin x = 1.$$

$$\sin x = \frac{1}{2}.$$

$$\therefore x = 30^\circ \text{ or } 150^\circ.$$

$$18. \tan x \sec x = \sqrt{2}.$$

$$\tan x \sec x = \sqrt{2}.$$

$$\tan^2 x \sec^2 x = 2.$$

By [20],

$$\tan^2 x (1 + \tan^2 x) = 2.$$

$$\tan^2 x + \tan^4 x = 2.$$

$$\tan^4 x + \tan^2 x - 2 = 0.$$

$$(\tan^2 x - 1)(\tan^2 x + 2) = 0.$$

(i)

$$\tan^2 x - 1 = 0.$$

$$\tan^2 x = 1.$$

$$\tan x = \pm 1.$$

$$\therefore x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

(ii)

$$\tan^2 x + 2 = 0.$$

$$\tan^2 x = -2.$$

$$\tan x = \pm \sqrt{-2}.$$

$\therefore x$ is impossible.

$$\therefore x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

But the values $x = 225^\circ$ and $x = 315^\circ$ do not satisfy the given equation

$$\therefore x = 45^\circ \text{ or } 135^\circ.$$

$$19. \cos x - \cos 2x = 1.$$

$$\cos x - \cos 2x = 1.$$

By [32],

$$\cos x - \cos^2 x + \sin^2 x = 1.$$

$$\cos x - \cos^2 x + 1 - \cos^2 x = 1.$$

$$2 \cos^2 x - \cos x = 0.$$

$$\cos x (2 \cos x - 1) = 0.$$

(i)

$$\cos x = 0.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)

$$2 \cos x - 1 = 0.$$

$$2 \cos x = 1.$$

$$\cos x = \frac{1}{2}.$$

$$\therefore x = 60^\circ \text{ or } 300^\circ.$$

$$\therefore x = 60^\circ, 90^\circ, 270^\circ, \text{ or } 300^\circ.$$

$$20. \cos 3x + 8 \cos^3 x = 0.$$

$$\cos 3x + 8 \cos^3 x = 0.$$

By Ex. 16, Exercise 44,

$$4 \cos^3 x - 3 \cos x + 8 \cos^3 x = 0.$$

$$12 \cos^3 x - 3 \cos x = 0.$$

$$\cos x (4 \cos^2 x - 1) = 0.$$

(i)

$$\cos x = 0.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)

$$4 \cos^2 x - 1 = 0.$$

$$4 \cos^2 x = 1.$$

$$\cos^2 x = \frac{1}{4}.$$

$$\cos x = \pm \frac{1}{2}.$$

$$\therefore x = 60^\circ, 120^\circ, 240^\circ, \text{ or } 300^\circ.$$

$$\therefore x = 60^\circ, 90^\circ, 120^\circ, 240^\circ, 270^\circ, \text{ or } 300^\circ.$$

$$21. \tan x + \cot x = \tan 2x.$$

$$\tan x + \cot x = \tan 2x.$$

By [31],

$$\tan x + \frac{1}{\tan x} = \frac{2 \tan x}{1 - \tan^2 x}.$$

$$\frac{\tan^2 x + 1}{\tan x} = \frac{2 \tan x}{1 - \tan^2 x}.$$

$$1 - \tan^4 x = 2 \tan^2 x.$$

$$\tan^4 x + 2 \tan^2 x = 1.$$

$$\tan^4 x + 2 \tan^2 x + 1 = 2.$$

$$\tan^2 x + 1 = \pm \sqrt{2}.$$

$$\tan^2 x = -1 \pm \sqrt{2}.$$

$$\tan^2 x = 0.4142 \text{ or } -2.4142.$$

$$\tan x = \pm 0.6436.$$

$$\therefore x = 32^\circ 46', 147^\circ 14', 212^\circ 46', \text{ or } 327^\circ 14'.$$

$$22. \tan x + \sec x = a.$$

$$\tan x + \sec x = a.$$

$$\sec x = a - \tan x.$$

$$\sec^2 x = a^2 - 2a \tan x + \tan^2 x.$$

By [20],

$$1 + \tan^2 x = a^2 - 2a \tan x + \tan^2 x.$$

$$2a \tan x = a^2 - 1.$$

$$\tan x = \frac{a^2 - 1}{2a}.$$

$$\therefore x = \tan^{-1} \frac{a^2 - 1}{2a}.$$

$$23. \cos 2x = a(1 - \cos x).$$

$$\cos 2x = a(1 - \cos x).$$

By [32],

$$\cos^2 x - \sin^2 x = a(1 - \cos x).$$

$$\cos^2 x - 1 + \cos^2 x = a - a \cos x.$$

$$2 \cos^2 x + a \cos x = a + 1.$$

$$16 \cos^2 x + () + a^2 = a^2 + 8a + 8.$$

$$4 \cos x + a = \pm \sqrt{a^2 + 8a + 8}.$$

$$\cos x = \frac{-a \pm \sqrt{a^2 + 8a + 8}}{4}.$$

$$x = \cos^{-1} \left(\frac{-a \pm \sqrt{a^2 + 8a + 8}}{4} \right).$$

$$24. \sin^{-1} \frac{1}{2} x = 30^\circ.$$

$$\sin^{-1} \frac{1}{2} x = 30^\circ.$$

$$\sin(\sin^{-1} \frac{1}{2} x) = \sin 30^\circ.$$

$$\frac{1}{2} x = \frac{1}{2}.$$

$$\therefore x = 1.$$

$$25. \tan^{-1} x + 2 \cot^{-1} x = 135^\circ.$$

$$\tan^{-1} x + 2 \cot^{-1} x = 135^\circ.$$

$$\text{But } \tan^{-1} x + \cot^{-1} x = 90^\circ.$$

$$\text{Subtract, } \cot^{-1} x = 45^\circ.$$

$$\therefore x = 1.$$

$$26. \sec x - \cot x = \csc x - \tan x.$$

$$\sec x - \cot x = \csc x - \tan x.$$

$$\frac{1}{\cos x} - \frac{\cos x}{\sin x} = \frac{1}{\sin x} - \frac{\sin x}{\cos x}.$$

$$\sin x - \cos^2 x = \cos x - \sin^2 x.$$

$$(\sin x - \cos x) + (\sin^2 x - \cos^2 x) = 0.$$

$$(\sin x - \cos x)(1 + \sin x + \cos x) = 0.$$

$$(i) \quad \sin x - \cos x = 0.$$

$$\sin x = \cos x.$$

$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

$$(ii) \quad 1 + \sin x + \cos x = 0.$$

$$\sin x + \cos x = -1.$$

$$\sin^2 x + 2 \sin x \cos x + \cos^2 x = 1.$$

$$1 + 2 \sin x \cos x = 1.$$

$$2 \sin x \cos x = 0.$$

By [30],

$$\sin 2x = 0.$$

$$\therefore 2x = 0^\circ, 180^\circ, 360^\circ, \text{ or } 540^\circ.$$

$$\therefore x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.$$

$$\therefore x = 0^\circ, 45^\circ, 90^\circ, 180^\circ, 225^\circ, \text{ or } 270^\circ$$

$$27. \tan 2x \tan x = 1.$$

$$\tan 2x \tan x = 1.$$

$$\left(\frac{2 \tan x}{1 - \tan^2 x} \right) \tan x = 1.$$

$$2 \tan^2 x = 1 - \tan^2 x.$$

$$3 \tan^2 x = 1.$$

$$\tan x = \pm \sqrt{\frac{1}{3}}.$$

$$= \pm \frac{1}{3} \sqrt{3}.$$

$$\therefore x = 30^\circ, 150^\circ, 210^\circ, \text{ or } 330^\circ.$$

$$28. \tan^2 x + \cot^2 x = \frac{10}{3}.$$

$$\tan^2 x + \cot^2 x = \frac{10}{3}.$$

$$\tan^2 x + \frac{1}{\tan^2 x} = \frac{10}{3}.$$

$$3 \tan^4 x + 3 = 10 \tan^2 x.$$

$$3 \tan^4 x - 10 \tan^2 x + 3 = 0.$$

$$(\tan^2 x - 3)(3 \tan^2 x - 1) = 0.$$

$$(i) \quad \tan^2 x - 3 = 0.$$

$$\tan^2 x = 3.$$

$$\tan x = \pm \sqrt{3}.$$

$$\therefore x = 60^\circ, 120^\circ, 240^\circ, \text{ or } 300^\circ.$$

$$\begin{aligned}
 \text{(ii)} \quad & 3 \tan^2 x - 1 = 0. \\
 & 3 \tan^2 x = 1. \\
 & \tan^2 x = \frac{1}{3}. \\
 & \tan x = \pm \sqrt{\frac{1}{3}} = \pm \frac{1}{\sqrt{3}} \sqrt{3}. \\
 & \therefore x = 30^\circ, 150^\circ, 210^\circ, \text{ or } 330^\circ. \\
 \therefore x = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ, \text{ or } 330^\circ.
 \end{aligned}$$

$$29. \sin x + \sin 2x = 1 - \cos 2x.$$

$$\begin{aligned}
 & \sin x + \sin 2x = 1 - \cos 2x. \\
 \text{By [30],} \quad & \sin x + 2 \sin x \cos x = 1 - \cos 2x. \\
 \text{By [32],} \quad & \sin x + 2 \sin x \cos x = 2 \sin^2 x. \\
 & \sin x (1 + 2 \cos x - 2 \sin x) = 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(i)} \quad & \sin x = 0. \\
 & \therefore x = 0^\circ \text{ or } 180^\circ.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & 1 + 2 \cos x - 2 \sin x = 0. \\
 & \sin x - \cos x = \frac{1}{2}. \\
 \sin^2 x + \cos^2 x - 2 \sin x \cos x &= \frac{1}{4}. \\
 1 - 2 \sin x \cos x &= \frac{1}{4}. \\
 2 \sin x \cos x &= \frac{3}{4}.
 \end{aligned}$$

$$\begin{aligned}
 \text{By [30],} \quad & \sin 2x = \frac{3}{4} = 0.75. \\
 \therefore 2x &= 48^\circ 36', 131^\circ 24', 408^\circ 36', \text{ or } 491^\circ 24'. \\
 \therefore x &= 24^\circ 18', 65^\circ 42', 204^\circ 18', \text{ or } 245^\circ 42'. \\
 \therefore x &= 0^\circ, 24^\circ 18', 65^\circ 42', 180^\circ, 204^\circ 18', \text{ or } 245^\circ 42'.
 \end{aligned}$$

But the values $x = 24^\circ 18'$ and $x = 245^\circ 42'$ do not satisfy the given equation.

$$\therefore x = 0^\circ, 65^\circ 42', 180^\circ, \text{ or } 204^\circ 18'.$$

$$30. 4 \cos 2x + 6 \sin x = 5.$$

$$\begin{aligned}
 & 4 \cos 2x + 6 \sin x = 5. \\
 \text{By [32],} \quad & 4(1 - 2 \sin^2 x) + 6 \sin x = 5. \\
 & 4 - 8 \sin^2 x + 6 \sin x = 5. \\
 & 8 \sin^2 x - 6 \sin x + 1 = 0. \\
 & (2 \sin x - 1)(4 \sin x - 1) = 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(i)} \quad & 2 \sin x - 1 = 0. \\
 & 2 \sin x = 1. \\
 & \sin x = \frac{1}{2}. \\
 & \therefore x = 30^\circ \text{ or } 150^\circ.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & 4 \sin x - 1 = 0. \\
 & 4 \sin x = 1. \\
 & \sin x = \frac{1}{4} = 0.25. \\
 & \therefore x = 14^\circ 29' \text{ or } 165^\circ 31'. \\
 & \therefore x = 14^\circ 29', 30^\circ, 150^\circ, \text{ or } 165^\circ 31'.
 \end{aligned}$$

$$31. \sin 4x - \sin 2x = \sin x.$$

$$\sin 4x - \sin 2x = \sin x.$$

By [43],

$$2 \cos 3x \sin x = \sin x$$

$$\sin x (2 \cos 3x - 1) = 0.$$

(i)

$$\sin x = 0.$$

$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

(ii)

$$2 \cos 3x - 1 = 0.$$

$$2 \cos 3x = 1.$$

$$\cos 3x = \frac{1}{2}.$$

$$\therefore 3x = 60^\circ + n360^\circ \text{ or } 300^\circ + n360^\circ.$$

$$\therefore x = 20^\circ, 140^\circ, \text{ or } 260^\circ; \text{ or } 100^\circ, 220^\circ, \text{ or } 340^\circ.$$

$$\therefore x = 0^\circ, 20^\circ, 100^\circ, 140^\circ, 180^\circ, 220^\circ, 260^\circ, \text{ or } 340^\circ.$$

$$32. 2 \sin^2 x + \sin^2 2x = 2.$$

$$2 \sin^2 x + \sin^2 2x = 2.$$

By [30],

$$2 \sin^2 x + 4 \sin^2 x \cos^2 x = 2.$$

$$\sin^2 x + 2 \sin^2 x (1 - \sin^2 x) = 1.$$

$$\sin^2 x + 2 \sin^2 x - 2 \sin^4 x = 1.$$

$$2 \sin^4 x - 3 \sin^2 x + 1 = 0.$$

$$(\sin^2 x - 1)(2 \sin^2 x - 1) = 0.$$

(i)

$$\sin^2 x - 1 = 0.$$

$$\sin^2 x = 1.$$

$$\sin x = \pm 1.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)

$$2 \sin^2 x - 1 = 0.$$

$$2 \sin^2 x = 1.$$

$$\sin^2 x = \frac{1}{2}.$$

$$\sin x = \pm \sqrt{\frac{1}{2}} = \pm \frac{1}{2} \sqrt{2}.$$

$$\therefore x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

$$\therefore x = 45^\circ, 90^\circ, 135^\circ, 225^\circ, 270^\circ, \text{ or } 315^\circ.$$

$$33. \sin x \sec 2x = 1.$$

$$\sin x \sec 2x = 1.$$

$$\frac{\sin x}{\cos 2x} = 1.$$

$$\sin x = \cos 2x.$$

By [32],

$$\sin x = \cos^2 x - \sin^2 x.$$

$$\sin x = 1 - 2 \sin^2 x.$$

$$2 \sin^2 x + \sin x - 1 = 0.$$

$$(\sin x + 1)(2 \sin x - 1) = 0.$$

$$\begin{aligned} \text{(i)} \quad & \sin x + 1 = 0. \\ & \sin x = -1. \\ & \therefore x = 270^\circ. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & 2 \sin x - 1 = 0. \\ & 2 \sin x = 1. \\ & \sin x = \frac{1}{2}. \\ & \therefore x = 30^\circ \text{ or } 150^\circ. \\ & \therefore x = 30^\circ, 150^\circ, \text{ or } 270^\circ \end{aligned}$$

$$34. \sin^2 x + \sin 2x = 1.$$

$$\begin{aligned} & \sin^2 x + \sin 2x = 1. \\ \text{By [30],} \quad & \sin^2 x + 2 \sin x \cos x = 1. \\ & \sin^2 x + 2 \sin x \sqrt{1 - \sin^2 x} = 1. \\ & 2 \sin x \sqrt{1 - \sin^2 x} = 1 - \sin^2 x. \\ & 4 \sin^2 x - 4 \sin^4 x = 1 - 2 \sin^2 x + \sin^4 x. \\ & 5 \sin^4 x - 6 \sin^2 x + 1 = 0. \\ & (\sin^2 x - 1)(5 \sin^2 x - 1) = 0. \end{aligned}$$

$$\begin{aligned} \text{(i)} \quad & \sin^2 x - 1 = 0. \\ & \sin^2 x = 1. \\ & \sin x = \pm 1. \\ & \therefore x = 90^\circ \text{ or } 270^\circ. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & 5 \sin^2 x - 1 = 0. \\ & 5 \sin^2 x = 1. \\ & \sin^2 x = \frac{1}{5}. \\ & \sin x = \pm \sqrt{\frac{1}{5}} = \pm \frac{1}{5} \sqrt{5} = \pm 0.4472. \\ & \therefore x = 26^\circ 34', 153^\circ 26', 206^\circ 34', \text{ or } 333^\circ 26'. \\ & \therefore x = 26^\circ 34', 90^\circ, 153^\circ 26', 206^\circ 34', 270^\circ, \text{ or } 333^\circ 26' \end{aligned}$$

But the values $x = 153^\circ 26'$ and $x = 333^\circ 26'$ do not satisfy the given equation.

$$\therefore x = 26^\circ 34', 90^\circ, 206^\circ 34', \text{ or } 270^\circ.$$

$$35. \cos x \sin 2x \csc x = 1.$$

$$\cos x \sin 2x \csc x = 1.$$

$$\begin{aligned} \text{By [30],} \quad & \cos x (2 \sin x \cos x) \times \frac{1}{\sin x} = 1. \\ & 2 \cos^2 x = 1. \\ & \cos^2 x = \frac{1}{2}. \\ & \cos x = \pm \sqrt{\frac{1}{2}} = \pm \frac{1}{2} \sqrt{2}. \\ & \therefore x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ \end{aligned}$$

$$36. \cot x \tan 2x = \sec 2x. \quad \cot x \tan 2x = \sec 2x.$$

$$\text{By [31] and [32],} \quad \frac{1}{\tan x} \times \frac{2 \tan x}{1 - \tan^2 x} = \frac{1}{\cos^2 x - \sin^2 x}.$$

$$\frac{2}{1 - \tan^2 x} = \frac{1}{\cos^2 x - \sin^2 x}.$$

$$2 \cos^2 x - 2 \sin^2 x = 1 - \frac{\sin^2 x}{\cos^2 x}.$$

$$2 - 2 \sin^2 x - 2 \sin^2 x = 1 - \frac{\sin^2 x}{1 - \sin^2 x}.$$

$$1 - 4 \sin^2 x = - \frac{\sin^2 x}{1 - \sin^2 x}.$$

$$1 - 5 \sin^2 x + 4 \sin^4 x = - \sin^2 x.$$

$$1 - 4 \sin^2 x + 4 \sin^4 x = 0.$$

$$\text{Extract root,} \quad 1 - 2 \sin^2 x = 0.$$

$$2 \sin^2 x = 1.$$

$$\sin^2 x = \frac{1}{2}.$$

$$\sin x = \pm \sqrt{\frac{1}{2}} = \pm \frac{1}{2} \sqrt{2}.$$

$$\therefore x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

$$37. \sin 2x = \cos 4x.$$

$$\sin 2x = \cos 4x.$$

$$\text{By [32],}$$

$$\sin 2x = \cos^2 2x - \sin^2 2x.$$

$$\sin 2x = 1 - \sin^2 2x - \sin^2 2x.$$

$$2 \sin^2 2x + \sin 2x - 1 = 0.$$

$$(\sin 2x + 1)(2 \sin 2x - 1) = 0.$$

(i)

$$\sin 2x + 1 = 0.$$

$$\sin 2x = -1.$$

$$\therefore 2x = 270^\circ \text{ or } 630^\circ.$$

$$\therefore x = 135^\circ \text{ or } 315^\circ.$$

(ii)

$$2 \sin 2x - 1 = 0.$$

$$2 \sin 2x = 1.$$

$$\sin 2x = \frac{1}{2}.$$

$$\therefore 2x = 30^\circ, 150^\circ, 390^\circ, \text{ or } 510^\circ.$$

$$\therefore x = 15^\circ, 75^\circ, 195^\circ, \text{ or } 255^\circ.$$

$$\therefore x = 15^\circ, 75^\circ, 135^\circ, 195^\circ, 255^\circ, \text{ or } 315^\circ$$

$$38. \sin 2z \cot z - \sin^2 z = \frac{1}{2}.$$

$$\sin 2z \cot z - \sin^2 z = \frac{1}{2}.$$

$$\text{By [30],} \quad 2 \sin z \cos z \times \frac{\cos z}{\sin z} - \sin^2 z = \frac{1}{2}.$$

$$2 \cos^2 z - \sin^2 z = \frac{1}{2}.$$

$$2 - 2 \sin^2 z - \sin^2 z = \frac{1}{2}.$$

$$3 \sin^2 z = \frac{3}{2}.$$

$$\sin^2 z = \frac{1}{2}.$$

$$\sin z = \pm \sqrt{\frac{1}{2}} = \pm \frac{1}{2} \sqrt{2}.$$

$$\therefore z = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

$$39. \tan^2 x = \sin 2x.$$

$$\tan^2 x = \sin 2x.$$

By [30],

$$\tan^2 x = 2 \sin x \cos x.$$

$$\tan^2 x = 2 \tan x \cos^2 x.$$

$$\tan^2 x = \frac{2 \tan x}{\sec^2 x}.$$

By [20],

$$\tan^2 x = \frac{2 \tan x}{1 + \tan^2 x}.$$

$$\tan^2 x + \tan^4 x = 2 \tan x.$$

$$\tan x (\tan^3 x + \tan x - 2) = 0.$$

$$\tan x (\tan x - 1) (\tan^2 x + \tan x + 2) = 0.$$

(i)

$$\tan x = 0.$$

$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

(ii)

$$\tan x - 1 = 0.$$

$$\tan x = 1.$$

$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

(iii)

$$\tan^2 x + \tan x + 2 = 0.$$

$$4 \tan^2 x + 4 \tan x + 1 = -7.$$

$$2 \tan x + 1 = \pm \sqrt{-7}.$$

$$\tan x = \frac{1}{2} (-1 \pm \sqrt{-7}).$$

$$\therefore x \text{ is impossible.}$$

$$\therefore x = 0^\circ, 45^\circ, 180^\circ, \text{ or } 225^\circ.$$

$$40. \sec 2x + 1 = 2 \cos x.$$

$$\sec 2x + 1 = 2 \cos x.$$

$$\frac{1}{\cos 2x} + 1 = 2 \cos x.$$

$$1 + \cos 2x = 2 \cos x \cos 2x.$$

By [32],

$$1 + \cos^2 x - \sin^2 x = 2 \cos x (\cos^2 x - \sin^2 x).$$

$$1 + \cos^2 x - 1 + \cos^2 x = 2 \cos x (\cos^2 x - 1 + \cos^2 x).$$

$$2 \cos^2 x = 2 \cos x (2 \cos^2 x - 1).$$

$$\cos x (2 \cos^2 x - \cos x - 1) = 0.$$

$$\cos x (\cos x - 1) (2 \cos x + 1) = 0.$$

(i)

$$\cos x = 0.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)

$$\cos x - 1 = 0.$$

$$\cos x = 1.$$

$$\therefore x = 0^\circ.$$

(iii)

$$2 \cos x + 1 = 0.$$

$$\cos x = -\frac{1}{2}.$$

$$\therefore x = 120^\circ \text{ or } 240^\circ.$$

$$\therefore x = 0^\circ, 90^\circ, 120^\circ, 240^\circ, \text{ or } 270^\circ.$$

$$41 \quad \tan 2x + \tan 3x = 0.$$

$$\tan 2x + \tan 3x = 0.$$

$$\tan 2x = -\tan 3x = \tan(-3x).$$

$$\therefore 2x = -3x \text{ or } 180^\circ - 3x.$$

$$(i) \quad 5x = 0^\circ + n360^\circ.$$

$$\therefore x = 0^\circ, 72^\circ, 144^\circ, 216^\circ, \text{ or } 288^\circ.$$

$$(ii) \quad 5x = 180^\circ + n360^\circ.$$

$$\therefore x = 36^\circ, 108^\circ, 180^\circ, 252^\circ, \text{ or } 324^\circ.$$

$$\therefore x = 0^\circ, 36^\circ, 72^\circ, 108^\circ, 144^\circ, 180^\circ, 216^\circ, 252^\circ, 288^\circ, \text{ or } 324^\circ.$$

$$42. \csc x = \cot x + \sqrt{3}.$$

$$\csc x = \cot x + \sqrt{3}.$$

$$\csc^2 x = \cot^2 x + 2\sqrt{3}\cot x + 3.$$

By [21],

$$1 + \cot^2 x = \cot^2 x + 2\sqrt{3}\cot x + 3.$$

$$2\sqrt{3}\cot x = -2.$$

$$\cot x = -\frac{1}{\sqrt{3}} = -\frac{1}{3}\sqrt{3}.$$

$$\therefore x = 120^\circ \text{ or } 300^\circ.$$

But the value $x = 300^\circ$ does not satisfy the given equation.

$$\therefore x = 120^\circ.$$

$$43. \tan x \tan 3x = -\frac{2}{3}.$$

$$\tan x \tan 3x = -\frac{2}{3}.$$

By Ex. 6, Exercise 76,

$$\tan x \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x} = -\frac{2}{3}.$$

$$\frac{3 \tan^2 x - \tan^4 x}{1 - 3 \tan^2 x} = -\frac{2}{3}.$$

$$15 \tan^2 x - 5 \tan^4 x = -2 + 6 \tan^2 x.$$

$$5 \tan^4 x - 9 \tan^2 x - 2 = 0.$$

$$(\tan^2 x - 2)(5 \tan^2 x + 1) = 0.$$

$$(i) \quad \tan^2 x - 2 = 0.$$

$$\tan^2 x = 2.$$

$$\tan x = \pm \sqrt{2} = \pm 1.4142.$$

$$\therefore x = 54^\circ 44', 125^\circ 16', 234^\circ 44', \text{ or } 305^\circ 16'$$

$$(ii) \quad 5 \tan^2 x + 1 = 0.$$

$$5 \tan^2 x = -1.$$

$$\tan^2 x = -\frac{1}{5}.$$

$$\tan x = \pm \sqrt{-\frac{1}{5}}.$$

$\therefore x$ is impossible.

$$\therefore x = 54^\circ 44', 125^\circ 16', 234^\circ 44', \text{ or } 305^\circ 16'.$$

$$44. \cos 5x + \cos 3x + \cos x = 0.$$

$$\cos 5x + \cos 3x + \cos x = 0.$$

$$\text{By [44],} \quad 2 \cos 4x \cos x + \cos x = 0.$$

$$\cos x (2 \cos 4x + 1) = 0.$$

$$(i) \quad \cos x = 0.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

$$(ii) \quad 2 \cos 4x + 1 = 0.$$

$$2 \cos 4x = -1.$$

$$\cos 4x = -\frac{1}{2}.$$

$$\therefore 4x = 120^\circ + n360^\circ \text{ or } 240^\circ + n360^\circ.$$

$$\therefore x = 30^\circ, 120^\circ, 210^\circ \text{ or } 300^\circ; \text{ or } 60^\circ, 150^\circ, 240^\circ, \text{ or } 330^\circ.$$

$$\therefore x = 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 270^\circ, 300^\circ, \text{ or } 330^\circ$$

$$45. \sin^2 x - \cos^2 x = k.$$

$$\sin^2 x - \cos^2 x = k.$$

$$\sin^2 x - 1 + \sin^2 x = k.$$

$$2 \sin^2 x - 1 = k.$$

$$2 \sin^2 x = k + 1.$$

$$\sin x = \pm \sqrt{\frac{k+1}{2}}.$$

$$x = \sin^{-1} \pm \sqrt{\frac{k+1}{2}}.$$

$$46. \sin x + 2 \cos x = 1.$$

$$\sin x + 2 \cos x = 1.$$

$$\sin x + 2 \sqrt{1 - \sin^2 x} = 1.$$

$$2 \sqrt{1 - \sin^2 x} = 1 - \sin x.$$

$$4 - 4 \sin^2 x = 1 - 2 \sin x + \sin^2 x.$$

$$5 \sin^2 x - 2 \sin x - 3 = 0.$$

$$(5 \sin x + 3)(\sin x - 1) = 0.$$

$$(i) \quad 5 \sin x = -3.$$

$$\sin x = -\frac{3}{5}.$$

$$x = 216^\circ 52', \text{ or } 323^\circ 8'.$$

$$(ii) \quad \sin x = 1.$$

$$x = 90^\circ.$$

$$x = 90^\circ, 216^\circ 52', \text{ or } 323^\circ 8'.$$

$$47. \sin 4x - \cos 3x = \sin 2x.$$

$$\sin 4x - \cos 3x = \sin 2x.$$

$$\sin 4x - \sin 2x - \cos 3x = 0.$$

By [43],

$$2 \cos 3x \sin x - \cos 3x = 0.$$

$$\cos 3x (2 \sin x - 1) = 0.$$

(i)

$$\cos 3x = 0.$$

$$\therefore 3x = 90^\circ + n360^\circ \text{ or } 270^\circ + n360^\circ.$$

$$\therefore x = 30^\circ, 150^\circ, \text{ or } 270^\circ; \text{ or } 90^\circ, 210^\circ, \text{ or } 330^\circ.$$

(ii)

$$2 \sin x - 1 = 0.$$

$$2 \sin x = 1.$$

$$\sin x = \frac{1}{2}.$$

$$\therefore x = 30^\circ \text{ or } 150^\circ.$$

$$\therefore x = 30^\circ, 90^\circ, 150^\circ, 210^\circ, 270^\circ, \text{ or } 330^\circ.$$

$$48. \sin x + \cos x = \sec x.$$

$$\sin x + \cos x = \sec x.$$

$$\sin x + \cos x = \frac{1}{\cos x}.$$

$$\sin x \cos x + \cos^2 x = 1.$$

$$\sin x \cos x + (\cos^2 x - 1) = 0.$$

$$\sin x \cos x - \sin^2 x = 0.$$

$$\sin x (\cos x - \sin x) = 0.$$

(i)

$$\sin x = 0.$$

$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

(ii)

$$\cos x - \sin x = 0.$$

$$\cos x = \sin x.$$

$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

$$\therefore x = 0^\circ, 45^\circ, 180^\circ, \text{ or } 225^\circ.$$

$$49. 2 \cos x \cos 3x + 1 = 0.$$

$$2 \cos x \cos 3x + 1 = 0.$$

By Ex. 16, Exercise 44,

$$2 \cos x (4 \cos^2 x - 3 \cos x) + 1 = 0.$$

$$8 \cos^4 x - 6 \cos^2 x + 1 = 0.$$

$$(4 \cos^2 x - 1)(2 \cos^2 x - 1) = 0.$$

(i)

$$4 \cos^2 x - 1 = 0.$$

$$4 \cos^2 x = 1.$$

$$\cos^2 x = \frac{1}{4}.$$

$$\cos x = \pm \frac{1}{2}.$$

$$\therefore x = 60^\circ, 120^\circ, 240^\circ, \text{ or } 300^\circ.$$

(ii)

$$2 \cos^2 x - 1 = 0.$$

$$2 \cos^2 x = 1.$$

$$\cos^2 x = \frac{1}{2}.$$

$$\cos x = \pm \sqrt{\frac{1}{2}} = \pm \frac{1}{2} \sqrt{2}.$$

$$\therefore x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

$$\therefore x = 45^\circ, 60^\circ, 120^\circ, 135^\circ, 225^\circ, 240^\circ, 300^\circ, \text{ or } 315^\circ.$$

$$50. \cos 3x - 2 \cos 2x + \cos x = 0.$$

$$\cos 3x - 2 \cos 2x + \cos x = 0.$$

$$(\cos 3x + \cos x) - 2 \cos 2x = 0.$$

By [44],

$$2 \cos 2x \cos x - 2 \cos 2x = 0.$$

$$\cos 2x (\cos x - 1) = 0.$$

(i)

$$\cos 2x = 0.$$

$$\therefore 2x = 90^\circ, 270^\circ, 450^\circ, \text{ or } 630^\circ.$$

$$\therefore x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

(ii)

$$\cos x - 1 = 0.$$

$$\cos x = 1.$$

$$\therefore x = 0^\circ.$$

$$\therefore x = 0^\circ, 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ$$

$$51. \sin(x - 30^\circ) = \frac{1}{2} \sqrt{3} \sin x.$$

$$\sin(x - 30^\circ) = \frac{1}{2} \sqrt{3} \sin x.$$

By [23],

$$\sin x \cos 30^\circ - \cos x \sin 30^\circ = \frac{1}{2} \sqrt{3} \sin x.$$

$$\frac{1}{2} \sqrt{3} \sin x - \frac{1}{2} \cos x = \frac{1}{2} \sqrt{3} \sin x.$$

$$-\frac{1}{2} \cos x = 0.$$

$$\cos x = 0.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

$$52. \sin^{-1} x + 2 \cos^{-1} x = \frac{2}{3} \pi.$$

$$\sin^{-1} x + 2 \cos^{-1} x = \frac{2}{3} \pi.$$

But

$$\sin^{-1} x + \cos^{-1} x = \frac{1}{2} \pi.$$

Subtract,

$$\cos^{-1} x = \frac{1}{6} \pi.$$

$$\therefore x = \frac{1}{2} \sqrt{3}.$$

$$53. \sin^{-1} x + 3 \cos^{-1} x = 210^\circ.$$

$$\sin^{-1} x + 3 \cos^{-1} x = 210^\circ.$$

But

$$\sin^{-1} x + \cos^{-1} x = 90^\circ.$$

Subtract,

$$2 \cos^{-1} x = 120^\circ.$$

$$\cos^{-1} x = 60^\circ.$$

$$\therefore x = \frac{1}{2}.$$

$$54. \frac{1 - \tan x}{1 + \tan x} = \cos 2x.$$

$$\frac{1 - \tan x}{1 + \tan x} = \cos 2x.$$

By [32],

$$\frac{\cos x - \sin x}{\cos x + \sin x} = \cos^2 x - \sin^2 x.$$

$$\frac{\cos x - \sin x}{\cos x + \sin x} = (\cos x - \sin x)(\cos x + \sin x).$$

$$\cos x - \sin x = (\cos x - \sin x)(\cos x + \sin x)^2.$$

$$(\cos x - \sin x)[1 - (\cos x + \sin x)^2] = 0.$$

$$(i) \quad \cos x - \sin x = 0.$$

$$\cos x = \sin x.$$

$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

$$(ii) \quad 1 - (\cos x + \sin x)^2 = 0.$$

$$1 - (\cos^2 x + \sin^2 x + 2 \sin x \cos x) = 0.$$

$$1 - (1 + 2 \sin x \cos x) = 0.$$

$$2 \sin x \cos x = 0.$$

By [30],

$$\sin 2x = 0.$$

$$\therefore 2x = 0^\circ, 180^\circ, 360^\circ, 540^\circ.$$

$$\therefore x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.$$

$$\therefore x = 0^\circ, 45^\circ, 90^\circ, 180^\circ, 225^\circ, \text{ or } 270^\circ.$$

$$55. \tan\left(\frac{1}{4}\pi + x\right) + \tan\left(\frac{1}{4}\pi - x\right) = 4.$$

$$\tan\left(\frac{1}{4}\pi + x\right) + \tan\left(\frac{1}{4}\pi - x\right) = 4.$$

$$\text{By [26] and [27], } \frac{1 + \tan x}{1 - \tan x} + \frac{1 - \tan x}{1 + \tan x} = 4.$$

$$1 + 2 \tan x + \tan^2 x + 1 - 2 \tan x + \tan^2 x = 4 - 4 \tan^2 x.$$

$$6 \tan^2 x = 2.$$

$$\tan^2 x = \frac{1}{3}.$$

$$\tan x = \pm \sqrt{\frac{1}{3}} = \pm \frac{1}{\sqrt{3}} \sqrt{3}.$$

$$\therefore x = 30^\circ, 150^\circ, 210^\circ, \text{ or } 330^\circ.$$

$$56. \sqrt{1 + \sin x} - \sqrt{1 - \sin x} = 2 \cos x.$$

$$\sqrt{1 + \sin x} - \sqrt{1 - \sin x} = 2 \cos x.$$

$$\text{Square, } 1 + \sin x - 2\sqrt{1 - \sin^2 x} + 1 - \sin x = 4 \cos^2 x.$$

$$2 - 2\sqrt{1 - \sin^2 x} = 4 \cos^2 x.$$

$$1 - \sqrt{\cos^2 x} = 2 \cos^2 x.$$

$$1 - \cos x = 2 \cos^2 x.$$

$$2 \cos^2 x + \cos x - 1 = 0.$$

$$(\cos x + 1)(2 \cos x - 1) = 0.$$

$$\begin{aligned} \text{(i)} \quad & \cos x + 1 = 0. \\ & \cos x = -1. \\ & \therefore x = 180^\circ. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & 2 \cos x - 1 = 0. \\ & \cos x = \frac{1}{2}. \\ & \therefore x = 60^\circ \text{ or } 300^\circ. \\ & \therefore x = 60^\circ, 180^\circ, \text{ or } 300^\circ. \end{aligned}$$

But the values $x = 180^\circ$ and $x = 300^\circ$ do not satisfy the given equation.

$$\therefore x = 60^\circ.$$

$$\begin{aligned} 57. \quad & \sin(45^\circ + x) + \cos(45^\circ - x) = 1. \\ & \sin(45^\circ + x) + \cos(45^\circ - x) = 1. \\ & \quad 45^\circ + x = 90^\circ - (45^\circ - x). \\ \therefore & \cos(45^\circ - x) + \cos(45^\circ - x) = 1. \\ & 2 \cos(45^\circ - x) = 1. \\ & \cos(45^\circ - x) = \frac{1}{2}. \\ & 45^\circ - x = 60^\circ \text{ or } 300^\circ. \\ & \therefore x = -15^\circ \text{ or } -255^\circ. \\ & \therefore x = 105^\circ \text{ or } 345^\circ. \end{aligned}$$

$$\begin{aligned} 58. \quad & (1 - \tan x) \cos 2x = a(1 + \tan x). \\ & (1 - \tan x) \cos 2x = a(1 + \tan x). \\ & \cos 2x = a \frac{1 + \tan x}{1 - \tan x}. \\ & \cos 2x = \frac{a(\cos x + \sin x)}{\cos x - \sin x}. \\ \text{By [32],} \quad & \cos^2 x - \sin^2 x = \frac{a(\cos x + \sin x)}{\cos x - \sin x}. \\ & (\cos x - \sin x)^2 (\cos x + \sin x) - a(\cos x + \sin x) = 0. \\ & (\cos x + \sin x) [(\cos x - \sin x)^2 - a] = 0. \end{aligned}$$

$$\begin{aligned} \text{(i)} \quad & \cos x + \sin x = 0. \\ & \sin x = -\cos x. \\ & \therefore x = 135^\circ \text{ or } 315^\circ. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & (\cos x - \sin x)^2 - a = 0. \\ & \cos^2 x + \sin^2 x - 2 \sin x \cos x = a. \\ & 1 - 2 \sin x \cos x = a. \end{aligned}$$

$$\begin{aligned} \text{By [30],} \quad & \sin 2x = 1 - a. \\ & \therefore 2x = \sin^{-1}(1 - a). \\ & \therefore x = \frac{1}{2} \sin^{-1}(1 - a). \\ & \therefore x = 135^\circ, 315^\circ, \text{ or } \frac{1}{2} \sin^{-1}(1 - a) \end{aligned}$$

$$59. \sin^6 x + \cos^6 x = \frac{7}{2} \sin^2 2x.$$

$$\sin^6 x + \cos^6 x = \frac{7}{2} \sin^2 2x.$$

$$(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x) = \frac{7}{2} \sin^2 2x.$$

$$\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x = \frac{7}{2} \sin^2 2x.$$

By [30],

$$(\sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x) - 3 \sin^2 x \cos^2 x = \frac{7}{2} \sin^2 2x.$$

$$(\sin^2 x + \cos^2 x)^2 = \frac{13}{2} \sin^2 x \cos^2 x.$$

$$1 = \frac{13}{2} \sin^2 x \cos^2 x.$$

$$16 \sin^2 x \cos^2 x = 3.$$

$$4 \sin^2 x \cos^2 x = \frac{3}{4}.$$

By [30],

$$\sin^2 2x = \pm \frac{1}{2} \sqrt{3}.$$

$$\therefore 2x = 60^\circ, 120^\circ, 240^\circ, 300^\circ, 420^\circ, 480^\circ, 600^\circ, \text{ or } 660^\circ.$$

$$\therefore x = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ, \text{ or } 330^\circ$$

$$60. \sec(x + 120^\circ) + \sec(x - 120^\circ) = 2 \cos x.$$

$$\sec(x + 120^\circ) + \sec(x - 120^\circ) = 2 \cos x.$$

$$\frac{1}{\cos(x + 120^\circ)} + \frac{1}{\cos(x - 120^\circ)} = 2 \cos x$$

$$\frac{\cos(x - 120^\circ) + \cos(x + 120^\circ)}{\cos(x + 120^\circ) \cos(x - 120^\circ)} = 2 \cos x.$$

$$\text{By [44],} \quad \frac{2 \cos x \cos 120^\circ}{\cos(x + 120^\circ) \cos(x - 120^\circ)} = 2 \cos x.$$

By [24] and [25],

$$\frac{2 \cos x \cos 120^\circ}{(\cos x \cos 120^\circ - \sin x \sin 120^\circ)(\cos x \cos 120^\circ + \sin x \sin 120^\circ)} = 2 \cos x.$$

$$\frac{2 \cos x \cos 120^\circ}{\cos^2 x \cos^2 120^\circ - \sin^2 x \sin^2 120^\circ} = 2 \cos x.$$

$$\text{But} \quad \sin 120^\circ = \frac{1}{2} \sqrt{3} \text{ and } \cos 120^\circ = -\frac{1}{2}.$$

$$\therefore \frac{-\cos x}{\frac{1}{4} \cos^2 x - \frac{3}{4} \sin^2 x} = 2 \cos x.$$

$$-\cos x = \frac{1}{2} \cos x (\cos^2 x - 3 \sin^2 x).$$

$$-2 \cos x = \cos x (4 \cos^2 x - 3).$$

$$\cos x (4 \cos^2 x - 1) = 0.$$

(i)

$$\cos x = 0.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)

$$4 \cos^2 x - 1 = 0.$$

$$4 \cos^2 x = 1.$$

$$\cos^2 x = \frac{1}{4}.$$

$$\cos x = \pm \frac{1}{2}.$$

$$\therefore x = 60^\circ, 120^\circ, 240^\circ, \text{ or } 300^\circ.$$

$$\therefore x = 60^\circ, 90^\circ, 120^\circ, 240^\circ, 270^\circ, \text{ or } 300^\circ$$

$$61. \sin^2 x \cos^2 x - \cos^2 x - \sin^2 x + 1 = 0.$$

$$\sin^2 x \cos^2 x - \cos^2 x - \sin^2 x + 1 = 0.$$

$$\sin^2 x \cos^2 x - (\cos^2 x + \sin^2 x) + 1 = 0.$$

$$\sin^2 x \cos^2 x - 1 + 1 = 0.$$

$$\sin^2 x \cos^2 x = 0.$$

$$\sin x \cos x = 0.$$

$$2 \sin x \cos x = 0.$$

By [30],

$$\sin 2x = 0.$$

$$\therefore 2x = 0^\circ, 180^\circ, 360^\circ, \text{ or } 540^\circ$$

$$\therefore x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ$$

$$62. \sin x + \sin 2x + \sin 3x = 0.$$

$$\sin x + \sin 2x + \sin 3x = 0.$$

$$(\sin 3x + \sin x) + \sin 2x = 0.$$

By [42],

$$2 \sin 2x \cos x + \sin 2x = 0.$$

$$\sin 2x (2 \cos x + 1) = 0.$$

(i)

$$\sin 2x = 0.$$

$$\therefore 2x = 0^\circ, 180^\circ, 360^\circ, \text{ or } 540^\circ$$

$$\therefore x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.$$

(ii)

$$2 \cos x + 1 = 0.$$

$$2 \cos x = -1.$$

$$\cos x = -\frac{1}{2}.$$

$$\therefore x = 120^\circ \text{ or } 240^\circ.$$

$$\therefore x = 0^\circ, 90^\circ, 120^\circ, 180^\circ, 240^\circ, \text{ or } 270^\circ.$$

$$63. \sin \theta + 2 \sin 2\theta + 3 \sin 3\theta = 0.$$

$$\sin \theta + 2 \sin 2\theta + 3 \sin 3\theta = 0.$$

By [30], and Ex. 15, Exercise 44,

$$\sin \theta + 4 \sin \theta \cos \theta + 9 \sin \theta - 12 \sin^3 \theta = 0.$$

$$10 \sin \theta + 4 \sin \theta \cos \theta - 12 \sin^3 \theta = 0.$$

$$\sin \theta (5 + 2 \cos \theta - 6 \sin^2 \theta) = 0.$$

(i)

$$\sin \theta = 0.$$

$$\therefore \theta = 0^\circ \text{ or } 180^\circ$$

(ii)

$$5 + 2 \cos \theta - 6 \sin^2 \theta = 0.$$

$$5 + 2 \cos \theta - 6 + 6 \cos^2 \theta = 0.$$

$$6 \cos^2 \theta + 2 \cos \theta = 1.$$

$$36 \cos^2 \theta + 12 \cos \theta + 1 = 7.$$

$$6 \cos \theta + 1 = \pm \sqrt{7}.$$

$$6 \cos \theta = -1 \pm \sqrt{7}.$$

$$\cos \theta = \frac{1}{6}(-1 \pm 2.64575)$$

$$= 0.2743 \text{ or } -0.6076.$$

$$\therefore \theta = 74^\circ 5' \text{ or } 285^\circ 55'; \text{ or } 127^\circ 25' \text{ or } 232^\circ 35'.$$

$$\therefore \theta = 0^\circ, 74^\circ 5', 127^\circ 25', 180^\circ, 232^\circ 35', \text{ or } 285^\circ 55'.$$

$$64. \sin 3x = \cos 2x - 1. \quad \sin 3x = \cos 2x - 1.$$

By Ex. 15, Exercise 44, and [32],

$$3 \sin x - 4 \sin^3 x = 1 - 2 \sin^2 x - 1.$$

$$4 \sin^3 x - 2 \sin^2 x - 3 \sin x = 0.$$

$$\sin x (4 \sin^2 x - 2 \sin x - 3) = 0.$$

$$(i) \quad \sin x = 0.$$

$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

$$(ii) \quad 4 \sin^2 x - 2 \sin x - 3 = 0.$$

$$16 \sin^2 x - 8 \sin x + 1 = 13.$$

$$4 \sin x - 1 = \pm \sqrt{13}.$$

$$\sin x = \frac{1}{4} (1 \pm \sqrt{13})$$

$$= 1.1514 \text{ or } -0.6514.$$

$$\therefore x = 220^\circ 39' \text{ or } 319^\circ 21'.$$

$$\therefore x = 0^\circ, 180^\circ, 220^\circ 39', \text{ or } 319^\circ 21'$$

$$65. \sin(x + 12^\circ) + \sin(x - 8^\circ) = \sin 20^\circ.$$

$$\sin(x + 12^\circ) + \sin(x - 8^\circ) = \sin 20^\circ.$$

$$\text{By [42],} \quad 2 \sin(x + 2^\circ) \cos 10^\circ = \sin 20^\circ.$$

$$\text{By [30],} \quad 2 \sin(x + 2^\circ) \cos 10^\circ = 2 \sin 10^\circ \cos 10^\circ.$$

$$2 \sin(x + 2^\circ) = 2 \sin 10^\circ.$$

$$\sin(x + 2^\circ) = \sin 10^\circ.$$

$$\therefore x + 2^\circ = 10^\circ \text{ or } 170^\circ.$$

$$\therefore x = 8^\circ \text{ or } 168^\circ.$$

$$66. \tan(60^\circ + x) \tan(60^\circ - x) = -2.$$

$$\tan(60^\circ + x) \tan(60^\circ - x) = -2.$$

By [26] and [27],

$$\frac{\tan 60^\circ + \tan x}{1 - \tan 60^\circ \tan x} \times \frac{\tan 60^\circ - \tan x}{1 + \tan 60^\circ \tan x} = -2.$$

$$\frac{\tan^2 60^\circ - \tan^2 x}{1 - \tan^2 60^\circ \tan^2 x} = -2.$$

$$\frac{3 - \tan^2 x}{1 - 3 \tan^2 x} = -2.$$

$$3 - \tan^2 x = -2 + 6 \tan^2 x.$$

$$7 \tan^2 x = 5.$$

$$\tan^2 x = \frac{5}{7} = 0.71428571.$$

$$\tan x = \pm 0.8451.$$

$$\therefore x = 40^\circ 12', 139^\circ 48', 220^\circ 12', \text{ or } 319^\circ 48'.$$

$$67. \tan x + \tan 2x = 0.$$

$$\tan x + \tan 2x = 0.$$

$$\text{By [31],} \quad \tan x + \frac{2 \tan x}{1 - \tan^2 x} = 0.$$

$$\tan x - \tan^3 x + 2 \tan x = 0.$$

$$\tan x (\tan^2 x - 3) = 0.$$

$$(i) \quad \tan x = 0.$$

$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

$$(ii) \quad \tan^2 x - 3 = 0.$$

$$\tan^2 x = 3.$$

$$\tan x = \pm \sqrt{3}.$$

$$\therefore x = 60^\circ, 120^\circ, 240^\circ, \text{ or } 300^\circ.$$

$$\therefore x = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, \text{ or } 300^\circ$$

$$68. \sin(x + 120^\circ) + \sin(x + 60^\circ) = \frac{3}{2}.$$

$$\sin(x + 120^\circ) + \sin(x + 60^\circ) = \frac{3}{2}.$$

$$\text{By [42],} \quad 2 \sin(x + 90^\circ) \cos 30^\circ = \frac{3}{2}.$$

$$2 \cos x \cos 30^\circ = \frac{3}{2}.$$

$$2 \cos x \left(\frac{1}{2} \sqrt{3}\right) = \frac{3}{2}.$$

$$\cos x = \frac{3}{2 \sqrt{3}} = \frac{1}{2} \sqrt{3}.$$

$$\therefore x = 30^\circ \text{ or } 330^\circ.$$

$$69. \sin(x + 30^\circ) \sin(x - 30^\circ) = \frac{1}{2}.$$

$$\sin(x + 30^\circ) \sin(x - 30^\circ) = \frac{1}{2}.$$

$$\text{By [45],} \quad -\frac{1}{2}(\cos 2x - \cos 60^\circ) = \frac{1}{2}.$$

$$\cos 2x - \cos 60^\circ = -1.$$

$$\cos 2x - \frac{1}{2} = -1.$$

$$\cos 2x = -\frac{1}{2}.$$

$$\therefore 2x = 120^\circ, 240^\circ, 480^\circ, \text{ or } 600^\circ.$$

$$\therefore x = 60^\circ, 120^\circ, 240^\circ, \text{ or } 300^\circ$$

$$70. \sin 2\theta = \cos 3\theta.$$

$$\sin 2\theta = \cos 3\theta.$$

$$\text{By [30], and Ex. 16, Exercise 44,}$$

$$2 \sin \theta \cos \theta = 4 \cos^3 \theta - 3 \cos \theta.$$

$$4 \cos^3 \theta - 3 \cos \theta - 2 \sin \theta \cos \theta = 0.$$

$$\cos \theta (4 \cos^2 \theta - 3 - 2 \sin \theta) = 0.$$

$$(i) \quad \cos \theta = 0.$$

$$\therefore \theta = 90^\circ \text{ or } 270^\circ.$$

$$(ii) \quad 4 \cos^2 \theta - 3 - 2 \sin \theta = 0.$$

$$4 - 4 \sin^2 \theta - 3 - 2 \sin \theta = 0.$$

$$4 \sin^2 \theta + 2 \sin \theta = 1.$$

$$16 \sin^2 \theta + 8 \sin \theta + 1 = 5.$$

$$4 \sin \theta + 1 = \pm \sqrt{5}.$$

$$4 \sin \theta = -1 \pm \sqrt{5}.$$

$$\sin \theta = \frac{1}{4}(-1 \pm \sqrt{5})$$

$$= \frac{1}{4}(-1 \pm 2.23606)$$

$$= 0.3090 \text{ or } -0.8090.$$

$$\therefore \theta = 18^\circ \text{ or } 162^\circ; \text{ or } 234^\circ \text{ or } 306^\circ.$$

$$\therefore \theta = 18^\circ, 90^\circ, 162^\circ, 234^\circ, 270^\circ, \text{ or } 306^\circ$$

$$71 \quad \sin^4 x + \cos^4 x = \frac{5}{8}.$$

$$\sin^4 x + \cos^4 x = \frac{5}{8}.$$

$$\sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x = 2 \sin^2 x \cos^2 x + \frac{5}{8}.$$

$$(\sin^2 x + \cos^2 x)^2 = 2 \sin^2 x \cos^2 x + \frac{5}{8}.$$

$$1 = 2 \sin^2 x \cos^2 x + \frac{5}{8}.$$

$$2 \sin^2 x \cos^2 x = \frac{3}{8}.$$

$$4 \sin^2 x \cos^2 x = \frac{3}{4}.$$

$$\text{By [30],} \quad \sin^2 2x = \frac{3}{4}.$$

$$\sin 2x = \pm \frac{1}{2} \sqrt{3}.$$

$$\therefore 2x = 60^\circ, 120^\circ, 240^\circ, 300^\circ, 420^\circ, 480^\circ, 600^\circ, \text{ or } 660^\circ$$

$$\therefore x = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ, \text{ or } 330^\circ.$$

$$72. \sin^4 x - \cos^4 x = \frac{7}{25}.$$

$$\sin^4 x - \cos^4 x = \frac{7}{25}.$$

$$(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) = \frac{7}{25}.$$

$$\sin^2 x - \cos^2 x = \frac{7}{25}.$$

$$\cos^2 x - \sin^2 x = -\frac{7}{25}.$$

$$\text{By [32],} \quad \cos 2x = -\frac{7}{25}.$$

$$\therefore 2x = 106^\circ 16', 253^\circ 44', 466^\circ 16', \text{ or } 613^\circ 44'.$$

$$\therefore x = 53^\circ 8', 126^\circ 52', 233^\circ 8', \text{ or } 306^\circ 52'.$$

$$73. \tan(x + 30^\circ) = 2 \cos x.$$

$$\tan(x + 30^\circ) = 2 \cos x.$$

Let

$$x + 30^\circ = y.$$

Then

$$x = y - 30^\circ.$$

Substitute,

$$\tan y = 2 \cos(y - 30^\circ).$$

By [25],

$$\tan y = 2 \cos y \cos 30^\circ + 2 \sin y \sin 30^\circ.$$

$$\frac{\sin y}{\cos y} = \sqrt{3} \cos y + \sin y.$$

$$\sin y = \sqrt{3} \cos^2 y + \sin y \cos y.$$

$$\sin y(1 - \cos y) = \sqrt{3} \cos^2 y.$$

$$\sin^2 y(1 - \cos y)^2 = 3 \cos^4 y.$$

$$(1 - \cos^2 y)(1 - 2 \cos y + \cos^2 y) = 3 \cos^4 y.$$

$$1 - 2 \cos y + 2 \cos^3 y - \cos^4 y = 3 \cos^4 y.$$

$$4 \cos^4 y - 2 \cos^3 y + 2 \cos y - 1 = 0.$$

$$(2 \cos y - 1)(2 \cos^3 y + 1) = 0.$$

(i)

$$2 \cos y - 1 = 0.$$

$$2 \cos y = 1.$$

$$\cos y = \frac{1}{2}.$$

$$\therefore y = 60^\circ \text{ or } 300^\circ.$$

$$\therefore x = 30^\circ \text{ or } 270^\circ.$$

$$\begin{aligned}
 \text{(ii)} \quad & 2 \cos^3 y + 1 = 0. \\
 & 2 \cos^3 y = -1. \\
 & \cos^3 y = -\frac{1}{2}. \\
 & \cos y = \sqrt[3]{-\frac{1}{2}}.
 \end{aligned}$$

The two other roots of equation (ii) are complex.

$$\begin{aligned}
 \cos y &= -0.7937. \\
 \therefore y &= 127^\circ 28' \text{ or } 232^\circ 32'. \\
 \therefore x &= 97^\circ 28' \text{ or } 202^\circ 32'. \\
 \therefore x &= 30^\circ, 97^\circ 28', 202^\circ 32', \text{ or } 270^\circ.
 \end{aligned}$$

But the values $x = 97^\circ 28'$, $x = 202^\circ 32'$, and $x = 270^\circ$ do not satisfy the given equation.

$$\therefore x = 30^\circ.$$

$$74. \sec x = 2 \tan x + \frac{1}{4}.$$

$$\sec x = 2 \tan x + \frac{1}{4}.$$

$$\sec^2 x = 4 \tan^2 x + \tan x + \frac{1}{16}.$$

By [20],

$$1 + \tan^2 x = 4 \tan^2 x + \tan x + \frac{1}{16}.$$

$$3 \tan^2 x + \tan x - \frac{1}{16} = 0.$$

$$48 \tan^2 x + 16 \tan x - 15 = 0.$$

$$(4 \tan x + 3)(12 \tan x - 5) = 0.$$

$$\text{(i)} \quad 4 \tan x + 3 = 0.$$

$$4 \tan x = -3.$$

$$\tan x = -\frac{3}{4} = -0.75.$$

$$\therefore x = 143^\circ 8' \text{ or } 323^\circ 8'.$$

$$\text{(ii)} \quad 12 \tan x - 5 = 0.$$

$$12 \tan x = 5.$$

$$\tan x = \frac{5}{12} = 0.4167.$$

$$\therefore x = 22^\circ 37' \text{ or } 202^\circ 37'.$$

$$\therefore x = 22^\circ 37', 143^\circ 8', 202^\circ 37', \text{ or } 323^\circ 8'$$

But the values $x = 202^\circ 37'$ and $x = 323^\circ 8'$ do not satisfy the given equation.

$$\therefore x = 22^\circ 37' \text{ or } 143^\circ 8'$$

$$75. \sin 11x \sin 4x + \sin 5x \sin 2x = 0.$$

$$\sin 11x \sin 4x + \sin 5x \sin 2x = 0.$$

By [45],

$$\sin 11x \sin 4x = -\frac{1}{2}(\cos 15x - \cos 7x),$$

and

$$\sin 5x \sin 2x = -\frac{1}{2}(\cos 7x - \cos 3x).$$

$$\therefore -\frac{1}{2}(\cos 15x - \cos 7x) - \frac{1}{2}(\cos 7x - \cos 3x) = 0.$$

$$\cos 15x - \cos 7x + \cos 7x - \cos 3x = 0.$$

$$\cos 15x - \cos 3x = 0.$$

By [45],

$$-2 \sin 9x \sin 6x = 0.$$

$$\sin 9x \sin 6x = 0.$$

By Ex. 15, Exercise 44, $\sin 9x = 3 \sin 3x - 4 \sin^3 3x$,

and by [34],
$$\sin 6x = \pm \sqrt{\frac{1 - \cos 12x}{2}}.$$

Substitute,
$$(3 \sin 3x - 4 \sin^3 3x) \left(\pm \sqrt{\frac{1 - \cos 12x}{2}} \right) = 0.$$

$$\sin 3x (3 - 4 \sin^2 3x) \left(\pm \sqrt{\frac{1 - \cos 12x}{2}} \right) = 0.$$

(i) $\sin 3x = 0.$

$$\therefore 3x = 0^\circ, 180^\circ, 360^\circ, 540^\circ, 720^\circ, \text{ or } 900^\circ.$$

$$\therefore x = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, \text{ or } 300^\circ.$$

(ii) $3 - 4 \sin^2 3x = 0.$

$$4 \sin^2 3x = 3.$$

$$2 \sin 3x = \pm \sqrt{3}.$$

$$\sin 3x = \pm \frac{1}{2} \sqrt{3}.$$

$$\therefore 3x = 60^\circ, 120^\circ, 240^\circ, 300^\circ, 420^\circ, 480^\circ, 600^\circ, 660^\circ, 780^\circ, 840^\circ, 960^\circ, \text{ or } 1020^\circ.$$

$$\therefore x = 20^\circ, 40^\circ, 80^\circ, 100^\circ, 140^\circ, 160^\circ, 200^\circ, 220^\circ, 260^\circ, 280^\circ, 320^\circ, \text{ or } 340^\circ.$$

(iii)
$$\pm \sqrt{\frac{1 - \cos 12x}{2}} = 0.$$

$$\frac{1 - \cos 12x}{2} = 0.$$

$$1 - \cos 12x = 0.$$

$$\cos 12x = 1.$$

$$\therefore 12x = 0^\circ, 360^\circ, 720^\circ, 1080^\circ, 1440^\circ, 1800^\circ, 2160^\circ, 2520^\circ, 2880^\circ, 3240^\circ, 3600^\circ, \text{ or } 3960^\circ.$$

$$\therefore x = 0^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 180^\circ, 210^\circ, 240^\circ, 270^\circ, 300^\circ, \text{ or } 330^\circ.$$

$$\therefore x = 0^\circ, 20^\circ, 30^\circ, 40^\circ, 60^\circ, 80^\circ, 90^\circ, 100^\circ, 120^\circ, 140^\circ, 150^\circ, 160^\circ, 180^\circ, 200^\circ, 210^\circ, 220^\circ, 240^\circ, 260^\circ, 270^\circ, 280^\circ, 300^\circ, 320^\circ, 330^\circ, \text{ or } 340^\circ.$$

76. $\cos x + \cos 3x + \cos 5x + \cos 7x = 0.$

$$\cos x + \cos 3x + \cos 5x + \cos 7x = 0.$$

$$(\cos 7x + \cos 5x) + (\cos 3x + \cos x) = 0.$$

By [44], $2 \cos 6x \cos x + 2 \cos 2x \cos x = 0.$

$$\cos x (\cos 6x + \cos 2x) = 0.$$

By [44], $\cos x (2 \cos 4x \cos 2x) = 0.$

$$\cos x \cos 2x \cos 4x = 0.$$

(i) $\cos x = 0.$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)

$$\cos 2x = 0.$$

$$\therefore 2x = 90^\circ, 270^\circ, 450^\circ, \text{ or } 630^\circ.$$

$$\therefore x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

(iii)

$$\cos 4x = 0.$$

$$\therefore 4x = 90^\circ, 270^\circ, 450^\circ, 630^\circ, 810^\circ, 990^\circ, 1170^\circ, \text{ or } 1350^\circ.$$

$$\therefore x = 22\frac{1}{2}^\circ, 67\frac{1}{2}^\circ, 112\frac{1}{2}^\circ, 157\frac{1}{2}^\circ, 202\frac{1}{2}^\circ, 247\frac{1}{2}^\circ, 292\frac{1}{2}^\circ, \text{ or } 337\frac{1}{2}^\circ.$$

$$\therefore x = 22\frac{1}{2}^\circ, 45^\circ, 67\frac{1}{2}^\circ, 90^\circ, 112\frac{1}{2}^\circ, 135^\circ, 157\frac{1}{2}^\circ, 202\frac{1}{2}^\circ, 225^\circ, 247\frac{1}{2}^\circ, 270^\circ, 292\frac{1}{2}^\circ, 315^\circ, \text{ or } 337\frac{1}{2}^\circ.$$

$$77. \sin(x + 12^\circ) \cos(x - 12^\circ) = \cos 33^\circ \sin 57^\circ.$$

$$\sin(x + 12^\circ) \cos(x - 12^\circ) = \cos 33^\circ \sin 57^\circ.$$

$$\text{By [42],} \quad \frac{1}{2}(\sin 2x + \sin 24^\circ) = \frac{1}{2}(\sin 90^\circ + \sin 24^\circ).$$

$$\sin 2x + \sin 24^\circ = \sin 90^\circ + \sin 24^\circ$$

$$\sin 2x = \sin 90^\circ.$$

$$\therefore 2x = 90^\circ + n360^\circ.$$

$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

$$78. \sin^{-1}x + \sin^{-1}\frac{1}{2}x = 120^\circ.$$

$$\sin^{-1}x + \sin^{-1}\frac{1}{2}x = 120^\circ.$$

$$\sin(\sin^{-1}x + \sin^{-1}\frac{1}{2}x) = \sin 120^\circ = \cos 30^\circ = \frac{1}{2}\sqrt{3}.$$

$$\text{By [22],} \quad \sin(\sin^{-1}x) \cos(\sin^{-1}\frac{1}{2}x) + \cos(\sin^{-1}x) \sin(\sin^{-1}\frac{1}{2}x) = \frac{1}{2}\sqrt{3}$$

$$x\sqrt{1 - \frac{1}{4}x^2} + \frac{1}{2}x\sqrt{1 - x^2} = \frac{1}{2}\sqrt{3}.$$

$$x\sqrt{4 - x^2} + x\sqrt{1 - x^2} = \sqrt{3}.$$

$$x\sqrt{4 - x^2} = \sqrt{3} - x\sqrt{1 - x^2}.$$

$$x^2(4 - x^2) = 3 - 2\sqrt{3}x\sqrt{1 - x^2} + x^2(1 - x^2).$$

$$4x^2 - x^4 = 3 - 2\sqrt{3}x\sqrt{1 - x^2} + x^2 - x^4.$$

$$3x^2 - 3 = -2\sqrt{3}x\sqrt{1 - x^2}.$$

Square,

$$9x^4 - 18x^2 + 9 = 12x^2 - 12x^4.$$

$$21x^4 - 30x^2 + 9 = 0.$$

$$7x^4 - 10x^2 + 3 = 0.$$

$$(x^2 - 1)(7x^2 - 3) = 0.$$

(i)

$$x^2 - 1 = 0.$$

$$x^2 = 1.$$

$$\therefore x = \pm 1.$$

(ii)

$$7x^2 - 3 = 0.$$

$$7x^2 = 3.$$

$$x^2 = \frac{3}{7}.$$

$$\therefore x = \pm \sqrt{\frac{3}{7}} = \pm \frac{1}{7}\sqrt{21}.$$

$$\therefore x = \pm 1 \text{ or } \pm \frac{1}{7}\sqrt{21}.$$

$$79. \tan^{-1} x + \tan^{-1} 2x = \tan^{-1} 3\sqrt{3}.$$

$$\tan^{-1} x + \tan^{-1} 2x = \tan^{-1} 3\sqrt{3}.$$

$$\tan(\tan^{-1} x + \tan^{-1} 2x) = 3\sqrt{3}.$$

By [26],

$$\frac{x + 2x}{1 - 2x^2} = 3\sqrt{3}.$$

$$\frac{3x}{1 - 2x^2} = 3\sqrt{3}.$$

$$\frac{x}{1 - 2x^2} = \sqrt{3}.$$

$$x = \sqrt{3} - 2\sqrt{3}x^2.$$

$$2\sqrt{3}x^2 + x - \sqrt{3} = 0.$$

$$x^2 + \frac{1}{6}\sqrt{3}x - \frac{1}{2} = 0.$$

$$(x - \frac{1}{3}\sqrt{3})(x + \frac{1}{2}\sqrt{3}) = 0.$$

$$\therefore x = \frac{1}{3}\sqrt{3} \text{ or } -\frac{1}{2}\sqrt{3}.$$

$$80. \tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1} 2x.$$

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1} 2x.$$

$$\tan[\tan^{-1}(x+1) + \tan^{-1}(x-1)] = 2x.$$

By [26],
$$\frac{x+1+x-1}{1-(x+1)(x-1)} = 2x.$$

$$\frac{2x}{2-x^2} = 2x.$$

$$x = 2x - x^3.$$

$$x^3 - x = 0.$$

$$x(x^2 - 1) = 0.$$

$$\therefore x = 0 \text{ or } \pm 1.$$

$$81. (3 - 4 \cos^2 x) \sin 2x = 0.$$

$$(3 - 4 \cos^2 x) \sin 2x = 0.$$

(i)

$$3 - 4 \cos^2 x = 0.$$

$$4 \cos^2 x = 3.$$

$$2 \cos x = \pm \sqrt{3}.$$

$$\cos x = \pm \frac{1}{2} \sqrt{3}.$$

$$\therefore x = 30^\circ, 150^\circ, 210^\circ, \text{ or } 330^\circ$$

(ii)

$$\sin 2x = 0.$$

$$\therefore 2x = 0^\circ, 180^\circ, 360^\circ, \text{ or } 540^\circ.$$

$$\therefore x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.$$

$$\therefore x = 0^\circ, 30^\circ, 90^\circ, 150^\circ, 180^\circ, 210^\circ, 270^\circ, \text{ or } 330^\circ.$$

$$82. \cos 2\theta \sec \theta + \sec \theta + 1 = 0.$$

$$\text{By [32],} \quad \frac{\cos 2\theta \sec \theta + \sec \theta + 1 = 0.}{\frac{\cos^2 \theta - \sin^2 \theta}{\cos \theta} + \sec \theta + 1 = 0.}$$

$$\frac{2 \cos^2 \theta - 1}{\cos \theta} + \frac{1}{\cos \theta} + 1 = 0.$$

$$2 \cos^2 \theta + \cos \theta = 0.$$

$$\cos \theta (2 \cos \theta + 1) = 0.$$

$$(i) \quad \cos \theta = 0.$$

$$\therefore \theta = 90^\circ \text{ or } 270^\circ.$$

$$(ii) \quad 2 \cos \theta + 1 = 0.$$

$$2 \cos \theta = -1.$$

$$\cos \theta = -\frac{1}{2}.$$

$$\therefore \theta = 120^\circ \text{ or } 240^\circ.$$

$$\therefore \theta = 90^\circ, 120^\circ, 240^\circ, \text{ or } 270^\circ.$$

But the values $\theta = 90^\circ$ and $\theta = 270^\circ$ do not satisfy the given equation

$$\therefore \theta = 120^\circ \text{ or } 240^\circ.$$

$$83. \sin x \cos 2x \tan x \cot 2x \sec x \csc 2x = 1.$$

$$\sin x \cos 2x \tan x \cot 2x \sec x \csc 2x = 1.$$

$$\text{By [32],} \quad \cos 2x = \cos^2 x - \sin^2 x.$$

$$\text{By [33],} \quad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x} = \frac{\frac{\cos^2 x}{\sin^2 x} - 1}{\frac{2 \cos x}{\sin x}} = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}.$$

$$\text{By [14] and [30],} \quad \csc 2x = \frac{1}{\sin 2x} = \frac{1}{2 \sin x \cos x}.$$

Substitute,

$$\sin x (\cos^2 x - \sin^2 x) \left(\frac{\sin x}{\cos x} \right) \left(\frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} \right) \left(\frac{1}{\cos x} \right) \left(\frac{1}{2 \sin x \cos x} \right) = 1.$$

$$\frac{(\cos^2 x - \sin^2 x)^2}{4 \cos^4 x} = 1.$$

$$\frac{(2 \cos^2 x - 1)^2}{4 \cos^4 x} = 1.$$

$$\frac{2 \cos^2 x - 1}{2 \cos^2 x} = \pm 1.$$

$$\therefore 2 \cos^2 x - 1 = \pm 2 \cos^2 x.$$

$$4 \cos^2 x = 1.$$

$$\cos^2 x = \frac{1}{4}.$$

$$\cos x = \pm \frac{1}{2}.$$

$$\therefore x = 60^\circ, 120^\circ, 240^\circ, \text{ or } 300^\circ.$$

$$84. \tan(\theta + 45^\circ) = 8 \tan \theta.$$

By [26],

$$\tan(\theta + 45^\circ) = 8 \tan \theta.$$

$$\frac{\tan \theta + \tan 45^\circ}{1 - \tan \theta \tan 45^\circ} = 8 \tan \theta.$$

$$\frac{\tan \theta + 1}{1 - \tan \theta} = 8 \tan \theta.$$

$$\tan \theta + 1 = 8 \tan \theta - 8 \tan^2 \theta.$$

$$8 \tan^2 \theta - 7 \tan \theta = -1.$$

$$256 \tan^2 \theta - () + 49 = 17.$$

$$16 \tan \theta - 7 = \pm \sqrt{17}.$$

$$16 \tan \theta = 7 \pm \sqrt{17}.$$

$$\tan \theta = \frac{1}{16} (7 \pm 4.1231)$$

$$= 0.6952 \text{ or } 0.1798.$$

$$\therefore \theta = 34^\circ 48' \text{ or } 214^\circ 48'; \text{ or } 10^\circ 12' \text{ or } 190^\circ 12'.$$

$$\therefore \theta = 10^\circ 12', 34^\circ 48', 190^\circ 12', \text{ or } 214^\circ 48'.$$

$$85. \tan(\theta + 45^\circ) \tan \theta = 2.$$

$$\tan(\theta + 45^\circ) \tan \theta = 2.$$

By [26],

$$\frac{\tan \theta + \tan 45^\circ}{1 - \tan \theta \tan 45^\circ} \tan \theta = 2.$$

$$\frac{(\tan \theta + 1) \tan \theta}{1 - \tan \theta} = 2.$$

$$\frac{\tan^2 \theta + \tan \theta}{1 - \tan \theta} = 2.$$

$$\tan^2 \theta + \tan \theta = 2 - 2 \tan \theta.$$

$$\tan^2 \theta + 3 \tan \theta = 2.$$

$$4 \tan^2 \theta + () + 9 = 17.$$

$$2 \tan \theta + 3 = \pm \sqrt{17}.$$

$$2 \tan \theta = -3 \pm \sqrt{17}.$$

$$\tan \theta = \frac{1}{2} (-3 \pm 4.1231)$$

$$= 0.5616 \text{ or } -3.5616.$$

$$\therefore \theta = 29^\circ 19' \text{ or } 209^\circ 19'; \text{ or } 105^\circ 41' \text{ or } 285^\circ 41'.$$

$$\therefore \theta = 29^\circ 19', 105^\circ 41', 209^\circ 19', \text{ or } 285^\circ 41'.$$

$$86. \sin x + \sin 3x = \cos x - \cos 3x.$$

$$\sin x + \sin 3x = \cos x - \cos 3x.$$

By Exs. 15 and 16, Exercise 44,

$$\sin x + 3 \sin x - 4 \sin^3 x = \cos x - 4 \cos^3 x + 3 \cos x.$$

$$4 \sin^3 x - 4 \cos^3 x - 4 \sin x + 4 \cos x = 0.$$

$$(\sin^3 x - \cos^3 x) - (\sin x - \cos x) = 0.$$

$$(\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x) - (\sin x - \cos x) = 0.$$

$$(\sin x - \cos x)(1 + \sin x \cos x) - (\sin x - \cos x) = 0.$$

$$(\sin x - \cos x)(1 + \sin x \cos x - 1) = 0.$$

$$(\sin x - \cos x)(\sin x \cos x) = 0.$$

- (i) $\sin x - \cos x = 0.$
 $\sin x = \cos x.$
 $\therefore x = 45^\circ \text{ or } 225^\circ.$
- (ii) $\sin x = 0.$
 $\therefore x = 0^\circ \text{ or } 180^\circ.$
- (iii) $\cos x = 0.$
 $\therefore x = 90^\circ \text{ or } 270^\circ.$
 $\therefore x = 0^\circ, 45^\circ, 90^\circ, 180^\circ, 225^\circ, \text{ or } 270^\circ.$

$$87. \sin \frac{1}{2}x (\cos 2x - 2) (1 - \tan^2 x) = 0.$$

$$\sin \frac{1}{2}x (\cos 2x - 2) (1 - \tan^2 x) = 0.$$

- (i) $\sin \frac{1}{2}x = 0.$
 $\therefore \frac{1}{2}x = 0^\circ \text{ or } 180^\circ.$
 $\therefore x = 0^\circ.$
- (ii) $\cos 2x - 2 = 0.$
 $\cos 2x = 2.$
 $\therefore x \text{ is impossible.}$
- (iii) $1 - \tan^2 x = 0.$
 $\tan^2 x = 1.$
 $\tan x = \pm 1.$
 $\therefore x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$
 $\therefore x = 0^\circ, 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$

$$88. \tan x + \tan 2x = \tan 3x.$$

$$\tan x + \tan 2x = \tan 3x.$$

$$\text{By [31] and [26], } \tan x + \frac{2 \tan x}{1 - \tan^2 x} = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$$

$$\tan x \left(1 + \frac{2}{1 - \tan^2 x} - \frac{3 - \tan^2 x}{1 - 3 \tan^2 x} \right) = 0.$$

- (i) $\tan x = 0.$
 $\therefore x = 0^\circ \text{ or } 180^\circ.$

$$(ii) \quad 1 + \frac{2}{1 - \tan^2 x} - \frac{3 - \tan^2 x}{1 - 3 \tan^2 x} = 0.$$

$$1 - 4 \tan^2 x + 3 \tan^4 x + 2 - 6 \tan^2 x - 3 + 4 \tan^2 x - \tan^4 x = 0.$$

$$2 \tan^4 x - 6 \tan^2 x = 0.$$

$$\tan^2 x (\tan^2 x - 3) = 0.$$

$$\tan^2 x = 0 \text{ or } 3.$$

$$\text{If } \tan x = 0,$$

$$x = 0^\circ \text{ or } 180^\circ.$$

$$\text{If } \tan x = \pm \sqrt{3},$$

$$x = 60^\circ, 120^\circ, 240^\circ, \text{ or } 300^\circ.$$

$$\therefore x = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, \text{ or } 300^\circ.$$

$$89. \cot x - \tan x = \sin x + \cos x.$$

$$\cot x - \tan x = \sin x + \cos x.$$

$$\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \sin x + \cos x.$$

$$\frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \sin x + \cos x.$$

$$\cos^2 x - \sin^2 x = \sin x \cos x (\sin x + \cos x).$$

$$(\sin x + \cos x)(\cos x - \sin x - \sin x \cos x) = 0.$$

$$(i) \quad \sin x + \cos x = 0.$$

$$\sin x = -\cos x.$$

$$\therefore x = 135^\circ \text{ or } 315^\circ.$$

$$(ii) \quad \cos x - \sin x - \sin x \cos x = 0.$$

$$\cos x - \sin x = \sin x \cos x.$$

$$\text{Square, } \cos^2 x + \sin^2 x - 2 \sin x \cos x = \sin^2 x \cos^2 x.$$

$$1 - 2 \sin x \cos x = \sin^2 x \cos^2 x.$$

$$\sin^2 x \cos^2 x + 2 \sin x \cos x = 1.$$

$$\sin^2 x \cos^2 x + 2 \sin x \cos x + 1 = 2.$$

$$\text{Extract the root, } \sin x \cos x + 1 = \pm \sqrt{2}.$$

$$\sin x \cos x = -1 \pm \sqrt{2}.$$

$$2 \sin x \cos x = -2 \pm 2\sqrt{2}.$$

$$\text{By [30], } \sin 2x = -2 \pm 2\sqrt{2}.$$

$$\sin 2x = 0.8284 \text{ or } -4.8284.$$

$$2x = 55^\circ 56', 124^\circ 4', 415^\circ 56', \text{ or } 484^\circ 4'.$$

$$\therefore x = 27^\circ 58', 62^\circ 2', 207^\circ 58', \text{ or } 242^\circ 2'.$$

$$\therefore x = 27^\circ 58', 62^\circ 2', 135^\circ, 207^\circ 58', 242^\circ 2', \text{ or } 315^\circ.$$

But the values $62^\circ 2'$ and $207^\circ 58'$ do not satisfy the given equation.

$$\therefore x = 27^\circ 58', 135^\circ, 242^\circ 2', \text{ or } 315^\circ.$$

Prove the following identities :

$$90. (1 + \cot x + \tan x)(\sin x - \cos x) = \frac{\sec x}{\csc^2 x} - \frac{\csc x}{\sec^2 x}.$$

By [18] and [19],

$$\begin{aligned} (1 + \cot x + \tan x)(\sin x - \cos x) &= \left(1 + \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}\right)(\sin x - \cos x) \\ &= \sin x - \cos x + \cos x - \frac{\cos^2 x}{\sin x} + \frac{\sin^2 x}{\cos x} - \sin x \\ &= \frac{\sin^2 x}{\cos x} - \frac{\cos^2 x}{\sin x} \\ &= \frac{\sec x}{\csc^2 x} - \frac{\csc x}{\sec^2 x}. \end{aligned}$$

$$91. 2 \csc 2x \cot x = 1 + \cot^2 x.$$

By [14] and [19],

$$2 \csc 2x \cot x = \frac{2}{\sin 2x} \cdot \frac{\cos x}{\sin x};$$

by [30],

$$\begin{aligned} &= \frac{\cos x}{\sin^2 x \cos x} \\ &= \frac{1}{\sin^2 x} = \csc^2 x; \end{aligned}$$

by [21],

$$= 1 + \cot^2 x.$$

$$92. \sin a + \sin b + \sin(a+b) = 4 \cos \frac{1}{2} a \cos \frac{1}{2} b \sin \frac{1}{2} (a+b).$$

By [42] and [38],

$$\begin{aligned} &\sin a + \sin b + \sin(a+b) \\ &= 2 \sin \frac{1}{2} (a+b) \cos \frac{1}{2} (a-b) + 2 \sin \frac{1}{2} (a+b) \cos \frac{1}{2} (a+b) \\ &= 2 \sin \frac{1}{2} (a+b) [\cos \frac{1}{2} (a+b) + \cos \frac{1}{2} (a-b)]; \end{aligned}$$

$$\begin{aligned} \text{By [24] and [25], } &= 2 \sin \frac{1}{2} (a+b) [\cos \frac{1}{2} a \cos \frac{1}{2} b - \sin \frac{1}{2} a \sin \frac{1}{2} b \\ &\quad + \cos \frac{1}{2} a \cos \frac{1}{2} b + \sin \frac{1}{2} a \sin \frac{1}{2} b] \\ &= (2 \cos \frac{1}{2} a \cos \frac{1}{2} b) [2 \sin \frac{1}{2} (a+b)] \\ &= 4 \cos \frac{1}{2} a \cos \frac{1}{2} b \sin \frac{1}{2} (a+b). \end{aligned}$$

$$93. \tan(45^\circ + x) - \tan(45^\circ - x) = 2 \tan 2x.$$

By [26] and [27],

$$\begin{aligned} \tan(45^\circ + x) - \tan(45^\circ - x) &= \frac{\tan 45^\circ + \tan x}{1 - \tan 45^\circ \tan x} - \frac{\tan 45^\circ - \tan x}{1 + \tan 45^\circ \tan x} \\ &= \frac{1 + \tan x}{1 - \tan x} - \frac{1 - \tan x}{1 + \tan x} \\ &= \frac{(1 + \tan x)^2 - (1 - \tan x)^2}{1 - \tan^2 x} \\ &= \frac{4 \tan x}{1 - \tan^2 x}; \\ \text{by [31], } &= 2 \tan 2x. \end{aligned}$$

$$94. \cot^2 x - \cos^2 x = \cot^2 x \cos^2 x.$$

By [19],

$$\begin{aligned} \cot^2 x - \cos^2 x &= \frac{\cos^2 x}{\sin^2 x} - \cos^2 x \\ &= \frac{\cos^2 x - \sin^2 x \cos^2 x}{\sin^2 x} \\ &= \frac{(1 - \sin^2 x) \cos^2 x}{\sin^2 x} \\ &= \frac{\cos^4 x}{\sin^2 x} \\ &= \frac{\cos^2 x}{\sin^2 x} \cos^2 x \\ &= \cot^2 x \cos^2 x. \end{aligned}$$

$$95. \tan^2 x - \sin^2 x = \tan^2 x \sin^2 x.$$

$$\begin{aligned} \text{By [18], } \tan^2 x - \sin^2 x &= \frac{\sin^2 x}{\cos^2 x} - \sin^2 x \\ &= \frac{\sin^2 x - \sin^2 x \cos^2 x}{\cos^2 x} \\ &= \frac{(1 - \cos^2 x) \sin^2 x}{\cos^2 x} \\ &= \frac{\sin^2 x}{\cos^2 x} \sin^2 x \\ &= \tan^2 x \sin^2 x. \end{aligned}$$

$$96. \cot^4 x + \cot^2 x = \csc^4 x - \csc^2 x.$$

$$\begin{aligned} \text{By [19], } \cot^4 x + \cot^2 x &= \frac{\cos^4 x}{\sin^4 x} + \frac{\cos^2 x}{\sin^2 x} \\ &= \frac{\cos^4 x + \sin^2 x \cos^2 x}{\sin^4 x} \\ &= \frac{(1 - \sin^2 x)(1 - \sin^2 x) + \sin^2 x(1 - \sin^2 x)}{\sin^4 x} \\ &= \frac{1 - \sin^2 x}{\sin^4 x} \\ &= \frac{1}{\sin^4 x} - \frac{1}{\sin^2 x} \\ &= \csc^4 x - \csc^2 x. \end{aligned}$$

$$\begin{aligned} 97. \cos^2 x + \sin^2 x \cos^2 y &= \cos^2 y + \sin^2 y \cos^2 x \\ \cos^2 x + \sin^2 x \cos^2 y &= (1 - \sin^2 x) + \sin^2 x \cos^2 y \\ &= 1 - (1 - \cos^2 y) \sin^2 x \\ &= 1 - \sin^2 y (1 - \cos^2 x) \\ &= \cos^2 y - \sin^2 y \cos^2 x. \end{aligned}$$

Exercise 78. Simultaneous Equations (Page 170)

Solve the following systems for x and y :

$$1. \sin x + \sin y = \sin a. \quad (1)$$

$$\cos x + \cos y = 1 + \cos a. \quad (2)$$

Transform (1) by [42],

$$2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y) = \sin a. \quad (3)$$

Transform (2) by [44],

$$2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y) = 1 + \cos a. \quad (4)$$

$$\text{Divide (3) by (4),} \quad \frac{\sin \frac{1}{2}(x + y)}{\cos \frac{1}{2}(x + y)} = \frac{\sin a}{1 + \cos a}.$$

$$\text{By [18],} \quad \tan \frac{1}{2}(x + y) = \frac{\sin a}{1 + \cos a}.$$

Square,

$$\begin{aligned}\tan^2 \frac{1}{2}(x+y) &= \frac{\sin^2 a}{(1+\cos a)^2} = \frac{1-\cos^2 a}{(1+\cos a)^2} \\ &= \frac{1-\cos a}{1+\cos a} = \tan^2 \frac{1}{2}a.\end{aligned}$$

$$\tan \frac{1}{2}(x+y) = \tan \frac{1}{2}a.$$

$$x+y = a.$$

Substitute a for $(x+y)$ in (3),

$$2 \sin \frac{1}{2}a \cos \frac{1}{2}(x-y) = \sin a.$$

$$\cos \frac{1}{2}(x-y) = \frac{\sin a}{2 \sin \frac{1}{2}a};$$

$$\text{by [38],} \quad = \frac{2 \sin \frac{1}{2}a \cos \frac{1}{2}a}{2 \sin \frac{1}{2}a} = \cos \frac{1}{2}a.$$

$$x-y = \pm a.$$

$$x-y = \pm a.$$

$$x+y = a.$$

$$x = a \text{ or } 0,$$

$$y = 0 \text{ or } a.$$

$$2. \sin^2 x + \sin^2 y = a. \quad (1)$$

$$\cos^2 x - \cos^2 y = b. \quad (2)$$

Subtract (2) from (1),

$$\sin^2 x - \cos^2 x + \sin^2 y + \cos^2 y^2 = a - b. \quad (3)$$

Add (1) and (2),

$$\sin^2 x + \cos^2 x + \sin^2 y - \cos^2 y = a + b. \quad (4)$$

From (3) by [16], $\sin^2 x - \cos^2 x = a - b - 1.$

$$\sin^2 x - 1 + \sin^2 x = a - b - 1.$$

$$2 \sin^2 x = a - b.$$

$$\sin x = \pm \sqrt{\frac{a-b}{2}}.$$

$$x = \sin^{-1} \pm \sqrt{\frac{a-b}{2}}.$$

From (4) by [16], $\sin^2 y - \cos^2 y = a + b - 1.$

$$\sin^2 y - 1 + \sin^2 y = a + b - 1.$$

$$2 \sin^2 y = a + b.$$

$$\sin y = \pm \sqrt{\frac{a+b}{2}}.$$

$$y = \sin^{-1} \pm \sqrt{\frac{a+b}{2}}.$$

$$\therefore x = \sin^{-1} \pm \sqrt{\frac{a-b}{2}},$$

$$y = \sin^{-1} \pm \sqrt{\frac{a+b}{2}}.$$

$$3. \sin x - \sin y = 0.7038. \quad (1)$$

$$\cos x - \cos y = -0.7245. \quad (2)$$

Expand (1) by [43],

$$2 \cos \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y) = 0.7038. \quad (3)$$

Expand (2) by [45],

$$-2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y) = -0.7245. \quad (4)$$

$$\text{Divide (4) by (3),} \quad \frac{\sin \frac{1}{2}(x+y)}{\cos \frac{1}{2}(x+y)} = \frac{0.7245}{0.7038}.$$

$$\frac{\sin \frac{1}{2}(x+y)}{\sqrt{1 - \sin^2 \frac{1}{2}(x+y)}} = \frac{0.7245}{0.7038}.$$

$$\frac{\sin^2 \frac{1}{2}(x+y)}{1 - \sin^2 \frac{1}{2}(x+y)} = \frac{0.7245^2}{0.7038^2}.$$

$$0.7038^2 \sin^2 \frac{1}{2}(x+y) = 0.7245^2 - 0.7245^2 \sin^2 \frac{1}{2}(x+y).$$

$$(0.7038^2 + 0.7245^2) \sin^2 \frac{1}{2}(x+y) = 0.7245^2.$$

$$\sqrt{0.7038^2 + 0.7245^2} \sin \frac{1}{2}(x+y) = \pm 0.7245.$$

$$\sin \frac{1}{2}(x+y) = \frac{\pm 0.7245}{\sqrt{0.7038^2 + 0.7245^2}}.$$

$$\sin \frac{1}{2}(x+y) = \frac{\pm 0.7245}{1.0100667}.$$

$$\sin \frac{1}{2}(x+y) = \pm 0.7172. \quad (5)$$

Substitute in (4) the value of $\sin \frac{1}{2}(x+y)$,

$$2(\pm 0.7172) \sin \frac{1}{2}(x-y) = 0.7245.$$

$$\sin \frac{1}{2}(x-y) = \frac{0.7245}{\pm 1.4344}.$$

$$\sin \frac{1}{2}(x-y) = \pm 0.5051. \quad (6)$$

$$\text{From (5),} \quad \frac{1}{2}(x+y) = 45^\circ 50', 134^\circ 10', 225^\circ 50', \text{ or } 314^\circ 10'. \quad (7)$$

$$\text{From (6),} \quad \frac{1}{2}(x-y) = 30^\circ 20', 149^\circ 40', 210^\circ 20', \text{ or } 329^\circ 40'. \quad (8)$$

$$\text{Add (7) and (8),} \quad x = 76^\circ 10', 283^\circ 50', 436^\circ 10', \text{ or } 643^\circ 50'.$$

$$\text{Subtract (8) from (7), } y = 15^\circ 30', 344^\circ 30', 15^\circ 30', \text{ or } 344^\circ 30'.$$

$$\therefore x = 76^\circ 10' \text{ or } 283^\circ 50',$$

$$y = 15^\circ 30' \text{ or } 344^\circ 30'.$$

But the values $x = 283^\circ 50'$ and $y = 344^\circ 30'$ do not satisfy the given system of equations.

$$\therefore x = 76^\circ 10',$$

$$y = 15^\circ 30'.$$

$$4. x \sin 21^\circ + y \cos 44^\circ = 179.70. \quad (1)$$

$$x \cos 21^\circ + y \sin 44^\circ = 232.30. \quad (2)$$

$$0.3584x + 0.7193y = 179.70. \quad (3)$$

$$0.9336x + 0.6947y = 232.30. \quad (4)$$

$$3584x + 7193y = 1,797,000. \quad (5)$$

$$9336x + 6947y = 2,323,000. \quad (6)$$

Multiply (5) by 1167,

$$4,182,528x + 8,394,231y = 2,097,099,000. \quad (7)$$

Multiply (6) by 448,

$$4,182,528x + 3,112,256y = 1,040,704,000. \quad (8)$$

Subtract (8) from (7), $5,281,975y = 1,056,395,000.$

$$y = 200.$$

Substitute the value of y in (5),

$$3584x + 1,438,600 = 1,797,000.$$

$$3584x = 358,400.$$

$$x = 100.$$

$$\therefore x = 100,$$

$$y = 200.$$

$$5. \sin^2 x + y = m. \quad (1)$$

$$\cos^2 x + y = n. \quad (2)$$

Add (1) and (2), $\sin^2 x + \cos^2 x + 2y = m + n.$

By [16], $1 + 2y = m + n.$

$$y = \frac{m + n - 1}{2}.$$

Subtract (2) from (1), $\sin^2 x - \cos^2 x = m - n.$

By [16], $\sin^2 x - 1 + \sin^2 x = m - n.$

$$2 \sin^2 x = m - n + 1.$$

$$\sin x = \pm \sqrt{\frac{m - n + 1}{2}}.$$

$$x = \sin^{-1} \pm \sqrt{\frac{m - n + 1}{2}}.$$

$$\therefore x = \sin^{-1} \pm \sqrt{\frac{m - n + 1}{2}}.$$

$$y = \frac{m + n - 1}{2}.$$

$$6. \sin x + \sin y = 1. \quad (1)$$

$$\sin x - \sin y = 1. \quad (2)$$

Add (1) and (2),

$$2 \sin x = 2.$$

$$\sin x = 1.$$

$$x = 90^\circ.$$

Subtract (2) from (1),

$$2 \sin y = 0.$$

$$\sin y = 0.$$

$$y = 0^\circ \text{ or } 180^\circ.$$

$$\therefore x = 90^\circ.$$

$$y = 0^\circ \text{ or } 180^\circ$$

$$7. \cos x + \cos y = a. \quad (1)$$

$$\cos 2x + \cos 2y = b. \quad (2)$$

Transform (2) by [32],

$$\cos^2 x - \sin^2 x + \cos^2 y - \sin^2 y = b.$$

$$\cos^2 x - 1 + \cos^2 x + \cos^2 y - 1 + \cos^2 y = b.$$

$$2 \cos^2 x + 2 \cos^2 y = b + 2. \quad (3)$$

From (1), $\cos x = a - \cos y.$

$$\cos^2 x = a^2 - 2a \cos y + \cos^2 y.$$

Substitute in (3),

$$2a^2 - 4a \cos y + 2 \cos^2 y + 2 \cos^2 y = b + 2.$$

$$4 \cos^2 y - 4a \cos y = b - 2a^2 + 2.$$

Complete the square,

$$4 \cos^2 y - 4a \cos y + a^2 = b - a^2 + 2.$$

$$2 \cos y - a = \pm \sqrt{b - a^2 + 2}.$$

$$\cos y = \frac{1}{2} (a \pm \sqrt{b - a^2 + 2}).$$

From (1),

$$\cos y = a - \cos x.$$

Substitute in (3),

$$2 \cos^2 x + 2a^2 - 4a \cos x + 2 \cos^2 x = b + 2.$$

$$4 \cos^2 x - 4a \cos x = b - 2a^2 + 2.$$

Complete the square,

$$4 \cos^2 x - 4a \cos x + a^2 = b - a^2 + 2.$$

$$2 \cos x - a = \pm \sqrt{b - a^2 + 2}.$$

$$\cos x = \frac{1}{2} (a \pm \sqrt{b - a^2 + 2}).$$

$$\therefore x = \cos^{-1} \frac{1}{2} (a \pm \sqrt{b - a^2 + 2}).$$

$$y = \cos^{-1} \frac{1}{2} (a \pm \sqrt{b - a^2 + 2}).$$

$$8. \sin x + \sin y = 2m \sin a. \quad (1)$$

$$\cos x + \cos y = 2n \cos a. \quad (2)$$

Square (1) and (2) and add,

$$2 \cos x \cos y + 2 \sin x \sin y = 4m^2 \sin^2 a + 4n^2 \cos^2 a - 2.$$

By [25],

$$\cos(x - y) = 2m^2 \sin^2 a + 2n^2 \cos^2 a - 1$$

$$= 2m^2 - 2m^2 \cos^2 a + 2n^2 \cos^2 a - 1$$

$$= 2m^2 - (2m^2 - 2n^2) \cos^2 a - 1.$$

$$x - y = \cos^{-1} [2m^2 - (2m^2 - 2n^2) \cos^2 a - 1].$$

Transform (1) by [42],

$$2 \sin \frac{1}{2} (x + y) \cos \frac{1}{2} (x - y) = 2m \sin a. \quad (3)$$

Transform (2) by [44],

$$2 \cos \frac{1}{2} (x + y) \cos \frac{1}{2} (x - y) = 2n \cos a. \quad (4)$$

Divide (3) by (4), $\tan \frac{1}{2} (x + y) = \frac{m}{n} \tan a.$

$$x + y = 2 \tan^{-1} \frac{m}{n} \tan a.$$

$$x - y = \cos^{-1}[2m^2 - (2m^2 - 2n^2)\cos^2 a - 1].$$

$$x = \tan^{-1} \frac{m}{n} \tan a + \frac{1}{2} \cos^{-1}[2m^2 - (2m^2 - 2n^2)\cos^2 a - 1].$$

$$y = \tan^{-1} \frac{m}{n} \tan a - \frac{1}{2} \cos^{-1}[2m^2 - (2m^2 - 2n^2)\cos^2 a - 1].$$

9. Find two angles, x and y , knowing that the sum of their sines is a and the sum of their cosines is b .

$$\sin x + \sin y = a. \quad (1)$$

$$\cos x + \cos y = b. \quad (2)$$

Transform (1) by [42],

$$2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) = a. \quad (3)$$

Transform (1) by [44],

$$2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) = b. \quad (4)$$

Divide (3) by (4), $\tan \frac{1}{2}(x+y) = \frac{a}{b}. \quad (5)$

By [18], $\frac{\sin \frac{1}{2}(x+y)}{\cos \frac{1}{2}(x+y)} = \frac{a}{b}. \quad (6)$

$$\frac{\sin \frac{1}{2}(x+y)}{\sqrt{1 - \sin^2 \frac{1}{2}(x+y)}} = \frac{a}{b}.$$

Square, $\frac{\sin^2 \frac{1}{2}(x+y)}{1 - \sin^2 \frac{1}{2}(x+y)} = \frac{a^2}{b^2}.$

$$b^2 \sin^2 \frac{1}{2}(x+y) = a^2 - a^2 \sin^2 \frac{1}{2}(x+y).$$

$$(a^2 + b^2) \sin^2 \frac{1}{2}(x+y) = a^2.$$

$$\sin^2 \frac{1}{2}(x+y) = \frac{a^2}{a^2 + b^2}.$$

$$\sin \frac{1}{2}(x+y) = \frac{a}{\sqrt{a^2 + b^2}}. \quad (7)$$

Substitute in (3) the value of $\sin \frac{1}{2}(x+y)$,

$$\frac{2a}{\sqrt{a^2 + b^2}} \cos \frac{1}{2}(x-y) = a.$$

$$\cos \frac{1}{2}(x-y) = \frac{1}{2} \sqrt{a^2 + b^2}. \quad (8)$$

From (5), $x+y = 2 \tan^{-1} \frac{a}{b}. \quad (9)$

From (8), $x-y = 2 \cos^{-1} \frac{1}{2} \sqrt{a^2 + b^2}. \quad (10)$

Add (9) and (10) and divide by 2, $x = \tan^{-1} \frac{a}{b} + \cos^{-1} \frac{1}{2} \sqrt{a^2 + b^2}.$

Subtract (10) from (9) and divide by 2,

$$y = \tan^{-1} \frac{a}{b} - \cos^{-1} \frac{1}{2} \sqrt{a^2 + b^2}.$$

Solve the following systems for r and x :

$$10. \quad r \sin x = 92.344 \quad (1)$$

$$r \cos x = 205.309 \quad (2)$$

Divide (1) by (2), $\tan x = \frac{92.344}{205.309} = 0.4498.$

$$\therefore x = 24^\circ 13' \text{ or } 204^\circ 13'. \quad (3)$$

From (1), $r = \frac{92.344}{\sin x} = \frac{92.344}{\pm 0.4102} = \pm 225.12.$

$$\therefore x = 24^\circ 13', \quad r = 225.12;$$

or $x = 204^\circ 13', \quad r = -225.12.$

$$11. \quad r \sin (x - 19^\circ 18') = 59.4034 \quad (1)$$

$$r \cos (x - 30^\circ 54') = 147.9347 \quad (2)$$

Expand (1) by [23],

$$r \sin x \cos 19^\circ 18' - r \cos x \sin 19^\circ 18' = 59.4034. \quad (3)$$

Expand (2) by [25],

$$r \cos x \cos 30^\circ 54' + r \sin x \sin 30^\circ 54' = 147.9347. \quad (4)$$

Multiply (3) by $\sin 30^\circ 54'$,

$$r \sin x \sin 30^\circ 54' \cos 19^\circ 18' - r \cos x \sin 19^\circ 18' \sin 30^\circ 54' \\ = 59.4034 \sin 30^\circ 54'. \quad (5)$$

Multiply (4) by $\cos 19^\circ 18'$,

$$r \sin x \sin 30^\circ 54' \cos 19^\circ 18' + r \cos x \cos 19^\circ 18' \cos 30^\circ 54' \\ = 147.9347 \cos 19^\circ 18'. \quad (6)$$

Subtract (5) from (6),

$$r \cos x (\cos 19^\circ 18' \cos 30^\circ 54' + \sin 19^\circ 18' \sin 30^\circ 54') \\ = 147.9347 \cos 19^\circ 18' - 59.4034 \sin 30^\circ 54'.$$

By [25], $r \cos x \cos (30^\circ 54' - 19^\circ 18')$

$$= 147.9347 \cos 19^\circ 18' - 59.4034 \sin 30^\circ 54'.$$

$$r \cos x = \frac{147.9347 \cos 19^\circ 18' - 59.4034 \sin 30^\circ 54'}{\cos 10^\circ 36'}. \quad (7)$$

Multiply (3) by $\cos 30^\circ 54'$,

$$r \sin x \cos 30^\circ 54' \cos 19^\circ 18' - r \cos x \cos 30^\circ 54' \sin 19^\circ 18' \\ = 59.4034 \cos 30^\circ 54'. \quad (8)$$

Multiply (4) by $\sin 19^\circ 18'$,

$$r \sin x \sin 30^\circ 54' \sin 19^\circ 18' + r \cos x \cos 30^\circ 54' \sin 19^\circ 18' \\ = 147.9347 \sin 19^\circ 18'. \quad (9)$$

Add (8) and (9), $r \sin x (\cos 30^\circ 54' \cos 19^\circ 18' + \sin 30^\circ 54' \sin 19^\circ 18')$

$$= 147.9347 \sin 19^\circ 18' + 59.4034 \cos 30^\circ 54'.$$

By [25], $r \sin x \cos (30^\circ 54' - 19^\circ 18')$

$$= 147.9347 \sin 19^\circ 18' + 59.4034 \cos 30^\circ 54'.$$

$$\therefore r \sin x = \frac{147.9347 \sin 19^\circ 18' + 59.4034 \cos 30^\circ 54'}{\cos 10^\circ 36'}. \quad (10)$$

Divide (10) by (7),

$$\begin{aligned}\tan x &= \frac{147.9347 \sin 19^\circ 18' + 59.4034 \cos 30^\circ 54'}{147.9347 \cos 19^\circ 18' - 59.4034 \sin 30^\circ 54'} \\ &= \frac{147.9347 \times 0.3305 + 59.4034 \times 0.8581}{147.9347 \times 0.9438 - 59.4034 \times 0.5135} \\ &= \frac{48.89241835 + 50.97405754}{139.62076986 - 30.50364590} \\ &= \frac{99.86647589}{109.11712396} \\ &= 0.9152.\end{aligned}$$

$$\therefore x = 42^\circ 28' \text{ or } 222^\circ 28'. \quad (11)$$

Substitute in (1) the values of x found in (11),

$$r \sin (42^\circ 28' - 19^\circ 18') = 59.4034 \quad \text{or} \quad r \sin (222^\circ 28' - 19^\circ 18') = 59.4034$$

$$r \sin 23^\circ 10' = 59.4034 \quad \text{or} \quad r \sin 203^\circ 10' = 59.4034.$$

$$r = \frac{59.4034}{\sin 23^\circ 10'} \quad \text{or} \quad r = \frac{59.4034}{\sin 203^\circ 10'}.$$

$$r = \frac{59.4034}{0.3934} \quad \text{or} \quad r = \frac{59.4034}{-0.3934}.$$

$$r = 151 \quad \text{or} \quad r = -151.$$

$$\therefore x = 42^\circ 28', \quad r = 151;$$

$$\text{or} \quad x = 222^\circ 28', \quad r = -151.$$

Exercise 79. Simultaneous Equations (Page 171)

1. Solve for ϕ and x :

$$\text{versin } \phi = x$$

$$1 - \sin \phi = 0.5$$

$$\sin \phi = 1 - 0.5 = 0.5.$$

$$\phi = 30^\circ \text{ or } 150^\circ.$$

$$x = \text{versin } \phi = 1 - \cos \phi$$

$$= 1 - \cos 30^\circ$$

$$= 1 - 0.8660$$

$$= 0.1340;$$

$$\text{or} \quad x = 1 - \cos 150^\circ = 1 + 0.8660 = 1.866$$

$$\therefore x = 0.134 \text{ or } 1.866.$$

$$\phi = 30^\circ \text{ or } 150^\circ.$$

2. Solve for θ and x :

$$1 - \sin \theta = x \quad (1)$$

$$1 + \sin \theta = a \quad (2)$$

$$\text{Add (1) and (2),} \quad x = 2 - a.$$

$$\text{From (1),} \quad \sin \theta = 1 - x = 1 - 2 + a = a - 1.$$

$$\theta = \sin^{-1}(a - 1).$$

3. Solve for λ and μ :

$$\sin \lambda = \sqrt{2} \sin \mu$$

$$\tan \lambda = \sqrt{3} \tan \mu$$

By [18],
$$\frac{\sin \lambda}{\cos \lambda} = \sqrt{3} \frac{\sin \mu}{\cos \mu}.$$

$$\frac{\sqrt{2} \sin \mu}{\cos \lambda} = \frac{\sqrt{3} \sin \mu}{\cos \mu}.$$

$$\cos \lambda = \frac{\sqrt{2} \sin \mu \cos \mu}{\sqrt{3} \sin \mu} = \frac{\sqrt{2} \cos \mu}{\sqrt{3}}.$$

$$\cos^2 \lambda = \frac{2}{3} \cos^2 \mu.$$

$$\sin^2 \lambda = 2 \sin^2 \mu.$$

By [16], $2 \sin^2 \mu + \frac{2}{3} \cos^2 \mu = 1.$

$$6 \sin^2 \mu + 2(1 - \sin^2 \mu) = 3.$$

$$4 \sin^2 \mu = 1.$$

$$\sin \mu = \pm \frac{1}{2}.$$

$$\mu = \sin^{-1} \pm \frac{1}{2} = 30^\circ, 150^\circ, 210^\circ, \text{ or } 330^\circ.$$

$$\sin \lambda = \sqrt{2} \sin \mu = \pm \frac{1}{2} \sqrt{2}.$$

$$\lambda = \sin^{-1} \pm \frac{1}{2} \sqrt{2} = 45^\circ, 135^\circ, 225^\circ \text{ or } 315^\circ.$$

4. Solve for θ and ϕ :

$$\sin \theta + \cos \phi = a \quad (1)$$

$$\sin \phi + \cos \theta = b \quad (2)$$

Square (1) and (2) and add,

$$\sin^2 \theta + \cos^2 \theta + \sin^2 \phi + \cos^2 \phi + 2 \sin \theta \cos \phi + 2 \cos \theta \sin \phi = a^2 + b^2.$$

By [16] and [22], $2 \sin(\theta + \phi) = a^2 + b^2 - 2.$

$$\sin(\theta + \phi) = \frac{a^2 + b^2 - 2}{2}.$$

Subtract square of (2) from square of (1),

$$\sin^2 \theta - \cos^2 \theta - \sin^2 \phi + \cos^2 \phi + 2 \sin \theta \cos \phi - 2 \cos \theta \sin \phi = a^2 - b^2$$

By [32], $-\cos 2\theta + \cos 2\phi + 2 \sin(\theta - \phi) = a^2 - b^2.$

$$-(\cos 2\theta - \cos 2\phi) + 2 \sin(\theta - \phi) = a^2 - b^2.$$

By [45], $2 \sin(\theta + \phi) \sin(\theta - \phi) + 2 \sin(\theta - \phi) = a^2 - b^2.$

Substitute the value of $2 \sin(\theta + \phi)$,

$$(a^2 + b^2 - 2) [\sin(\theta - \phi)] + 2 \sin(\theta - \phi) = a^2 - b^2.$$

$$\sin(\theta - \phi) (a^2 + b^2 - 2 + 2) = a^2 - b^2.$$

$$\sin(\theta - \phi) = \frac{a^2 - b^2}{a^2 + b^2}.$$

$$\theta + \phi = \sin^{-1} \frac{a^2 + b^2 - 2}{2}.$$

$$\theta - \phi = \sin^{-1} \frac{a^2 - b^2}{a^2 + b^2}.$$

$$2\theta = \sin^{-1} \frac{a^2 + b^2 - 2}{2} + \sin^{-1} \frac{a^2 - b^2}{a^2 + b^2}.$$

$$\theta = \frac{1}{2} \sin^{-1} \left(\frac{a^2 + b^2}{2} - 1 \right) + \frac{1}{2} \sin^{-1} \frac{a^2 - b^2}{a^2 + b^2}.$$

$$2\phi = \sin^{-1} \frac{a^2 + b^2 - 2}{2} - \sin^{-1} \frac{a^2 - b^2}{a^2 + b^2}.$$

$$\phi = \frac{1}{2} \sin^{-1} \left(\frac{a^2 + b^2}{2} - 1 \right) - \frac{1}{2} \sin^{-1} \frac{a^2 - b^2}{a^2 + b^2}.$$

5. Solve for θ and ϕ :

$$a \sin^4 \theta - b \sin^4 \phi = a \quad (1)$$

$$a \cos^4 \theta - b \cos^4 \phi = b \quad (2)$$

Subtract (2) from (1),

$$a \sin^4 \theta - a \cos^4 \theta - b \sin^4 \phi + b \cos^4 \phi = a - b.$$

$$a (\sin^4 \theta - \cos^4 \theta) - b (\sin^4 \phi - \cos^4 \phi) = a - b.$$

$$a (\sin^2 \theta - \cos^2 \theta) (\sin^2 \theta + \cos^2 \theta) - b (\sin^2 \phi - \cos^2 \phi) (\sin^2 \phi + \cos^2 \phi) = a - b.$$

$$a (\sin^2 \theta - \cos^2 \theta) - b (\sin^2 \phi - \cos^2 \phi) = a - b.$$

$$a (1 - 2 \cos^2 \theta) - b (1 - 2 \cos^2 \phi) = a - b.$$

$$2a \cos^2 \theta = 2b \cos^2 \phi.$$

$$\cos^2 \theta = \frac{b \cos^2 \phi}{a}. \quad (3)$$

Substitute in (2),

$$\frac{b^2}{a} \cos^4 \phi - b \cos^4 \phi = b.$$

$$\frac{b^2 - ab}{a} \cos^4 \phi = b.$$

$$\cos^4 \phi = \frac{a}{b - a}.$$

$$\phi = \cos^{-1} \pm \sqrt[4]{\frac{a}{b - a}}.$$

From (3),

$$\cos^4 \theta = \frac{b^2}{a^2} \cos^4 \phi$$

$$= \frac{b^2}{a^2} \cdot \frac{a}{b - a}$$

$$= \frac{b^2}{a(b - a)}.$$

$$\theta = \cos^{-1} \pm \sqrt[4]{\frac{b^2}{a(b - a)}}.$$

6. Solve for θ :

$$\sin^2 \theta + 2 \cos \theta = 2$$

$$\cos \theta - \cos^2 \theta = 0$$

Subtract second equation from first,

$$\sin^2 \theta + \cos^2 \theta + 2 \cos \theta - \cos \theta = 2.$$

By [16],

$$1 + \cos \theta = 2.$$

$$\cos \theta = 1.$$

$$\therefore \theta = 0^\circ.$$

Exercise 80. Elimination (Page 172)

Find the eliminant with respect to $\alpha, \theta, \lambda, \mu$, or ϕ of the following equations:

$$1. \sin \phi + 1 = a \quad (1)$$

$$\cos \phi - 1 = b \quad (2)$$

Square (1) and (2) and add,

$$\sin^2 \phi + \cos^2 \phi = a^2 - 2a + 1 + b^2 + 2b + 1.$$

By [16],

$$1 = a^2 + b^2 - 2(a - b) + 2.$$

$$\therefore a^2 + b^2 - 2(a - b) = -1 \text{ is the eliminant.}$$

$$2. \tan \lambda - a = 0$$

$$\cot \lambda - b = 0$$

$$\tan \lambda = a.$$

$$\frac{1}{\tan \lambda} = b.$$

$$\tan \lambda = \frac{1}{b}.$$

$$a = \frac{1}{b}.$$

$$\therefore ab = 1 \text{ is the eliminant.}$$

$$3. \sin \alpha + m = n$$

$$\cos \alpha + p = q$$

$$\sin^2 \alpha = (n - m)^2.$$

$$\cos^2 \alpha = (q - p)^2.$$

$$\sin^2 \alpha + \cos^2 \alpha = (n - m)^2 + (q - p)^2.$$

$$\text{By [16], } (n - m)^2 + (q - p)^2 = 1.$$

$$\therefore (n - m)^2 + (q - p)^2 = 1 \text{ is the eliminant.}$$

$$4. a + \sec \phi = b$$

$$p + \cot \phi = q$$

$$\sec \phi = b - a. \quad (1)$$

$$\cot \phi = \frac{p}{q}.$$

$$\tan \phi = \frac{q}{p}. \quad (2)$$

Square (1) and (2) and subtract,

$$\sec^2 \phi - \tan^2 \phi = (b - a)^2 - \frac{q^2}{p^2}.$$

By [20],

$$(b - a)^2 = \frac{p^2 + q^2}{p^2}.$$

$$\therefore b - a = \frac{1}{p} \sqrt{p^2 + q^2} \text{ is the eliminant.}$$

$$5. \quad c \sin 2\phi + \cos 2\phi = 1 \quad (1)$$

$$b \sin 2\phi - \cos 2\phi = 1 \quad (2)$$

Add (1) and (2), $(b + c) \sin 2\phi = 2.$

$$\sin 2\phi = \frac{2}{b + c}.$$

By [16],

$$\cos^2 2\phi = 1 - \sin^2 2\phi.$$

$$\cos 2\phi = \sqrt{1 - \frac{4}{(b + c)^2}}.$$

Substitute these values in (1),

$$\frac{2c}{b + c} + \sqrt{1 - \frac{4}{(b + c)^2}} = 1.$$

$$\sqrt{1 - \frac{4}{(b + c)^2}} = 1 - \frac{2c}{b + c}.$$

$$1 - \frac{4}{(b + c)^2} = 1 - \frac{4c}{b + c} + \frac{4c^2}{(b + c)^2}.$$

$$\frac{(b + c)^2 - 4}{(b + c)^2} = \frac{(b + c)^2 - 4c(b + c) + 4c^2}{(b + c)^2}$$

$$(b + c)^2 - 4 = (b + c)^2 - 4bc - 4c^2 + 4c^2.$$

$$4bc = 4.$$

$\therefore bc = 1$ is the eliminant.

$$6. \quad x = r(\theta - \sin \theta) \quad (1)$$

$$y = r(1 - \cos \theta) \quad (2)$$

$$\theta = \text{versin}^{-1} \frac{y}{r}. \quad (3)$$

From (2),

$$1 - \cos \theta = \frac{y}{r}.$$

$$1 - \sqrt{1 - \sin^2 \theta} = \frac{y}{r}.$$

$$1 - \frac{y}{r} = \sqrt{1 - \sin^2 \theta}.$$

Square,

$$1 - \frac{2y}{r} + \frac{y^2}{r^2} = 1 - \sin^2 \theta.$$

$$\sin^2 \theta = \frac{2y}{r} - \frac{y^2}{r^2} = \frac{2ry - y^2}{r^2}.$$

$$\therefore \sin \theta = \pm \frac{1}{r} \sqrt{2ry - y^2}. \quad (4)$$

Substitute in (1) the value of θ given in (3) and the value of $\sin \theta$ found in (4),

$$x = r \text{versin}^{-1} \frac{y}{r} \mp \sqrt{2ry - y^2},$$

or

$$x = \pm \sqrt{2ry - y^2} + r \text{versin}^{-1} \frac{y}{r} \text{ is the eliminant.}$$

$$\begin{aligned} 7. \sin \phi + \sin 2\phi &= m & (1) \\ \cos \phi + \cos 2\phi &= n & (2) \end{aligned}$$

Square (1) and (2) and add,

$$\sin^2 \phi + 2 \sin \phi \sin 2\phi + \sin^2 2\phi = m^2.$$

$$\cos^2 \phi + 2 \cos \phi \cos 2\phi + \cos^2 2\phi = n^2.$$

$$1 + 2(\cos \phi \cos 2\phi + \sin \phi \sin 2\phi) + 1 = m^2 + n^2.$$

By [25], $\cos \phi \cos 2\phi + \sin \phi \sin 2\phi = \cos(2\phi - \phi) = \cos \phi$.

$$\therefore 1 + 2 \cos \phi + 1 = m^2 + n^2.$$

$$\cos \phi = \frac{m^2 + n^2 - 2}{2}.$$

By [32], $\cos 2\phi = 2 \cos^2 \phi - 1 = \frac{(m^2 + n^2 - 2)^2}{2} - 1.$

Substitute these values for $\cos \phi$ and $\cos 2\phi$ in (2),

$$\frac{m^2 + n^2 - 2}{2} + \frac{(m^2 + n^2 - 2)^2}{2} - 1 = n.$$

Clear of fractions, combine, and factor,

$$[m^2 + n^2 - 2][1 + m^2 + n^2 - 2] = 2n + 2.$$

$$(m^2 + n^2 - 1)^2 - m^2 - n^2 + 1 = 2n + 2.$$

$$\therefore (m^2 + n^2 - 1)^2 = (n + 1)^2 + m^2 \text{ is the eliminant.}$$

$$8. a + \sin \theta = \csc \theta$$

$$b + \cos \theta = \sec \theta$$

Putting $\frac{1}{\sin \theta}$ for $\csc \theta$, and $\frac{1}{\cos \theta}$ for $\sec \theta$, the equations become, on clearing of fractions,

$$a \sin \theta + \sin^2 \theta = 1. \quad (1)$$

$$b \cos \theta + \cos^2 \theta = 1. \quad (2)$$

Subtract (1) from (2),

$$b \cos \theta - a \sin \theta + \cos^2 \theta - \sin^2 \theta = 0. \quad (3)$$

From (1), $a \sin \theta = 1 - \sin^2 \theta$
 $= \cos^2 \theta.$

$$\therefore \sin \theta = \frac{\cos^2 \theta}{a}.$$

Substitute this value for $\sin \theta$ in (3),

$$b \cos \theta - \cos^2 \theta + \cos^2 \theta - \frac{\cos^4 \theta}{a^2} = 0.$$

$$b \cos \theta - \frac{\cos^4 \theta}{a^2} = 0.$$

$$\cos^3 \theta = a^2 b.$$

$$\cos \theta = (a^2 b)^{\frac{1}{3}}.$$

$$\cos^2 \theta = (a^2 b)^{\frac{2}{3}}.$$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$= 1 - (a^2 b)^{\frac{2}{3}}.$$

$$\sin \theta = (1 - (a^2 b)^{\frac{2}{3}})^{\frac{1}{2}}.$$

Substitute in (1) for $\sin \theta$ and $\sin^2 \theta$,

$$a(1 - (a^2 b)^{\frac{2}{3}})^{\frac{1}{2}} + 1 - (a^2 b)^{\frac{2}{3}} = 1.$$

$$a(1 - a^{\frac{4}{3}} b^{\frac{2}{3}})^{\frac{1}{2}} = a^{\frac{4}{3}} b^{\frac{2}{3}}.$$

$$(1 - a^{\frac{4}{3}} b^{\frac{2}{3}})^{\frac{1}{2}} = a^{\frac{1}{3}} b^{\frac{2}{3}}.$$

By squaring, we have $1 - a^{\frac{4}{3}} b^{\frac{2}{3}} = a^{\frac{2}{3}} b^{\frac{4}{3}}.$

$$\therefore a^{\frac{4}{3}} b^{\frac{2}{3}} + a^{\frac{2}{3}} b^{\frac{4}{3}} = 1 \text{ is the eliminant.}$$

$$9. \tan \alpha + \sin \alpha = m \quad (1)$$

$$\tan \alpha - \sin \alpha = n \quad (2)$$

Add (1) and (2),

$$2 \tan \alpha = m + n.$$

$$\tan \alpha = \frac{m + n}{2}.$$

Subtract (2) from (1), $2 \sin \alpha = m - n.$

$$\sin \alpha = \frac{m - n}{2}.$$

$$\cos \alpha = \sqrt{1 - \sin^2 \alpha}$$

$$= \sqrt{1 - \frac{(m - n)^2}{4}}.$$

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\frac{m - n}{2}}{\frac{\sqrt{4 - (m - n)^2}}{2}}$$

$$= \frac{m - n}{\sqrt{4 - (m - n)^2}}.$$

$$\frac{m + n}{2} = \frac{m - n}{\sqrt{4 - (m - n)^2}}.$$

$$\therefore (m + n) \sqrt{4 - (m - n)^2} = 2(m - n) \text{ is the eliminant.}$$

$$10. p \sin^2 \mu - p \cos^2 \mu = r \quad (1)$$

$$p' \cos^2 \mu - p' \sin^2 \mu = r' \quad (2)$$

$$-p(\cos^2 \mu - \sin^2 \mu) = r. \quad (3)$$

$$p'(\cos^2 \mu - \sin^2 \mu) = r'. \quad (4)$$

Divide (3) by (4),

$$\frac{-p}{p'} = \frac{r}{r'}.$$

$$\therefore p'r = -r'p \text{ is the eliminant.}$$

$$11. \sin 2\phi + \tan 2\phi = k \quad (1)$$

$$\sin 2\phi - \tan 2\phi = l \quad (2)$$

$$\text{Add (1) and (2),} \quad 2 \sin 2\phi = k + l. \quad (3)$$

$$\text{Subtract (1) and (2),} \quad 2 \tan 2\phi = k - l.$$

$$\text{From (3),} \quad 2 \cos 2\phi = \sqrt{4 - (k + l)^2}.$$

$$\frac{2 \sin 2\phi}{2 \cos 2\phi} = \tan 2\phi.$$

$$\text{Substituting,} \quad \frac{k + l}{\sqrt{4 - (k + l)^2}} = \frac{k - l}{2}; \quad \frac{(k + l)^2}{4 - (k + l)^2} = \frac{(k - l)^2}{4}.$$

$$\frac{4(k + l)^2}{(k - l)^2} = 4 - (k + l)^2.$$

$$\frac{(k + l)^2}{(k - l)^2} + \frac{(k + l)^2}{4} = 1.$$

$$\text{Simplifying,} \quad (k^2 - l^2)^2 + 16kl = 0.$$

$$12. p = a \cos \theta \cos \phi \quad (1)$$

$$q = b \cos \theta \sin \phi \quad (2)$$

$$r = c \sin \theta \quad (3)$$

$$\text{From (3),} \quad \sin \theta = \frac{r}{c}.$$

$$\cos \theta = \sqrt{\frac{c^2 - r^2}{c^2}}.$$

Substitute in (1) the value of $\cos \theta$,

$$p = \sqrt{\frac{a^2(c^2 - r^2)}{c^2}} \cos \phi.$$

$$\cos \phi = \sqrt{\frac{c^2 p^2}{a^2(c^2 - r^2)}}.$$

$$\sin \phi = \sqrt{\frac{a^2(c^2 - r^2) - c^2 p^2}{a^2(c^2 - r^2)}}.$$

Substitute in (2) the values of $\cos \theta$ and $\sin \phi$,

$$q = b \sqrt{\frac{c^2 - r^2}{c^2}} \cdot \sqrt{\frac{a^2(c^2 - r^2) - c^2 p^2}{a^2(c^2 - r^2)}}.$$

$$\begin{aligned} q^2 &= b^2 \left(\frac{c^2 - r^2}{c^2} \right) \left[\frac{a^2(c^2 - r^2) - c^2 p^2}{a^2(c^2 - r^2)} \right] \\ &= \frac{b^2(c^2 - r^2)[a^2(c^2 - r^2) - c^2 p^2]}{a^2 c^2 (c^2 - r^2)} \\ &= \frac{a^2 b^2 (c^2 - r^2)^2 - b^2 c^2 p^2 (c^2 - r^2)}{a^2 c^2 (c^2 - r^2)}. \end{aligned}$$

$$a^2 c^2 q^2 = a^2 b^2 (c^2 - r^2) - b^2 c^2 p^2.$$

$$\therefore a^2 b^2 r^2 + a^2 c^2 q^2 + b^2 c^2 p^2 = a^2 b^2 c^2 \text{ is the eliminant.}$$

Exercise 81. Roots of Unity (Page 176)

1. Find by De Moivre's Theorem the two square roots of 1.

If $\cos \phi + i \sin \phi = 1$, the given number,

then $\phi = 0, 2\pi, 4\pi, \dots$

Also $(\cos \phi + i \sin \phi)^{\frac{1}{2}} = 1^{\frac{1}{2}} =$ the two square roots of 1.

But $(\cos \phi + i \sin \phi)^{\frac{1}{2}} = \cos \frac{k(2\pi) + \phi}{2} + i \sin \frac{k(2\pi) + \phi}{2},$

where $k = 0$ or 1 , and $\phi = 0, 2\pi, 4\pi, \dots$

Therefore $1^{\frac{1}{2}} = \cos 0 + i \sin 0 = 1,$

$1^{\frac{1}{2}} = \cos \pi + i \sin \pi = -1.$

2. Find by De Moivre's Theorem the four 4th roots of 1.

If $\cos \phi + i \sin \phi = 1$, the given number,

then $\phi = 0, 2\pi, 4\pi, \dots$

Also $(\cos \phi + i \sin \phi)^{\frac{1}{4}} = 1^{\frac{1}{4}} =$ the four 4th roots of 1.

But $(\cos \phi + i \sin \phi)^{\frac{1}{4}} = \cos \frac{k(2\pi) + \phi}{4} + i \sin \frac{k(2\pi) + \phi}{4},$

where $k = 0, 1, 2$, or 3 , and $\phi = 0, 2\pi, 4\pi, \dots$

Therefore $1^{\frac{1}{4}} = \cos 0 + i \sin 0 = 1,$

$1^{\frac{1}{4}} = \cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} = i = \sqrt{-1},$

$1^{\frac{1}{4}} = \cos \frac{4\pi}{4} + i \sin \frac{4\pi}{4} = -1,$

$1^{\frac{1}{4}} = \cos \frac{6\pi}{4} + i \sin \frac{6\pi}{4} = -i = -\sqrt{-1}.$

3. Find three of the nine 9th roots of 1.

If $\cos \phi + i \sin \phi = 1$, the given number,

then $\phi = 0, 2\pi, 4\pi, \dots$

Also $(\cos \phi + i \sin \phi)^{\frac{1}{9}} = 1^{\frac{1}{9}} =$ the nine 9th roots of 1.

But $(\cos \phi + i \sin \phi)^{\frac{1}{9}} = \cos \frac{k(2\pi) + \phi}{9} + i \sin \frac{k(2\pi) + \phi}{9},$

where $k = 0, 1, 2, 3, 4, 5, 6, 7$, or 8 ,

and $\phi = 0, 2\pi, 4\pi, \dots$

Therefore

$$1^{\frac{1}{9}} = \cos 0 + i \sin 0 = 1,$$

$$1^{\frac{1}{9}} = \cos \frac{2\pi}{9} + i \sin \frac{2\pi}{9} = \cos 40^\circ + i \sin 40^\circ = 0.7660 + 0.6428 i$$

$$1^{\frac{1}{9}} = \cos \frac{4\pi}{9} + i \sin \frac{4\pi}{9} = \cos 80^\circ + i \sin 80^\circ = 0.1736 + 0.9848 i.$$

4. Find the five 5th roots of 1.

If $\cos \phi + i \sin \phi = 1$, the given number,
then $\phi = 0, 2\pi, 4\pi, \dots$

Also $(\cos \phi + i \sin \phi)^{\frac{1}{5}} = 1^{\frac{1}{5}} =$ the five 5th roots of 1.

But $(\cos \phi + i \sin \phi)^{\frac{1}{5}} = \cos \frac{k(2\pi) + \phi}{5} + i \sin \frac{k(2\pi) + \phi}{5},$

where $k = 0, 1, 2, 3, \text{ or } 4,$

and $\phi = 0, 2\pi, 4\pi, \dots$

Therefore

$$1^{\frac{1}{5}} = \cos 0^\circ + i \sin 0^\circ = 1,$$

$$\begin{aligned} 1^{\frac{1}{5}} &= \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = \cos 72^\circ + i \sin 72^\circ = \sin 18^\circ + i \cos 18^\circ \\ &= \frac{\sqrt{5}-1}{4} + i \frac{\sqrt{10+2\sqrt{5}}}{4}, \end{aligned}$$

$$\begin{aligned} 1^{\frac{1}{5}} &= \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5} = \cos 144^\circ + i \sin 144^\circ = -\cos 36^\circ + i \sin 36^\circ \\ &= -\frac{1+\sqrt{5}}{4} + i \frac{\sqrt{10-2\sqrt{5}}}{4}, \end{aligned}$$

$$\begin{aligned} 1^{\frac{1}{5}} &= \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5} = \cos 216^\circ + i \sin 216^\circ = -\cos 36^\circ - i \sin 36^\circ \\ &= -\frac{1+\sqrt{5}}{4} - i \frac{\sqrt{10-2\sqrt{5}}}{4}, \end{aligned}$$

$$\begin{aligned} 1^{\frac{1}{5}} &= \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5} = \cos 288^\circ - i \sin 288^\circ = \sin 18^\circ - i \cos 18^\circ \\ &= \frac{\sqrt{5}-1}{4} - i \frac{\sqrt{10+2\sqrt{5}}}{4}. \end{aligned}$$

5. Find the six 6th roots of +1 and of -1.

If $\cos \phi + i \sin \phi = 1$, the given number,
then $\phi = 0, 2\pi, 4\pi, \dots$

Also $(\cos \phi + i \sin \phi)^{\frac{1}{6}} = 1^{\frac{1}{6}} =$ the six 6th roots of 1.

But $(\cos \phi + i \sin \phi)^{\frac{1}{6}} = \cos \frac{k(2\pi) + \phi}{6} + i \sin \frac{k(2\pi) + \phi}{6},$

where $k = 0, 1, 2, 3, 4, \text{ or } 5,$

and $\phi = 0, 2\pi, 4\pi, \dots$

Therefore

$$1^{\frac{1}{6}} = \cos 0^\circ + i \sin 0^\circ = 1,$$

$$1^{\frac{1}{6}} = \cos \frac{2\pi}{6} + i \sin \frac{2\pi}{6} = \cos 60^\circ + i \sin 60^\circ = \frac{1}{2} + \frac{1}{2}\sqrt{-3},$$

$$1^{\frac{1}{6}} = \cos \frac{4\pi}{6} + i \sin \frac{4\pi}{6} = \cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + \frac{1}{2}\sqrt{-3},$$

$$1^{\frac{1}{6}} = \cos \frac{6\pi}{6} + i \sin \frac{6\pi}{6} = \cos 180^\circ + i \sin 180^\circ = -1,$$

$$1^{\frac{1}{6}} = \cos \frac{8\pi}{6} + i \sin \frac{8\pi}{6} = \cos 240^\circ + i \sin 240^\circ = -\frac{1}{2} - \frac{1}{2}\sqrt{-3},$$

$$1^{\frac{1}{6}} = \cos \frac{10\pi}{6} + i \sin \frac{10\pi}{6} = \cos 300^\circ + i \sin 300^\circ = \frac{1}{2} - \frac{1}{2}\sqrt{-3}.$$

If $\cos \phi + i \sin \phi = -1$, the given number,
then $\phi = \pi, 3\pi, 5\pi, \dots$

Also $(\cos \phi + i \sin \phi)^{\frac{1}{6}} = (-1)^{\frac{1}{6}} =$ the six 6th roots of -1 .

But $(\cos \phi + i \sin \phi)^{\frac{1}{6}} = \cos \frac{k(2\pi) + \phi}{6} + i \sin \frac{k(2\pi) + \phi}{6}$

where $k = 0, 1, 2, 3, 4$, or 5 , and $\phi = \pi, 3\pi, 5\pi, \dots$

Therefore $(\cos \phi + i \sin \phi)^{\frac{1}{6}} = \cos \frac{(2k+1)\pi}{6} + i \sin \frac{(2k+1)\pi}{6}$,

where $k = 0, 1, 2, 3, 4$, or 5 .

Therefore

$$(-1)^{\frac{1}{6}} = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = \cos 30^\circ + i \sin 30^\circ = \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{-1},$$

$$(-1)^{\frac{1}{6}} = \cos \frac{3\pi}{6} + i \sin \frac{3\pi}{6} = \cos 90^\circ + i \sin 90^\circ = \sqrt{-1},$$

$$(-1)^{\frac{1}{6}} = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = \cos 150^\circ + i \sin 150^\circ = -\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{-1},$$

$$(-1)^{\frac{1}{6}} = \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} = \cos 210^\circ + i \sin 210^\circ = -\frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{-1},$$

$$(-1)^{\frac{1}{6}} = \cos \frac{9\pi}{6} + i \sin \frac{9\pi}{6} = \cos 270^\circ + i \sin 270^\circ = -\sqrt{-1},$$

$$(-1)^{\frac{1}{6}} = \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} = \cos 330^\circ + i \sin 330^\circ = \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{-1}.$$

6. Find the four 4th roots of -1 .

If $\cos \phi + i \sin \phi = -1$, the given number,
then $\phi = \pi, 3\pi, 5\pi, \dots$

Also $(\cos \phi + i \sin \phi)^{\frac{1}{4}} = (-1)^{\frac{1}{4}} =$ the four 4th roots of -1 .

But $(\cos \phi + i \sin \phi)^{\frac{1}{4}} = \cos \frac{k(2\pi) + \phi}{4} + i \sin \frac{k(2\pi) + \phi}{4}$,

where $k = 0, 1, 2$, or 3 , and $\phi = \pi, 3\pi, 5\pi, \dots$

Therefore $(\cos \phi + i \sin \phi)^{\frac{1}{4}} = \cos \frac{(2k+1)\pi}{4} + i \sin \frac{(2k+1)\pi}{4}$.

Therefore

$$(-1)^{\frac{1}{4}} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \cos 45^\circ + i \sin 45^\circ = \frac{1}{2} \sqrt{2} + \frac{1}{2} \sqrt{-2},$$

$$(-1)^{\frac{1}{4}} = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} = \cos 135^\circ + i \sin 135^\circ = -\frac{1}{2} \sqrt{2} + \frac{1}{2} \sqrt{-2},$$

$$(-1)^{\frac{1}{4}} = \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} = \cos 225^\circ + i \sin 225^\circ = -\frac{1}{2} \sqrt{2} - \frac{1}{2} \sqrt{-2},$$

$$(-1)^{\frac{1}{4}} = \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} = \cos 315^\circ + i \sin 315^\circ = \frac{1}{2} \sqrt{2} - \frac{1}{2} \sqrt{-2}.$$

7. Show that the sum of the three cube roots of 1 is 0.

In § 157 the three cube roots of 1 are given:

$$1, -\frac{1}{2} + \frac{1}{2} \sqrt{-3}, -\frac{1}{2} - \frac{1}{2} \sqrt{-3}.$$

By addition, $1 - \frac{1}{2} + \frac{1}{2} \sqrt{-3} - \frac{1}{2} - \frac{1}{2} \sqrt{-3} = 0.$

8. Show that the sum of the five 5th roots of 1 is zero.

In Ex. 4 the five 5th roots of 1 were found to be

$$1, \frac{\sqrt{5}-1}{4} + i \frac{\sqrt{(10+2\sqrt{5})}}{4}, -\frac{1+\sqrt{5}}{4} + i \frac{\sqrt{(10-2\sqrt{5})}}{4},$$

$$-\frac{1+\sqrt{5}}{4} - i \frac{\sqrt{10-2(\sqrt{5})}}{4}, \frac{\sqrt{5}-1}{4} - i \frac{\sqrt{(10+2\sqrt{5})}}{4},$$

and the sum of these numbers equals zero.

9. From Exs. 7 and 8 infer the law as to the sum of the n th roots of 1 and prove this law.

The sum of the n n th roots of 1 is 0.

As shown in § 157, if we have the binomial equation

$$x^n - 1 = 0,$$

x = the n th root of 1.

But the second term, bx^{n-1} , does not appear in the equation $x^n - 1 = 0$, that is, $b = 0$. The absolute value of b is the sum of the roots of the equation, that is, the sum of the n n th roots of 1 is zero.

10. From Ex. 9 infer the law as to the sum of the n th roots of k and prove this law.

The sum of the n th roots of k is equal to zero.

The binomial equation

$$x^n - k = 0$$

has for its roots the n n th roots of k . The second term, bx^{n-1} , does not appear, hence $b = 0$. The absolute value of b is the sum of the roots of the equation, that is, the sum of the n n th roots of k is zero.

11. Show that any power of any one of the three cube roots of 1 is one of these three roots.

From § 154 the three cube roots of 1 are

$$\cos 2\pi + i \sin 2\pi = \cos 360^\circ + i \sin 360^\circ, \quad (1)$$

$$\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = \cos 120^\circ + i \sin 120^\circ, \quad (2)$$

$$\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = \cos 240^\circ + i \sin 240^\circ. \quad (3)$$

If one of these roots, say the second, be raised to the n th power, we have

$$\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)^n = \cos \frac{n \cdot 2\pi}{3} + i \sin \frac{n \cdot 2\pi}{3} = \cos n \cdot 120^\circ + i \sin n \cdot 120^\circ.$$

If $n = 2$ we get root (3), if $n = 3$ we get root (1), if $n = 4$ we get root (2), and as we increase the value of n , by making it 5, 6, 7 etc., the roots keep repeating.

12. Investigate the law implied in the statement of Ex. 11 for the four 4th roots and the five 5th roots of 1.

From Ex. 2 the four 4th roots of 1 are 1, i , -1 , and $-i$.

Any power of 1 = + 1.

Any even power of i = - 1 or + 1.

Any odd power of i = - i or + i .

Any even power of -1 = + 1.

Any odd power of -1 = - 1.

Any even power of $-i$ = - 1 or + 1.

Any odd power of $-i$ = + i or - i .

From Ex. 4 the five 5th roots of 1 are

$$\cos 0^\circ + i \sin 0^\circ = \cos 2\pi + i \sin 2\pi, \quad (1)$$

$$\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}, \quad (2)$$

$$\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}, \quad (3)$$

$$\cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}, \quad (4)$$

$$\cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}. \quad (5)$$

If we raise (2) to the n th power we have

$$\left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)^n = \cos \frac{n \cdot 2\pi}{5} + i \sin \frac{n \cdot 2\pi}{5}.$$

If

- $n = 2$ we get (3),
- $n = 3$ we get (4),
- $n = 4$ we get (5),
- $n = 5$ we get (1),
- $n = 6$ we get (2).

By giving n the values 7, 8, 9, ..., the values repeat.

Exercise 82. Roots of Numbers (Page 177)

1. Find the three cube roots of 125.

Let ω , ω^2 , ω^3 be the three cube roots of 1.

Then, by § 158, the three cube roots of 125 are

$$5\omega, 5\omega^2, 5\omega^3.$$

But

$$\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3},$$

$$\omega^2 = -\frac{1}{2} - \frac{1}{2}\sqrt{-3},$$

$$\omega^3 = 1.$$

$$\therefore 5\omega = -\frac{5}{2} + \frac{5}{2}\sqrt{-3},$$

$$5\omega^2 = -\frac{5}{2} - \frac{5}{2}\sqrt{-3},$$

$$5\omega^3 = 5.$$

2. Find the four 4th roots of -81 and verify the results.

$$-81 = 81(-1).$$

Then, by § 158, the four 4th roots of -1 may be represented by ω , ω^2 , ω^3 , ω^4 .

Then the four 4th roots of -81 will be 3ω , $3\omega^2$, $3\omega^3$, and $3\omega^4$.

From Ex. 6, Exercise 81, the four 4th roots of -1 are

$$\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{-2}, \quad -\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{-2}, \quad -\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{-2}, \quad \text{and} \quad \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{-2},$$

$$\therefore 3\omega = \frac{3}{2}\sqrt{2} + \frac{3}{2}\sqrt{-2}, \quad 3\omega^3 = -\frac{3}{2}\sqrt{2} - \frac{3}{2}\sqrt{-2},$$

$$3\omega^2 = -\frac{3}{2}\sqrt{2} + \frac{3}{2}\sqrt{-2}, \quad 3\omega^4 = \frac{3}{2}\sqrt{2} - \frac{3}{2}\sqrt{-2}.$$

Verification :

$$(3\omega)^4 = \left(\frac{3}{2}\right)^4 (\sqrt{2})^4 (1+i)^4 = \frac{81}{16} \cdot 4 \cdot (2i)^2 = \frac{81}{16} \cdot 4 \cdot (-4) = -81.$$

$$(3\omega^2)^4 = \left(\frac{3}{2}\right)^4 (\sqrt{2})^4 (-1+i)^4 = \frac{81}{16} \cdot 4 \cdot (-2i)^2 = \frac{81}{16} \cdot 4 \cdot (-4) = -81.$$

$$(3\omega^3)^4 = \left(\frac{3}{2}\right)^4 (\sqrt{2})^4 (-1-i)^4 = \frac{81}{16} \cdot 4 \cdot (2i)^2 = \frac{81}{16} \cdot 4 \cdot (-4) = -81.$$

$$(3\omega^4)^4 = \left(\frac{3}{2}\right)^4 (\sqrt{2})^4 (1-i)^4 = \frac{81}{16} \cdot 4 \cdot (-2i)^2 = \frac{81}{16} \cdot 4 \cdot (-4) = -81.$$

3. Find three of the 6th roots of 729 and verify the results.

Let ω , ω^2 , ω^3 denote three of the 6th roots of 1, then 3ω , $3\omega^2$, $3\omega^3$ will denote three of the 6th roots of 729.

From Ex. 5, Exercise 81, three of the 6th roots of 1 are

$$\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \quad -\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \quad -1.$$

Hence

$$3\omega = \frac{3}{2} + \frac{3}{2}\sqrt{-3},$$

$$3\omega^2 = -\frac{3}{2} + \frac{3}{2}\sqrt{-3},$$

$$3\omega^3 = -3.$$

Verification :

$$(3\omega)^6 = \left(\frac{3}{2}\right)^6 (1 + i\sqrt{3})^6 = \frac{729}{2^6} [1 + 3i\sqrt{3} + 3i^2(\sqrt{3})^2 + i^3(\sqrt{3})^3]^2 \\ = \frac{729}{2^6} (-8)^2 = 729.$$

$$(3\omega^2)^6 = \left(\frac{3}{2}\right)^6 (-1 + i\sqrt{3})^6 = \frac{729}{2^6} [1 - 3i\sqrt{3} + 3i^2(\sqrt{3})^2 - i^3(\sqrt{3})^3]^2 \\ = \frac{729}{2^6} (-8)^2 = 729.$$

$$(3\omega^3)^6 = (-3)^6 = 729.$$

4. Find three of the 10th roots of 1024 and verify the results.

Let $\omega, \omega^2, \omega^3$ denote three of the 10th roots of 1; then $2\omega, 2\omega^2, 2\omega^3$ will denote three of the 10th roots of 1024.

It will now be necessary to find ω, ω^2 , and ω^3 .

If $\cos \phi + i \sin \phi = 1$, the given number,
then $\phi = 0, 2\pi, 4\pi, \dots$

Also $(\cos \phi + i \sin \phi)^{\frac{1}{10}} = 1^{\frac{1}{10}} = 1$ the ten 10th roots of 1.

But $(\cos \phi + i \sin \phi)^{\frac{1}{10}} = \cos \frac{k(2\pi) + \phi}{10} + i \sin \frac{k(2\pi) + \phi}{10},$

where $k = 0, 1, 2, \dots$, or 9, and $\phi = 0, 2\pi, 4\pi, \dots$

Therefore $1^{\frac{1}{10}} = \cos \frac{2\pi}{10} + i \sin \frac{2\pi}{10} = \cos 36^\circ + i \sin 36^\circ.$

$$1^{\frac{1}{10}} = \cos \frac{4\pi}{10} + i \sin \frac{4\pi}{10} = \cos 72^\circ + i \sin 72^\circ.$$

$$1^{\frac{1}{10}} = \cos \frac{6\pi}{10} + i \sin \frac{6\pi}{10} = \cos 108^\circ + i \sin 108^\circ.$$

$$\therefore 2\omega = 2(\cos 36^\circ + i \sin 36^\circ),$$

$$2\omega^2 = 2(\cos 72^\circ + i \sin 72^\circ),$$

$$2\omega^3 = 2(\cos 108^\circ + i \sin 108^\circ).$$

Verification : $(2\omega)^{10} = [2(\cos 36^\circ + i \sin 36^\circ)]^{10} \\ = 1024(\cos 360^\circ + i \sin 360^\circ) \\ = 1024.$

$$(2\omega^2)^{10} = [2(\cos 72^\circ + i \sin 72^\circ)]^{10} \\ = 1024(\cos 720^\circ + i \sin 720^\circ) \\ = 1024.$$

$$(2\omega^3)^{10} = [2(\cos 108^\circ + i \sin 108^\circ)]^{10} \\ = 1024(\cos 1080^\circ + i \sin 1080^\circ) \\ = 1024$$

5. Find three of the 100th roots of 1.

If $\cos \phi + i \sin \phi = 1$, the given number,
then $\phi = 0, 2\pi, 4\pi, \dots$

Also $(\cos \phi + i \sin \phi)^{\frac{1}{100}} = 1^{\frac{1}{100}} =$ the hundred 100th roots of 1.

But $(\cos \phi + i \sin \phi)^{\frac{1}{100}} = \cos \frac{k(2\pi) + \phi}{100} + i \sin \frac{k(2\pi) + \phi}{100}$,

where $k = 0, 1, 2, \dots$, or 99, and $\phi = 0, 2\pi, 4\pi, \dots$.

Therefore $1^{\frac{1}{100}} = \cos \frac{2\pi}{100} + i \sin \frac{2\pi}{100}$
 $= \cos 3.6^\circ + i \sin 3.6^\circ$
 $= \cos 3^\circ 36' + i \sin 3^\circ 36'$
 $= 0.9980 + 0.0628i$,

$1^{\frac{1}{100}} = \cos \frac{4\pi}{100} + i \sin \frac{4\pi}{100}$
 $= \cos 7.2^\circ + i \sin 7.2^\circ$
 $= \cos 7^\circ 12' + i \sin 7^\circ 12'$
 $= 0.9921 + 0.1253i$,

$1^{\frac{1}{100}} = \cos \frac{6\pi}{100} + i \sin \frac{6\pi}{100}$
 $= \cos 10.8^\circ + i \sin 10.8^\circ$
 $= \cos 10^\circ 48' + i \sin 10^\circ 48'$
 $= 0.9823 + 0.1874i$.

6. Show that, if 2ω is one of the complex 7th roots of 128, two of the other roots are $2\omega^2$ and $2\omega^3$.

Let $\omega, \omega^2, \omega^3$ denote three of the seven 7th roots of 1. Then $2\omega, 2\omega^2, 2\omega^3$ will denote three of the seven 7th roots of 128.

Then $(2\omega^2)^7 = 2^7 \omega^{14} = 128$,
and $(2\omega^3)^7 = 2^7 \omega^{21} = 128$.

7. Show that either of the two complex cube roots of 1 is at the same time the square and the square root of the other.

From § 157, we have given the two complex cube roots of 1,

$$\cos 120^\circ + i \sin 120^\circ, \quad (1)$$

$$\cos 240^\circ + i \sin 240^\circ. \quad (2)$$

$$(\cos 120^\circ + i \sin 120^\circ)^2 = \cos 240^\circ + i \sin 240^\circ.$$

$\therefore$ (2) is the square of (1).

$$(\cos 120^\circ + i \sin 120^\circ)^{\frac{1}{2}} = \pm (\cos 60^\circ + i \sin 60^\circ).$$

But $\cos 240^\circ + i \sin 240^\circ = -\cos 60^\circ - i \sin 60^\circ$.

$\therefore$ (2) is the square root of (1).

In like manner,

$$\begin{aligned} (\cos 240^\circ + i \sin 240^\circ)^2 &= \cos 480^\circ + i \sin 480^\circ \\ &= \cos 120^\circ + i \sin 120^\circ. \end{aligned}$$

$\therefore$ (1) is the square of (2).

$$(\cos 240^\circ + i \sin 240^\circ)^{\frac{1}{2}} = \pm (\cos 120^\circ + i \sin 120^\circ).$$

$\therefore$ (1) is the square root of (2).

8. Show that a result similar to the one stated in Ex. 7 can be found with respect to the four 4th roots of 1.

In Ex. 2, Exercise 81, the two complex 4th roots of 1 are given:

$$\cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} = i, \quad (1)$$

$$\cos \frac{6\pi}{4} + i \sin \frac{6\pi}{4} = -i. \quad (2)$$

$$(i)^3 = -i.$$

$$(-i)^3 = i.$$

$$\therefore (-i)^{\frac{1}{3}} = i.$$

$$(-i)^3 = i.$$

9. Show that the sum of all the n th roots of 1 is zero.

$x^n = 1$, or $x^n - 1 = 0$, contains the n n th roots of 1. In this equation the term bx^{n-1} is wanting, hence $b = 0$, and b in absolute value equals the sum of the n n th roots of 1.

10. Show that the sum of the products of all the n th roots of 1, taken two by two, is zero.

The equation $x^n - 1 = 0$ gives the n n th roots of 1. In this equation the term $b_1 x^{n-2}$ is absent, hence $b_1 = 0$, and b_1 is the sum of the product of all the n th roots of 1, taken two by two.

Exercise 83. Properties of Logarithms (Page 183)

Prove the following properties of logarithms as given in § 159, using b as the base:

1. Properties 1 and 2.

The logarithm of 1 is 0.

Since $b^0 = 1$, from the definition of a logarithm, $\log_b 1 = 0$.

The logarithm of the base itself is 1.

$$b^1 = b.$$

$$\therefore \log_b b = 1.$$

2. Property 3.

The logarithm of the reciprocal of a positive number is the negative of the logarithm of the number.

Let N be any positive number and n its logarithm.

$$b^n = N.$$

$$\frac{1}{N} = \frac{1}{b^n} = b^{-n}.$$

$$\therefore \log_b \frac{1}{N} = -n = -\log_b N.$$

3. Property 4.

The logarithm of the product of two or more positive numbers is found by adding the logarithms of the several factors.

Let A and B be the numbers and x and y their logarithms.

Let

$$x = \log_b A,$$

$$y = \log_b B.$$

Then

$$A = b^x,$$

$$B = b^y.$$

$$AB = b^{x+y}.$$

$$\begin{aligned}\therefore \log_b AB &= x + y \\ &= \log_b A + \log_b B.\end{aligned}$$

4. Property 5.

The logarithm of the quotient of two positive numbers is found by subtracting the logarithm of the divisor from the logarithm of the dividend.

Let

$$x = \log_b A,$$

$$y = \log_b B.$$

Then

$$A = b^x,$$

$$B = b^y.$$

$$\begin{aligned}\therefore \log_b \frac{A}{B} &= x - y \\ &= \log_b A - \log_b B.\end{aligned}$$

5. Property 6.

The logarithm of a power of a positive number is found by multiplying the logarithm of the number by the exponent of the power.

Let A be the number and x the logarithm. Find A^p .

Let

$$x = \log_b A,$$

then

$$A = b^x.$$

Raise both sides to the p th power,

$$A^p = b^{px}.$$

$$\begin{aligned}\therefore \log_b A^p &= px \\ &= p \log_b A.\end{aligned}$$

6. Property 7.

The logarithm of the real positive value of a root of a positive number is found by dividing the logarithm of the number by the index of the root.

$$\begin{array}{ll} \text{Let} & x = \log_b A, \\ \text{then} & A = b^x. \end{array}$$

$$\begin{array}{ll} \text{Take the } r\text{th root,} & A^{\frac{1}{r}} = b^{\frac{x}{r}}. \\ & \therefore \log_b A^{\frac{1}{r}} = \frac{x}{r} \\ & = \frac{\log_b A}{r}. \end{array}$$

Find the value of each of the following :

7. $5!$

$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120.$$

8. $7!$

$$7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040.$$

9. $6!$

$$6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720.$$

10. $8!$

$$8! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40,320.$$

11. $10!$

$$10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = 3,628,800$$

Simplify the following :

12. $\frac{10!}{3!}$

$$\frac{10!}{3!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{1 \times 2 \times 3} = 604,800.$$

13. $\frac{10!}{8!}$

$$\frac{10!}{8!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} = 90.$$

14. $\frac{7!}{5!}$

$$\frac{7!}{5!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}{1 \times 2 \times 3 \times 4 \times 5} = 42.$$

15. $\frac{15!}{14!}$

$$\frac{15!}{14!} = \frac{14! \times 15}{14!} = 15.$$

16. $\frac{20!}{17!}$

$$\frac{20!}{17!} = \frac{17! \times 18 \times 19 \times 20}{17!} = 6840.$$

17. Find to five decimal places the value of $\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots\right)^2$.

	1.00000	
2	1.00000	
3	0.50000	
4	0.16666	
5	0.04166	
6	0.00833	
7	0.00138	
8	0.00019	
	0.00002	
	2.71824	$(2.71824)^2 = 7.38883.$

18. Find to five decimal places the value of $\left(2 + \frac{1}{2!} + \frac{1}{3!} + \dots\right)^{\frac{1}{2}}$.

From Prob. 17, $\left(2 + \frac{1}{2!} + \frac{1}{3!} + \dots\right) = 2.71824.$

$$(2.71824)^{\frac{1}{2}} = 1.64871.$$

By the use of the series for $\cos x$ find the following:

19. $\cos \frac{1}{2}.$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

$$\cos \frac{1}{2} = 1 - 0.125 + 0.0026 - 0.00002 = 0.8776 = \cos 28^\circ 39'$$

20. $\cos \frac{1}{8}.$

$$\cos \frac{1}{8} = 1 - 0.0078 + 0.00001 = 0.9922 = \cos 7^\circ 10'.$$

21. $\cos 2.$

$$\cos 2 = 1 - 2 + 0.66666 - 0.08888 + 0.00634 - 0.00020$$

$$= -0.41616$$

$$= \cos(180^\circ - x) = \cos[180^\circ - (65^\circ 24' 28'')]$$

$$= \cos 114^\circ 35' 32''.$$

22. $\cos 0.$

$$\cos 0 = 1 - \frac{0}{2} + \frac{0}{4!} \dots = 1 = \cos 0^\circ.$$

By the use of the series for $\sin x$ find the following:

23. $\sin 1.$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \dots$$

$$= 1 - 0.16666 + 0.00833 - 0.00019$$

$$= 0.84148$$

$$= \sin 57^\circ 17' 48''.$$

24. $\sin \frac{1}{2}$.

$$\begin{aligned}
 \sin \frac{1}{2} &= \frac{1}{2} - \frac{(\frac{1}{2})^3}{3!} + \frac{(\frac{1}{2})^5}{5!} \dots \\
 &= 0.50000 - 0.02083 + 0.00026 \\
 &= 0.4794 \\
 &= \sin 28^\circ 38' 40''.
 \end{aligned}$$

25. $\sin 2$.

$$\begin{aligned}
 \sin 2 &= 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \frac{2^9}{9!} \dots \\
 &= 2 - 1.33333 + 0.26666 - 0.02539 + 0.00141 \\
 &= 0.90935 \\
 &= \sin 65^\circ 24' 45'' \text{ or } \sin 114^\circ 35' 15''.
 \end{aligned}$$

26. $\sin 0$.

$$\sin 0 = 0 - \frac{0}{3!} + \frac{0}{5!} - \frac{0}{7!} \dots = 0 = \sin 0^\circ \text{ or } \sin 180^\circ.$$

*By the use of the series for $\tan x$ find the following:***27.** $\tan 0$.

$$\begin{aligned}
 \tan x &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} \dots \\
 \tan 0 &= 0 + \frac{0}{3} + \frac{0}{15} + \frac{0}{315} \dots = 0 \\
 &= \tan 0^\circ.
 \end{aligned}$$

28. $\tan 1$.

$$\begin{aligned}
 \tan 1 &= 1 + \frac{1}{3} + \frac{1^2}{15} + \frac{1^3}{315} + \frac{1^4}{2835} \dots \\
 &= 1 + 0.33333 + 0.13333 + 0.05396 + 0.02186 \\
 &= 1.5206 \\
 &= \tan 56^\circ 40' 12''.
 \end{aligned}$$

29. $\tan \frac{1}{2}$.

$$\begin{aligned}
 \tan \frac{1}{2} &= \frac{1}{2} + \frac{(\frac{1}{2})^3}{3} + \frac{2(\frac{1}{2})^5}{15} + \frac{17(\frac{1}{2})^7}{315} \dots \\
 &= 0.5 + 0.04166 + 0.00416 + 0.00042 \\
 &= 0.54614 \\
 &= \tan 28^\circ 38' 20''.
 \end{aligned}$$

30. $\tan 2$.

$$\begin{aligned}
 \tan 2 &= 2 + \frac{2^3}{3} + \frac{2 \times 2^5}{15} + \frac{17 \times 2^7}{315} \\
 &= 2 + 2.6666 + 4.2666 + 6.9079 \\
 &= 15.8411 \\
 &= \tan 86^\circ 23' 16''.
 \end{aligned}$$

Prove the following statements :

31. $e^{2\pi i} = 1$.

From § 163,

$$e^{ix} = \cos x + i \sin x.$$

Let

$$x = 2\pi.$$

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = \cos 360^\circ + i \sin 360^\circ = 1.$$

32. $e^{-\frac{\pi}{2}} = i$.

From § 164,

$$e^{i\pi} = -1.$$

Then $\frac{e^{i\pi}}{e^2} = \sqrt{-1} = i$.

$$\frac{e^{i\pi}}{e^2} = \frac{e^{2\pi i}}{e^{2i}} = e^{-\frac{\pi}{2i}}.$$

$$\therefore e^{-\frac{\pi}{2i}} = i.$$

$$\therefore e^{-\frac{\pi}{2}} = i^i.$$

33. $e^\pi = \sqrt[4]{-1}$.

From § 164,

$$e^{i\pi} = -1.$$

Take the i th root of this equation, $e^\pi = \sqrt[4]{-1}$.

34. $e^i = \sqrt[4]{-1}$.

From § 164,

$$e^{i\pi} = -1.$$

Take the π th root of this equation, $e^i = \sqrt[4]{-1}$.

Given $\log_e 2 = 0.6931$, find two logarithms (to the base e) of :

35. 2.

$$\log N = \log_e N + 2k\pi i.$$

$$\log 2 = 0.6931 + 2k\pi i = 0.6931 + 2\pi i \text{ or } 0.6931 + 4\pi i.$$

36. 4.

$$\log N = \log_e N + 2k\pi i.$$

$$\log 4 = \log 2^2 = 2 \log 2$$

$$= 2(0.6931) + 2k\pi i$$

$$= 1.3862 + 2\pi i$$

$$\text{or } 1.3862 + 4\pi i.$$

37. $\sqrt{2}$.

$$\log N = \log_e N + 2k\pi i.$$

$$\log \sqrt{2} = \frac{1}{2} \log 2$$

$$= \frac{1}{2}(0.6931) + 2k\pi i$$

$$= 0.3465 + 2\pi i$$

$$\text{or } 0.3465 + 4\pi i.$$

38. -2.

$$-2 = 2[\cos(k2\pi + \pi) + i \sin(k2\pi + \pi)].$$

$$\therefore e^{\log(-2)} = e^{\log_e 2} \cdot e^{(2k+1)\pi i}$$

$$= e^{\log_e 2 + (2k+1)\pi i}.$$

$$\therefore \log(-2) = \log_e 2 + (2k+1)\pi i.$$

$$\log(-2) = 0.6931 + (2k+1)\pi i.$$

Give k the values 0 and 1.

Then

$$\log(-2) = 0.6931 + \pi i$$

$$\text{or } 0.6931 + 3\pi i.$$

Given $\log_e 5 = 1.609$, find three logarithms (to the base e) of :

39. 5.

$$\log N = \log_e N + 2k\pi i.$$

$$\log 5 = \log_e 5 + 2k\pi i = 1.609 + 2\pi i, \text{ or } 1.609 + 4\pi i, \text{ or } 1.609 + 6\pi i.$$

40. 25.

$$\log N = \log_e N + 2k\pi i.$$

$$\log 25 = \log 5^2 = 2 \log 5 = 2(1.609) + 2k\pi i = 3.218 + 2\pi i,$$

$$\text{or } 3.218 + 4\pi i, \text{ or } 3.218 + 6\pi i.$$

41. 125.

$$\begin{aligned}
 \log N &= \log_e N + 2k\pi i. \\
 \log 125 &= \log 5^3 \\
 &= 3 \log 5 \\
 &= 3(1.609) + 2k\pi i \\
 &= 4.827 + 2\pi i, \\
 \text{or } 4.827 + 4\pi i, \\
 &\quad \text{or } 4.827 + 6\pi i.
 \end{aligned}$$

42. -5.

$$\begin{aligned}
 -5 &= 5[\cos(k2\pi + \pi) \\
 &\quad + i \sin(k2\pi + \pi)] \\
 \therefore e^{\log -5} &= e^{\log_e 5} \cdot e^{(2k+1)\pi i} \\
 &= e^{\log_e 5 + (2k+1)\pi i} \\
 \therefore \log(-5) &= \log_e 5 + (2k+1)\pi i \\
 &= 1.609 + \pi i, \\
 &\quad \text{or } 1.609 + 3\pi i, \\
 &\quad \text{or } 1.609 + 5\pi i.
 \end{aligned}$$

Given $\log_e 10 = 2.302585$, find two logarithms (to the base e) of:

43. 100.

$$\begin{aligned}
 \log N &= \log_e N + 2k\pi i. \\
 \log 100 &= \log 10^2 = 2 \log 10 \\
 &= 2(2.302585) + 2k\pi i \\
 &= 4.605170 + 2\pi i \text{ or } 4.605170 + 4\pi i.
 \end{aligned}$$

44. -10.

$$\begin{aligned}
 -10 &= 10[\cos(k2\pi + \pi) + i \sin(k2\pi + \pi)]. \\
 \therefore e^{\log(-10)} &= e^{\log_e 10} \cdot e^{(2k+1)\pi i} \\
 &= e^{\log_e 10 + (2k+1)\pi i} \\
 \therefore \log(-10) &= \log_e 10 + (2k+1)\pi i \\
 &= 2.302585 + \pi i \text{ or } 2.302585 + 3\pi i.
 \end{aligned}$$

45. 1000.

$$\begin{aligned}
 \log N &= \log_e N + 2k\pi i. \\
 \log 1000 &= \log 10^3 = 3 \log 10 \\
 &= 3(2.302585) + 2k\pi i \\
 &= 6.907755 + 2\pi i \text{ or } 6.907755 + 4\pi i.
 \end{aligned}$$

46. $\sqrt{10}$.

$$\begin{aligned}
 \log N &= \log_e N + 2k\pi i. \\
 \log \sqrt{10} &= \log 10^{\frac{1}{2}} = \frac{1}{2} \log 10 \\
 &= \frac{1}{2}(2.302585) + 2k\pi i \\
 &= 1.151292 + 2\pi i \text{ or } 1.151292 + 4\pi i.
 \end{aligned}$$

47. From the series of § 162 show that

$$\begin{aligned}
 \sin(-\phi) &= -\sin \phi. \\
 \sin \phi &= \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} \dots \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \sin(-\phi) &= -\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} \dots \\
 &= -\phi + \frac{\phi^3}{3!} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} \dots \quad (2)
 \end{aligned}$$

Add (1) and (2), $\sin(-\phi) + \sin \phi = 0$.

$$\sin(-\phi) = -\sin \phi.$$

48. Prove that the ratio of the circumference of a circle to the diameter equals $-2 \log(i) = -2i \log i$.

$$\text{From § 164,} \quad e^{\frac{i\pi}{2}} = i.$$

$$\text{Hence} \quad e^{\frac{i^2\pi}{2i}} = i.$$

$$e^{-\frac{\pi}{2i}} = i.$$

$$e^{-\frac{\pi}{2}} = i^i.$$

$$\text{Take logarithm,} \quad -\frac{\pi}{2} = \log(i^i).$$

$$\begin{aligned} \therefore \pi &= -2 \log(i^i) \\ &= -2i \log i. \end{aligned}$$

Exercise 84. Review Problems (Page 184)

1. The angle of elevation of the top of a vertical cliff at a point 575 ft. from the foot is $32^\circ 15'$. Find the height of the cliff.

$$\frac{a}{b} = \tan 32^\circ 15'. \quad a = 575 \tan 32^\circ 15'.$$

$$\log 575 = 2.75967$$

$$\log \tan 32^\circ 15' = 9.80000$$

$$\log a = 2.55967$$

$$a = 362.8.$$

The height of the cliff is 362.8 ft.

2. An aeroplane is above a straight road on which are two observers 1640 ft. apart. At a given signal the observers take the angles of elevation of the aeroplane, finding them to be 58° and 63° respectively. Find the height of the aeroplane and its distance from each observer.

$$A = 63^\circ, \quad B = 58^\circ, \quad c = 1640. \quad C = 180^\circ - (63^\circ + 58^\circ) = 59^\circ.$$

$$\frac{a}{c} = \frac{\sin A}{\sin C}, \quad a = \frac{1640 \sin 63^\circ}{\sin 59^\circ}.$$

$$\log 1640 = 3.21484$$

$$\log \sin 63^\circ = 9.94988$$

$$\log \sin 59^\circ = 0.06693$$

$$\log a = 3.23165$$

$$a = 1704.7.$$

$$\frac{b}{c} = \frac{\sin B}{\sin C}, \quad b = \frac{1640 \sin 58^\circ}{\sin 59^\circ}.$$

$$\log 1640 = 3.21484$$

$$\log \sin 58^\circ = 9.92842$$

$$\text{colog } \sin 59^\circ = 0.06693$$

$$\log b = 3.21019$$

$$b = 1622.5.$$

$$\frac{h}{a} = \sin 58^\circ.$$

$$h = 1704.7 \sin 58^\circ.$$

$$\log 1704.7 = 3.23165$$

$$\log \sin 58^\circ = 9.92842$$

$$\log h = 3.16007$$

$$h = 1445.67.$$

The height of the aeroplane is 1445.67 ft., and its distance from the two observers 1704.7 ft. and 1622.5 ft. respectively.

3. Prove that $(\sqrt{\csc x + \cot x} - \sqrt{\csc x - \cot x})^2 = 2(\csc x - 1)$.

$$\begin{aligned} (\sqrt{\csc x + \cot x} - \sqrt{\csc x - \cot x})^2 &= \csc x + \cot x - 2\sqrt{\csc^2 x - \cot^2 x} \\ &\quad + \csc x - \cot x \\ &= 2\csc x - 2\sqrt{\csc^2 x - \cot^2 x}; \\ \text{by [21],} \quad &= 2\csc x - 2\sqrt{1 + \cot^2 x - \cot^2 x} \\ &= 2\csc x - 2 \\ &= 2(\csc x - 1). \end{aligned}$$

4. Given $\sin x = 2m/(m^2 + 1)$ and $\sin y = 2n/(n^2 + 1)$, find the value of $\tan(x + y)$.

$$\sin x = \frac{2m}{m^2 + 1}, \quad \cos x = \sqrt{1 - \frac{4m^2}{(m^2 + 1)^2}}.$$

$$\sin y = \frac{2n}{n^2 + 1}, \quad \cos y = \sqrt{1 - \frac{4n^2}{(n^2 + 1)^2}}.$$

By [18],

$$\tan x = \frac{2m}{m^2 + 1} \bigg/ \left[1 - \frac{4m^2}{(m^2 + 1)^2} \right]^{\frac{1}{2}} = \frac{2m}{m^2 - 1},$$

$$\tan y = \frac{2n}{n^2 + 1} \bigg/ \left[1 - \frac{4n^2}{(n^2 + 1)^2} \right]^{\frac{1}{2}} = \frac{2n}{n^2 - 1}.$$

By [26],

$$\begin{aligned} \tan(x + y) &= \frac{\frac{2m}{m^2 - 1} + \frac{2n}{n^2 - 1}}{1 - \frac{2m}{m^2 - 1} \cdot \frac{2n}{n^2 - 1}} \\ &= \frac{2m(n^2 - 1) + 2n(m^2 - 1)}{(m^2 - 1)(n^2 - 1) - 4mn}. \end{aligned}$$

5. Find the least value of $\cos^2 x + \sec^2 x$.

Whatever the value of x , $\cos^2 x$ and $\sec^2 x$ are positive. $\sec^2 x$ is never less than 1, and $\cos^2 x$ is never less than 0. From the graphs in Ex. 25, Exercise 75, the smallest value of $\cos x + \sec x$ is when $x = 0$. Then $\cos x = 1$, and $\sec x = 1$.

When $x = 0^\circ$, $\cos^2 x + \sec^2 x = 2$.

6. Prove that $1 - \sin^2 x / \sin^2 y = \cos^2 x (1 - \tan^2 x / \tan^2 y)$.

$$\begin{aligned}
 & 1 - \sin^2 x / \sin^2 y \\
 &= 1 - \frac{\csc^2 y}{\csc^2 x} \\
 &= 1 - \frac{1 + \cot^2 y}{1 + \cot^2 x} \\
 &= 1 - \frac{1 + \frac{1}{\tan^2 y}}{1 + \frac{1}{\tan^2 x}} = 1 - \frac{1 + \tan^2 y}{\tan^2 y} \cdot \frac{\tan^2 x}{1 + \tan^2 x} \\
 &= \frac{\tan^2 y - \tan^2 x}{\tan^2 y (1 + \tan^2 x)} = \frac{1}{1 + \tan^2 x} \left(1 - \frac{\tan^2 x}{\tan^2 y} \right) \\
 &= \frac{1}{\sec^2 x} \left(1 - \frac{\tan^2 x}{\tan^2 y} \right) \\
 &= \cos^2 x \left(1 - \frac{\tan^2 x}{\tan^2 y} \right).
 \end{aligned}$$

7. Prove this formula, due to Euler :

$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{1}{4} \pi.$$

$$\text{Let } \alpha = \tan^{-1} \frac{1}{2}, \beta = \tan^{-1} \frac{1}{3}.$$

$$\text{Then } \tan \alpha = \frac{1}{2}, \tan \beta = \frac{1}{3}.$$

$$\text{By [26], } \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1.$$

$$\therefore \alpha + \beta = \tan^{-1} 1 = \frac{1}{4} \pi.$$

8. Prove that $\cot \frac{1}{2} x - \cot x = \csc x$.

$$\begin{aligned}
 \text{By [33], } \cot \frac{1}{2} x - \cot x &= \cot \frac{1}{2} x - \frac{\cot^2 \frac{1}{2} x - 1}{2 \cot \frac{1}{2} x} \\
 &= \frac{\cot^2 \frac{1}{2} x + 1}{2 \cot \frac{1}{2} x} = \frac{\csc^2 \frac{1}{2} x}{2 \cot \frac{1}{2} x} \\
 &= \frac{1}{2 \frac{\cos \frac{1}{2} x}{\sin \frac{1}{2} x} \cdot \sin^2 \frac{1}{2} x} = \frac{1}{2 \sin \frac{1}{2} x \cos \frac{1}{2} x}; \\
 &= \frac{1}{\sin x} = \csc x,
 \end{aligned}$$

by [30],

$$\begin{aligned}
 9. \text{ Prove that } (\sin x + i \cos x)^n &= \cos n\left(\frac{1}{2}\pi - x\right) + i \sin n\left(\frac{1}{2}\pi - x\right). \\
 \sin x &= \cos\left(\frac{1}{2}\pi - x\right). \\
 \cos x &= \sin\left(\frac{1}{2}\pi - x\right).
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence } \sin x + i \cos x &= \cos\left(\frac{1}{2}\pi - x\right) + i \sin\left(\frac{1}{2}\pi - x\right). \\
 \therefore (\sin x + i \cos x)^n &= [\cos\left(\frac{1}{2}\pi - x\right) + i \sin\left(\frac{1}{2}\pi - x\right)]^n \\
 &= \cos n\left(\frac{1}{2}\pi - x\right) + i \sin n\left(\frac{1}{2}\pi - x\right).
 \end{aligned}$$

10. Show that $\log i = \frac{1}{2}\pi i$, and that

$$\log(-i) = -\frac{1}{2}\pi i.$$

From § 164,

$$e^{i\pi} = -1.$$

Take square root of each side, $e^{\frac{i\pi}{2}} = \sqrt{-1} = i.$

Take logarithm of each side, $\frac{i\pi}{2} = \log i.$

$$\therefore \log i = \frac{1}{2}\pi i.$$

$$-i = \cos\left(2k\pi - \frac{\pi}{2}\right) + i \sin\left(2k\pi - \frac{\pi}{2}\right) = e^{\left(2k\pi - \frac{\pi}{2}\right)i}.$$

Take logarithms, $\log(-i) = \left(2k\pi - \frac{\pi}{2}\right)i.$

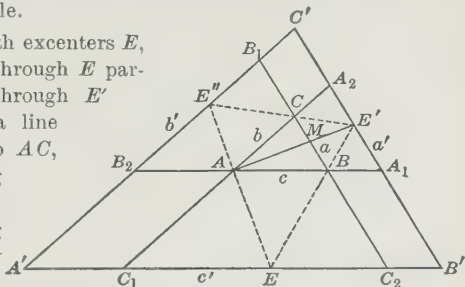
If $k = 0$, we get the principal value of $\log(-i)$; that is,

$$\log(-i) = -\frac{1}{2}\pi i.$$

11. Through the excenters of a triangle ABC lines are drawn parallel to the three sides, thus forming another triangle $A'B'C'$. Prove that the perimeter of $\triangle A'B'C'$ is $4R \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C$, where R is the radius of the circumcircle.

Given the $\triangle ABC$ with excenters E, E', E'' . Draw a line through E parallel to AB , a line through E' parallel to BC , and a line through E'' parallel to AC , these three lines forming the $\triangle A'B'C'$.

Draw AE' intersecting BC at M . By Geometry AE' bisects the $\angle A$.



By § 118, $\triangle ABM = \frac{1}{2}cAM \sin \frac{1}{2}A$, and $\triangle ACM = \frac{1}{2}bAM \sin \frac{1}{2}A$.

$$\therefore \triangle ABM + \triangle ACM = S = \frac{1}{2}(b+c)AM \sin \frac{1}{2}A.$$

$$\therefore AM = \frac{2S}{(b+c) \sin \frac{1}{2}A}.$$

$$\text{By § 118, } S = \frac{abc}{4R}.$$

$$\therefore AM = \frac{abc}{2R(b+c) \sin \frac{1}{2}A}.$$

Now $\frac{AE'}{r'} = \csc \frac{1}{2} A.$

$$\therefore AE' = r' \csc \frac{1}{2} A = 4R \cos \frac{1}{2} B \cos \frac{1}{2} C.$$

Since $B'C'$ and BC are parallel, $\triangle AA_1A_2$ and ABC are similar.

$$\begin{aligned} \therefore \frac{A_1A_2}{a} &= \frac{AE'}{AM} = \frac{4R \cos \frac{1}{2} B \cos \frac{1}{2} C}{\frac{abc}{2R(b+c) \sin \frac{1}{2} A}} \\ &= \frac{4R \cos \frac{1}{2} B \cos \frac{1}{2} C \cdot 2R(b+c) \sin \frac{1}{2} A}{abc} = \frac{2R \cos \frac{1}{2} B \cos \frac{1}{2} C \cdot (b+c)}{bc \cos \frac{1}{2} A}. \end{aligned}$$

$$\begin{aligned} \therefore A_1A_2 &= \frac{2Ra \cos \frac{1}{2} B \cos \frac{1}{2} C \cdot (b+c)}{bc \cos \frac{1}{2} A} \\ &= \frac{2Ra(b+c)}{bc} \sqrt{\frac{s(s-b)}{ac}} \times \frac{s(s-c)}{ab} \times \frac{bc}{s(s-a)} \\ &= \frac{2R(b+c)}{bc} \sqrt{\frac{s(s-b)(s-c)}{s-a}} \\ &= \frac{2abc(b+c)}{4bc \sqrt{s(s-a)(s-b)(s-c)}} \sqrt{\frac{s(s-b)(s-c)}{s-a}} = \frac{a(b+c)}{2(s-a)}. \end{aligned}$$

Similarly, $B_1B_2 = \frac{b(a+c)}{2(s-b)},$ and $C_1C_2 = \frac{c(a+b)}{2(s-c)}.$

By means of equal angles and parallel lines it is easy to prove that

$$A_1E' = A_1B = B'C_2,$$

and

$$A_2E' = A_2C = C'B_1.$$

$$\begin{aligned} \therefore A_1A_2 &= A_1E' + A_2E' \\ &= B'C_2 + C'B_1. \end{aligned}$$

Similarly,

$$\begin{aligned} B_1B_2 &= B_1E'' + B_2E'' \\ &= C'A_2 + A'C_1, \end{aligned}$$

and $C_1C_2 = C_1E + C_2E$

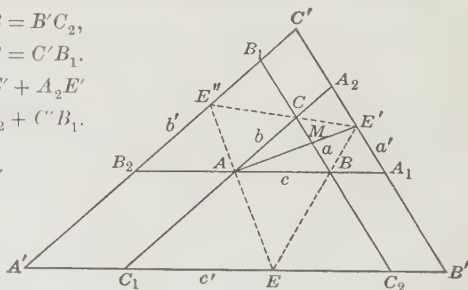
$$= A'B_2 + B'A_1.$$

$$\begin{aligned} \therefore A_1A_2 + B_1B_2 + C_1C_2 &= B'C_2 + C'B_1 + C'A_2 + A'C_1 + A'B_2 + B'A_1 \\ &= B'C_2 + A'C_1 + B'A_1 + C'A_2 + C'B_1 + A'B_2. \end{aligned}$$

$$\therefore A_1A_2 + B_1B_2 + C_1C_2 = \frac{1}{2}(a' + b' + c').$$

$$\therefore a' + b' + c' = 2(A_1A_2 + B_1B_2 + C_1C_2)$$

$$= 2 \left[\frac{a(b+c)}{2(s-a)} + \frac{b(a+c)}{2(s-b)} + \frac{c(a+b)}{2(s-c)} \right] = \frac{a(b+c)}{s-a} + \frac{b(a+c)}{s-b} + \frac{c(a+b)}{s-c}.$$



Putting $\frac{1}{2}(a+b+c)$ for s , we have

$$a' + b' + c' = \frac{2a(b+c)}{b+c-a} + \frac{2b(a+c)}{a-b+c} + \frac{2c(a+b)}{a+b-c}$$

$$= \frac{4a^2bc + 4ab^2c + 4abc^2}{(b+c-a)(a-b+c)(a+b-c)} = \frac{4abc(a+b+c)}{(b+c-a)(a-b+c)(a+b-c)}$$

$$= \frac{4abc \cdot 2s}{2(s-a) \cdot 2(s-b) \cdot 2(s-c)} = \frac{abc s}{(s-a)(s-b)(s-c)}.$$

Now, by § 118, $4R = \frac{abc}{\sqrt{s(s-a)(s-b)(s-c)}}.$

Also, by § 115, $\cot \frac{1}{2}A = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}, \quad \cot \frac{1}{2}B = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}},$

and $\cot \frac{1}{2}C = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}.$

$$\therefore 4R \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C$$

$$= \frac{abc}{\sqrt{s(s-a)(s-b)(s-c)}} \cdot \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \cdot \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{abc \sqrt{s^2}}{\sqrt{(s-a)^2(s-b)^2(s-c)^2}} = \frac{abc s}{(s-a)(s-b)(s-c)}.$$

But, as above, $a' + b' + c' = \frac{abc s}{(s-a)(s-b)(s-c)}.$

$$\therefore a' + b' + c' = 4R \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C.$$

12. Given two sides and the included angle of a triangle, find the perpendicular drawn to the third side from the opposite vertex.

Given $\angle A$ and sides b and c . Find the $\perp AD$ to the side a .

$$B + C = 180^\circ - A.$$

By [49], $\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}.$

$\angle C$ can be found from this equation.

$$\frac{AD}{b} = \sin C.$$

$$\therefore AD = b \sin C.$$

13. To find the height of a mountain a north-and-south base line is taken 1000 yd. long. From one end of this line the summit bears N. 80° E., and has an angle of elevation of $13^\circ 14'$; from the other end it bears N. $43^\circ 30'$ E. Find the height of the mountain.

$$\begin{aligned} \text{In the oblique } \triangle ABC, \quad \angle A &= 43^\circ 30', \\ \angle B &= 180^\circ - 80^\circ = 100^\circ, \\ \therefore \angle C &= 36^\circ 30'. \\ c &= 1000. \\ \frac{a}{c} &= \frac{\sin A}{\sin C}. \\ a &= \frac{1000 \sin 43^\circ 30'}{\sin 36^\circ 30'}. \end{aligned}$$

$$\begin{aligned} \log 1000 &= 3.00000 \\ \log \sin 43^\circ 30' &= 9.83781 \\ \text{colog } \sin 36^\circ 30' &= 0.22561 \\ \log a &= 3.06342 \end{aligned}$$

In the right triangle BDC made by h and the line from B to the top of the perpendicular h ,

$$\begin{aligned} \angle B &= 13^\circ 14'. \\ \frac{h}{a} &= \tan 13^\circ 14'. \\ h &= a \tan 13^\circ 14'. \\ \log a &= 3.06342 \\ \log \tan 13^\circ 14' &= 9.37137 \\ \log h &= 2.43479 \\ h &= 272.14 \text{ yd.} \end{aligned}$$

The height of the mountain is 816.42 ft.

14. The angle of elevation of a wireless telegraph tower is observed from a point on the horizontal plane on which it stands. At a point a feet nearer, the angle of elevation is the complement of the former. At a point b feet nearer still, the angle of elevation is double the first. Show that the height of the tower is $[(a+b)^2 - \frac{1}{4}a^2]^{\frac{1}{2}}$.

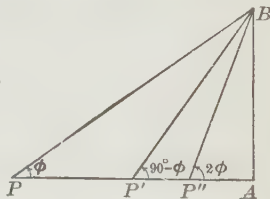
Let $AB = x$ denote height of tower.

$$P'P'' = a.$$

$$P'P'' = b.$$

$$P''A = y.$$

ϕ , $90^\circ - \phi$, and 2ϕ the three angles.



Then

$$\tan \phi = \frac{x}{a+b+y}, \quad (1)$$

$$\tan (90^\circ - \phi) = \frac{x}{b+y}, \quad (2)$$

$$\tan 2\phi = \frac{x}{y}. \quad (3)$$

From (2),

$$\cot \phi = \frac{x}{b+y}.$$

$$\therefore \tan \phi = \frac{b+y}{x}.$$

But

$$\tan \phi = \frac{x}{a+b+y}.$$

$$\therefore \frac{b+y}{x} = \frac{x}{a+b+y}$$

$$x^2 = (b+y)(a+b+y). \quad (4)$$

From (3) and (1),

$$\tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi} = \frac{\frac{2x}{a+b+y}}{1 - \left(\frac{x}{a+b+y}\right)^2} = \frac{x}{y}.$$

$$\frac{2x}{a+b+y} \cdot \frac{(a+b+y)^2}{(a+b+y)^2 - x^2} = \frac{x}{y}.$$

$$2(a+b+y)y = (a+b+y)^2 - x^2.$$

$$x^2 = (a+b+y)(a+b-y). \quad (5)$$

From (4) and (5),

$$(b+y)(a+b+y) = (a+b+y)(a+b-y).$$

$$(a+b+y)(a-2y) = 0.$$

$$\therefore y = \frac{a}{2}. \quad (6)$$

Substitute $\frac{a}{2}$ for y in (5),

$$x^2 = \left(a+b+\frac{a}{2}\right)\left(a+b-\frac{a}{2}\right)$$

$$= (a+b)^2 - \frac{a^2}{4}.$$

$$\therefore x = [(a+b)^2 - \frac{1}{4}a^2]^{\frac{1}{2}}.$$

The value $y = -(a+b)$ is inadmissible from the conditions of the problem.

Prove the following formulas :

15. $2 \cos^2 x = \cos 2x + 1.$

By [16],

$$2 \cos^2 x = \cos^2 x + (1 - \sin^2 x)$$

$$= \cos^2 x - \sin^2 x + 1;$$

by [32],

$$= \cos 2x + 1.$$

$$16. 2 \sin^2 x = -\cos 2x + 1.$$

$$\begin{aligned} \text{By [16],} \quad 2 \sin^2 x &= \sin^2 x + (1 - \cos^2 x) \\ &= -(\cos^2 x - \sin^2 x) + 1; \end{aligned}$$

$$\text{by [32],} \quad = -\cos 2x + 1.$$

$$17. 8 \cos^4 x = \cos 4x + 4 \cos 2x + 3.$$

$$8 \cos^4 x = 2(2 \cos^2 x)(2 \cos^2 x);$$

$$\begin{aligned} \text{by Ex. 15,} \quad &= 2(\cos 2x + 1)^2 \\ &= 2 \cos^2 2x + 4 \cos 2x + 2; \end{aligned}$$

$$\text{by Ex. 15,} \quad = \cos 4x + 4 \cos 2x + 3.$$

$$18. 4 \cos^3 x = \cos 3x + 3 \cos x.$$

$$4 \cos^3 x = 2 \cos x 2 \cos^2 x;$$

$$\begin{aligned} \text{by Ex. 15,} \quad &= 2 \cos x (\cos 2x + 1) \\ &= 2 \cos x \cos 2x + 2 \cos x; \end{aligned}$$

$$\begin{aligned} \text{by [44],} \quad &= \cos 3x + \cos x + 2 \cos x \\ &= \cos 3x + 3 \cos x. \end{aligned}$$

$$19. 4 \sin^3 x = -\sin 3x + 3 \sin x.$$

$$4 \sin^3 x = 2 \sin x (2 \sin^2 x);$$

$$\begin{aligned} \text{by Ex. 16,} \quad &= 2 \sin x (1 - \cos 2x) \\ &= 2 \sin x - 2 \sin x \cos 2x; \end{aligned}$$

$$\begin{aligned} \text{by [43],} \quad &= 2 \sin x - (\sin 3x - \sin x); \\ &= -\sin 3x + 3 \sin x. \end{aligned}$$

$$20. 8 \sin^4 x = \cos 4x - 4 \cos 2x + 3.$$

$$8 \sin^4 x = 2(2 \sin^2 x)(2 \sin^2 x);$$

$$\begin{aligned} \text{by Ex. 16,} \quad &= 2(1 - \cos 2x)^2 \\ &= 2 - 4 \cos 2x + 2 \cos^2 2x; \end{aligned}$$

$$\text{by Ex. 15,} \quad = \cos 4x - 4 \cos 2x + 3.$$

SPHERICAL TRIGONOMETRY

Exercise 85. Formulas of Right Triangles (Page 195)

1. From the formula $\cos c = \cos a \cos b$ show that the hypotenuse of a right spherical triangle is less than 90° if the two sides are both less than 90° or are both greater than 90° .

If a and b are both $< 90^\circ$ or both $> 90^\circ$, $\cos a$ and $\cos b$ have the same sign. Hence $\cos c$ is positive, and $c < 90^\circ$.

2. As in Ex. 1, show that the hypotenuse is greater than 90° if one side is greater than 90° and the other side less than 90° .

If a and b are unlike in kind, that is, if one is $> 90^\circ$ and one $< 90^\circ$, $\cos a$ and $\cos b$ have opposite signs. Hence $\cos c$ is negative, and $c > 90^\circ$.

3. From the formula $\cos A = \cos a \sin B$ show that in a right spherical triangle an oblique angle and the opposite side are either both greater than 90° or both less than 90° .

$$B < 180^\circ.$$

$\therefore \sin B$ is positive.

Hence the sign of $\cos A$ is the same as the sign of $\cos a$, and both must be greater than 90° or both less than 90° ; that is, alike in kind.

From the formulas on pages 193-195 state the inferences to be drawn respecting the values of the other parts when:

4. $c = 90^\circ$.

If $c = 90^\circ$,
(1) becomes $0 = \cos a \cos b$.
 $\therefore \cos a$ or $\cos b = 0$.
 $\therefore a$ or $b = 90^\circ$.

If $a = 90^\circ$, (6) becomes
 $\cos A = 0 \times \sin B = 0$.
 $\therefore A = 90^\circ$.

Hence, from § 172 (2),
since $C = 90^\circ$,
 $B = b$.

Similarly, if $b = 90^\circ$, $A = a$.

5. $a = 90^\circ$.

If $a = 90^\circ$,
(6) becomes
 $\cos A = 0 \times \sin B = 0$.
 $\therefore A = 90^\circ$.

Hence, from § 172 (2), since C is a right angle, and

$$\begin{aligned} A &= 90^\circ, \\ c &= a = 90^\circ, \\ B &= b. \end{aligned}$$

it follows that
and

6. $b = 90^\circ$.

If $b = 90^\circ$,

(7) becomes

$\cos B = 0 \times \sin A = 0.$

$\therefore B = 90^\circ.$

Hence, from § 172 (2),

since $C = 90^\circ$,

it follows that $c = b = 90$,

and $A = a.$

7. $c = a.$

If $c = a$,

(2) becomes $\sin a = \sin a \sin A.$

$\therefore \sin A = 1.$

$\therefore A = 90^\circ.$

Hence, from § 172 (2),

since $C = 90^\circ$,

it follows that $a = c = 90^\circ$,

and $B = b.$

8. $a = b.$

If $a = b$,

(2) becomes $\sin a = \sin c \sin A,$

(3) becomes $\sin a = \sin c \sin B.$

$\therefore \sin A = \sin B.$

$\therefore A = B.$

9. $A = 90^\circ.$

From § 172 (2),

since $C = 90^\circ$,

it follows that $a = c = 90^\circ$,

and $B = b.$

10. $c = 90^\circ$ and $a = 90^\circ.$

We have $a = c = 90^\circ$,

and $C = 90^\circ.$

Hence (2) becomes

$1 = 1 \times \sin A.$

$\therefore \sin A = 1.$

$\therefore A = 90^\circ.$

By § 172 (2), $B = b.$

11. $a = 90^\circ$ and $b = 90^\circ.$

We have $a = b = 90^\circ$,

and $C = 90^\circ.$

Since $a = 90^\circ$,

by Ex. 5 $A = 90^\circ.$

Since $b = 90^\circ$,

by Ex. 5 $B = 90^\circ,$

and $c = 90^\circ.$

That is, $a = b = c = 90^\circ$,

and $A = B = C = 90^\circ.$

Exercise 86. Spherical Triangles (Page 197)

Deduce eight of the formulas on page 191 by means of Napier's rules, taking for the middle part:

$$1. a. \quad \sin a = \cos(Co-c) \cos(Co-A). \\ \therefore \sin a = \sin c \sin A. \quad (2)$$

$$\sin a = \tan b \tan(Co-B). \\ \therefore \sin a = \tan b \cot B. \quad (9)$$

$$2. b. \quad \sin b = \cos(Co-c) \cos(Co-B). \\ \therefore \sin b = \sin c \sin B. \quad (3)$$

$$\sin b = \tan a \tan(Co-A). \\ \therefore \sin b = \tan a \cot A. \quad (8)$$

$$\begin{aligned}
 3. \text{ Co-}B. \quad & \sin (Co-B) = \cos b \cos (Co-A). \\
 & \therefore \cos B = \cos b \sin A. \quad (7) \\
 & \sin (Co-B) = \tan a \tan (Co-c). \\
 & \therefore \cos B = \tan a \cot c. \quad (5)
 \end{aligned}$$

$$\begin{aligned}
 4. \text{ Co-}A. \quad & \sin (Co-A) = \cos a \cos (Co-B). \\
 & \therefore \cos A = \cos a \sin B. \quad (6) \\
 & \sin (Co-A) = \tan b \tan (Co-c). \\
 & \therefore \cos A = \tan b \cot c. \quad (4)
 \end{aligned}$$

By Napier's rules deduce the following :

$$\begin{aligned}
 5. \cos B &= \tan a \cot c, & 6. \sin a &= \tan b \cot B. \\
 \sin (Co-B) &= \tan a \tan (Co-c), & \sin a &= \tan b \tan (Co-B). \\
 \therefore \cos B &= \tan a \cot c, & \therefore \sin a &= \tan b \cot B.
 \end{aligned}$$

7. What do Napier's rules become if we take as the five parts of the triangle the hypotenuse, the two oblique angles, and the *complements* of the two sides?

Each part will be replaced by its complement, and every function will be replaced by its complementary function.

Therefore, Napier's rules become

1. *The cosine of any middle part is equal to the product of the cotangents of the adjacent parts.*

2. *The cosine of any middle part is equal to the product of the sines of the opposite parts.*

8. Solve a spherical right triangle, given a , b , and c .

$$\text{From} \quad \sin a = \sin c \sin A, \text{ we find } A.$$

$$\text{From} \quad \cos B = \tan a \cot c, \text{ we find } B.$$

$$C = 90^\circ.$$

9. Solve a spherical right triangle, given A , B , and c .

$$\text{From} \quad \sin a = \sin c \sin A, \text{ we find } a.$$

$$\text{From} \quad \sin b = \sin c \sin B, \text{ we find } b.$$

$$C = 90^\circ.$$

10. Solve a spherical right triangle, given A , a , and b .

$$\text{From} \quad \cos B = \cos b \sin A, \text{ we find } B.$$

$$\text{From} \quad \cos c = \cos a \cos b, \text{ we find } c.$$

$$C = 90^\circ.$$

Find the number of degrees in the sides of a spherical triangle, given the angles of its polar triangle as follows :

11. 82° , 77° , 69° .

$$a = 180^\circ - A' = 180^\circ - 82^\circ = 98^\circ.$$

$$b = 180^\circ - B' = 180^\circ - 77^\circ = 103^\circ.$$

$$c = 180^\circ - C' = 180^\circ - 69^\circ = 111^\circ.$$

12. $84\frac{1}{2}^\circ$, $81\frac{3}{4}^\circ$, $72\frac{1}{6}^\circ$.

$$a = 180^\circ - A' = 180^\circ - 84\frac{1}{2}^\circ = 95\frac{1}{2}^\circ.$$

$$b = 180^\circ - B' = 180^\circ - 81\frac{3}{4}^\circ = 98\frac{1}{4}^\circ.$$

$$c = 180^\circ - C' = 180^\circ - 72\frac{1}{6}^\circ = 107\frac{5}{6}^\circ.$$

13. $78^\circ 30'$, 89° , 102° .

$$a = 180^\circ - A' = 180^\circ - 78^\circ 30' = 101^\circ 30'.$$

$$b = 180^\circ - B' = 180^\circ - 89^\circ = 91^\circ.$$

$$c = 180^\circ - C' = 180^\circ - 102^\circ = 78^\circ.$$

14. $83^\circ 40'$, $48^\circ 57'$, $103^\circ 43'$.

$$a = 180^\circ - A' = 180^\circ - 83^\circ 40' = 96^\circ 20'.$$

$$b = 180^\circ - B' = 180^\circ - 48^\circ 57' = 131^\circ 3'.$$

$$c = 180^\circ - C' = 180^\circ - 103^\circ 43' = 76^\circ 17'.$$

15. $96^\circ 37' 40''$, $82^\circ 29' 30''$, $68^\circ 47'$.

$$a = 180^\circ - A' = 180^\circ - 96^\circ 37' 40'' = 83^\circ 22' 20''.$$

$$b = 180^\circ - B' = 180^\circ - 82^\circ 29' 30'' = 97^\circ 30' 30''.$$

$$c = 180^\circ - C' = 180^\circ - 68^\circ 47' = 111^\circ 13'.$$

16. $43^\circ 29' 37''$, $98^\circ 22' 53''$, $87^\circ 36' 39''$.

$$a = 180^\circ - A' = 180^\circ - 43^\circ 29' 37'' = 136^\circ 30' 23''.$$

$$b = 180^\circ - B' = 180^\circ - 98^\circ 22' 53'' = 81^\circ 37' 7''.$$

$$c = 180^\circ - C' = 180^\circ - 87^\circ 36' 39'' = 92^\circ 23' 21''.$$

Find the number of degrees in the angles of a spherical triangle, given the sides of the polar triangle in Exs. 17-20 :

17. $68^\circ 42' 39''$, $93^\circ 48' 7''$, $89^\circ 38' 14''$.

$$A = 180^\circ - a' = 180^\circ - 68^\circ 42' 39'' = 111^\circ 17' 21''.$$

$$B = 180^\circ - b' = 180^\circ - 93^\circ 48' 7'' = 86^\circ 11' 53''.$$

$$C = 180^\circ - c' = 180^\circ - 89^\circ 38' 14'' = 90^\circ 21' 46''.$$

18. $78^\circ 47' 29''$, $106^\circ 36' 42''$, a quadrant.

$$A = 180^\circ - a' = 180^\circ - 78^\circ 47' 29'' = 101^\circ 12' 31''.$$

$$B = 180^\circ - b' = 180^\circ - 106^\circ 36' 42'' = 73^\circ 23' 18''.$$

$$C = 180^\circ - c' = 180^\circ - 90^\circ = 90^\circ.$$

19. $111^{\circ} 29' 43''$, a quadrant, a quadrant.

$$A = 180^{\circ} - a' = 180^{\circ} - 111^{\circ} 29' 43'' = 68^{\circ} 30' 17''.$$

$$B = 180^{\circ} - b' = 180^{\circ} - 90^{\circ} = 90^{\circ}.$$

$$C = 180^{\circ} - c' = 180^{\circ} - 90^{\circ} = 90^{\circ}.$$

20. A quadrant, half a quadrant, three fourths of a quadrant.

$$A = 180^{\circ} - a' = 180^{\circ} - 90^{\circ} = 90^{\circ}.$$

$$B = 180^{\circ} - b' = 180^{\circ} - 45^{\circ} = 135^{\circ}.$$

$$C = 180^{\circ} - c' = 180^{\circ} - 67\frac{1}{2}^{\circ} = 112\frac{1}{2}^{\circ}.$$

21. The angles of a spherical triangle are 70.5° , 80.7° , and 101.6° . Find the sides of the polar triangle.

$$a' = 180^{\circ} - A = 180^{\circ} - 70.5^{\circ} = 109.5^{\circ}.$$

$$b' = 180^{\circ} - B = 180^{\circ} - 80.7^{\circ} = 99.3^{\circ}.$$

$$c' = 180^{\circ} - C = 180^{\circ} - 101.6^{\circ} = 78.4^{\circ}.$$

22. The sides of a spherical triangle are 40.72° , 90° , and 127.83° . Find the angles of the polar triangle.

$$A' = 180^{\circ} - a = 180^{\circ} - 40.72^{\circ} = 139.28^{\circ}.$$

$$B' = 180^{\circ} - b = 180^{\circ} - 90^{\circ} = 90^{\circ}.$$

$$C' = 180^{\circ} - c = 180^{\circ} - 127.83^{\circ} = 52.17^{\circ}.$$

23. Show that, if a spherical triangle has three right angles, the sides of the triangle are quadrants.

Each vertex is the pole of the opposite side. Each side is therefore 90° .

24. Show that, if a spherical triangle has two right angles, the sides opposite these angles are quadrants, and the third angle is measured by the opposite side.

Let ABC be the triangle, and

$$B = C = 90^{\circ}.$$

Then A is the pole of a . Therefore b and c are quadrants, and the angle A is equal to the side BC measured in degrees.

25. How can the sides of a spherical triangle, measured in degrees, be found in units of length, when the length of the radius of the sphere is known?

Since the sides of the triangle are arcs of great circles, every degree of arc is $\frac{1}{360}$ of the circumference of a great circle, or $\frac{2\pi r}{360}$, where r is the radius of the sphere. Hence, to find the length of a side, multiply its measure in degrees by $\frac{2\pi r}{360}$ or $\frac{\pi r}{180}$.

Exercise 87. Given Two Sides (Page 199)*Solve the following right spherical triangles, given :*

1. $a = 30^\circ, b = 50^\circ.$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.93753$$

$$\log \cos b = \underline{9.80807}$$

$$\log \cos c = 9.74560$$

$$c = 56^\circ 10' 25''.$$

$$\sin b = \tan a \cot A.$$

$$\tan a = \sin b \tan A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 9.76144$$

$$\log \csc b = \underline{0.11575}$$

$$\log \tan A = 9.87719$$

$$A = 37^\circ 0' 18''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.07619$$

$$\log \csc a = \underline{0.30103}$$

$$\log \tan B = 10.37722$$

$$B = 67^\circ 14' 23''.$$

2. $a = 40^\circ, b = 60^\circ.$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.88425$$

$$\log \cos b = \underline{9.69897}$$

$$\log \cos c = 9.58322$$

$$c = 67^\circ 28' 45''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 9.92381$$

$$\log \csc b = \underline{0.06247}$$

$$\log \tan A = 9.98628$$

$$A = 44^\circ 5' 43''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.23856$$

$$\log \csc a = \underline{0.19193}$$

$$\log \tan B = 10.43049$$

$$B = 69^\circ 38' 22''$$

3. $a = 45^\circ, b = 72^\circ.$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.84949$$

$$\log \cos b = \underline{9.48998}$$

$$\log \cos c = 9.33947$$

$$c = 77^\circ 22' 43''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.00000$$

$$\log \csc b = \underline{0.02179}$$

$$\log \tan A = 10.02179$$

$$A = 46^\circ 26' 12''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.48822$$

$$\log \csc a = \underline{0.15051}$$

$$\log \tan B = 10.63873$$

$$B = 77^\circ 3' 37''.$$

4. $a = 56^\circ, b = 78^\circ.$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.74756$$

$$\log \cos b = \underline{9.31788}$$

$$\log \cos c = 9.06544$$

$$c = 83^\circ 19' 25''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.17101$$

$$\log \csc b = \underline{0.00960}$$

$$\log \tan A = 10.18061$$

$$A = 56^\circ 35' 4''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.67253$$

$$\log \csc a = \underline{0.08143}$$

$$\log \tan B = 10.75396$$

$$B = 80^\circ 0' 24''.$$

5. $a = 63^\circ, b = 87^\circ.$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.65705$$

$$\log \cos b = \underline{8.71880}$$

$$\log \cos c = 8.37585$$

$$c = 88^\circ 38' 19''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.29283$$

$$\log \csc b = \underline{0.00060}$$

$$\log \tan A = 10.29343$$

$$A = 63^\circ 1' 54''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 11.28060$$

$$\log \csc a = \underline{0.05012}$$

$$\log \tan B = 11.33072$$

$$B = 87^\circ 19' 35''.$$

6. $a = 68^\circ, b = 93^\circ.$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.57358$$

$$\log \cos b = \underline{8.71880} \text{ (n)}$$

$$\log \cos c = 8.29238 \text{ (n)}$$

$$c = 91^\circ 7' 24''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.39359$$

$$\log \csc b = \underline{0.00060}$$

$$\log \tan A = 10.39419$$

$$A = 68^\circ 1' 39''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 11.28060 \text{ (n)}$$

$$\log \csc a = \underline{0.03233}$$

$$\log \tan B = 11.31343 \text{ (n)}$$

$$B = 92^\circ 46' 55''.$$

7. $a = 75^\circ, b = 98^\circ.$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.41300$$

$$\log \cos b = \underline{9.14356} \text{ (n)}$$

$$\log \cos c = 8.55656 \text{ (n)}$$

$$c = 92^\circ 3' 52''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.57195$$

$$\log \csc b = \underline{0.00425}$$

$$\log \tan A = 10.57620$$

$$A = 75^\circ 8' 22''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.85220 \text{ (n)}$$

$$\log \csc a = \underline{0.01506}$$

$$\log \tan B = 10.86726 \text{ (n)}$$

$$B = 97^\circ 43' 51''.$$

8. $a = 82^\circ, b = 100^\circ.$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.14356$$

$$\log \cos b = \underline{9.23967} \text{ (n)}$$

$$\log \cos c = 8.38323 \text{ (n)}$$

$$c = 91^\circ 23' 5''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.85220$$

$$\log \csc b = \underline{0.00665}$$

$$\log \tan A = 10.85885$$

$$A = 82^\circ 7' 12''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.75368 \text{ (n)}$$

$$\log \csc a = \underline{0.00425}$$

$$\log \tan B = 10.75793 \text{ (n)}$$

$$B = 99^\circ 54' 17''.$$

9. $a = 95^\circ$, $b = 120^\circ$.

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 8.94030 \text{ (n)}$$

$$\log \cos b = \underline{9.69897 \text{ (n)}}$$

$$\log \cos c = 8.63927$$

$$c = 87^\circ 30' 8''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 11.05805 \text{ (n)}$$

$$\log \csc b = \underline{0.06247}$$

$$\log \tan A = 11.12052 \text{ (n)}$$

$$A = 94^\circ 19' 58''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.23856 \text{ (n)}$$

$$\log \csc a = \underline{0.00166}$$

$$\log \tan B = 10.24022 \text{ (n)}$$

$$B = 119^\circ 54' 19''.$$

10. $a = 120^\circ$, $b = 119^\circ$.

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.69897 \text{ (n)}$$

$$\log \cos b = \underline{9.68557 \text{ (n)}}$$

$$\log \cos c = 9.38454$$

$$c = 75^\circ 58' 18''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.23856 \text{ (n)}$$

$$\log \csc b = \underline{0.05818}$$

$$\log \tan A = 10.29674 \text{ (n)}$$

$$A = 116^\circ 47' 32''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.25625 \text{ (n)}$$

$$\log \csc a = \underline{0.06247}$$

$$\log \tan B = 10.31872 \text{ (n)}$$

$$B = 115^\circ 38' 35''.$$

11. $a = 36^\circ 27'$, $b = 43^\circ 32' 31''$.

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.90546$$

$$\log \cos b = \underline{9.86026}$$

$$\log \cos c = 9.76572$$

$$c = 54^\circ 20'.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 9.86842$$

$$\log \csc b = \underline{0.16185}$$

$$\log \tan A = 10.03027$$

$$A = 46^\circ 59' 43''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 9.97789$$

$$\log \csc a = \underline{0.22613}$$

$$\log \tan B = 10.20402$$

$$B = 57^\circ 59' 19''.$$

12. $a = 86^\circ 40'$, $b = 32^\circ 40'$.

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 8.76451$$

$$\log \cos b = \underline{9.92522}$$

$$\log \cos c = 8.68973$$

$$c = 87^\circ 11' 40''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 11.23475$$

$$\log \csc b = \underline{0.26781}$$

$$\log \tan A = 11.50256$$

$$A = 88^\circ 11' 58''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 9.80697$$

$$\log \csc a = \underline{0.00074}$$

$$\log \tan B = 9.80771$$

$$B = 32^\circ 42' 39''.$$

$$13. a = 50^\circ, b = 36^\circ 54' 49''.$$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.80807$$

$$\log \cos b = \underline{9.90284}$$

$$\log \cos c = 9.71091$$

$$c = 59^\circ 4' 26''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.07619$$

$$\log \csc b = \underline{0.22141}$$

$$\log \tan A = \underline{10.29760}$$

$$A = 63^\circ 15' 13''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 9.87575$$

$$\log \csc a = \underline{0.11575}$$

$$\log \tan B = \underline{9.99150}$$

$$B = 44^\circ 26' 22''.$$

$$14. a = 120^\circ 10', b = 150^\circ 59' 44''.$$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.70115 \ (n)$$

$$\log \cos b = \underline{9.94180 \ (n)}$$

$$\log \cos c = 9.64295$$

$$c = 63^\circ 55' 43''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.23565 \ (n)$$

$$\log \csc b = \underline{0.31437}$$

$$\log \tan A = \underline{10.55002 \ (n)}$$

$$A = 105^\circ 44' 21''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 9.74383 \ (n)$$

$$\log \csc a = \underline{0.06320}$$

$$\log \tan B = \underline{9.80703 \ (n)}$$

$$B = 147^\circ 19' 47''.$$

$$15. a = 22^\circ 15' 7'', b = 51^\circ 53'.$$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.96639$$

$$\log \cos b = \underline{9.79047}$$

$$\log \cos c = 9.75686$$

$$c = 55^\circ 9' 33''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 9.61188$$

$$\log \csc b = \underline{0.10416}$$

$$\log \tan A = \underline{9.71604}$$

$$A = 27^\circ 28' 35''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.10537$$

$$\log \csc a = \underline{0.42172}$$

$$\log \tan B = \underline{10.52709}$$

$$B = 73^\circ 27' 10''.$$

$$16. a = 14^\circ 16' 35'', b = 19^\circ 17'.$$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.98638$$

$$\log \cos b = \underline{9.97491}$$

$$\log \cos c = \underline{9.96129}$$

$$c = 23^\circ 50'.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 9.40562$$

$$\log \csc b = \underline{0.48117}$$

$$\log \tan A = \underline{9.88679}$$

$$A = 37^\circ 36' 55''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 9.54390$$

$$\log \csc a = \underline{0.60801}$$

$$\log \tan B = \underline{10.15191}$$

$$B = 54^\circ 49' 20''.$$

$$17. a = 32^\circ 9' 17'', b = 32^\circ 41'.$$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.92769$$

$$\log \cos b = \underline{9.92514}$$

$$\log \cos c = \underline{9.85283}$$

$$c = 44^\circ 33' 18''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 9.79840$$

$$\log \csc b = \underline{0.26761}$$

$$\log \tan A = \underline{10.06601}$$

$$A = 49^\circ 20' 16''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 9.80725$$

$$\log \csc a = \underline{0.27392}$$

$$\log \tan B = \underline{10.08117}$$

$$B = 50^\circ 19' 24''.$$

$$18. a = 132^\circ 14' 12'', b = 79^\circ 13' 38''.$$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.82750 \text{ (n)}$$

$$\log \cos b = \underline{9.27164}$$

$$\log \cos c = \underline{9.09914 \text{ (n)}}$$

$$c = 97^\circ 13' 4''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.04196 \text{ (n)}$$

$$\log \csc b = \underline{0.00772}$$

$$\log \tan A = \underline{10.04968 \text{ (n)}}$$

$$A = 131^\circ 43' 48''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.72064$$

$$\log \csc a = \underline{0.13055}$$

$$\log \tan B = \underline{10.85119}$$

$$B = 81^\circ 58' 54''.$$

$$19. a = 2^\circ 0' 55'', b = 0^\circ 27' 10''$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 8.54639$$

$$\log \csc b = \underline{2.10224}$$

$$\log \tan A = \underline{10.64863}$$

$$A = 77^\circ 20' 34''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 7.89777$$

$$\log \csc a = \underline{1.45388}$$

$$\log \tan B = \underline{9.35165}$$

$$B = 12^\circ 39' 55''.$$

$$\cos B = \tan a \cot c.$$

$$\tan c = \tan a \sec B.$$

$$\log \tan a = 8.54639$$

$$\log \sec B = \underline{0.01070}$$

$$\log \tan c = \underline{8.55709}$$

$$c = 2^\circ 3' 56''.$$

$$20. a = 20^\circ 20' 20'', b = 15^\circ 16' 50''$$

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.97204$$

$$\log \cos b = \underline{9.98437}$$

$$\log \cos c = \underline{9.95641}$$

$$c = 25^\circ 14' 40''.$$

$$\sin b = \tan a \cot A.$$

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 9.56900$$

$$\log \csc b = \underline{0.57915}$$

$$\log \tan A = \underline{10.14815}$$

$$A = 54^\circ 35' 18''.$$

$$\sin a = \tan b \cot B.$$

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 9.43649$$

$$\log \csc a = \underline{0.45896}$$

$$\log \tan B = \underline{9.89545}$$

$$B = 38^\circ 10' 9''.$$

21. How many degrees are there in the arc of a great circle drawn from a point on the equator in longitude 40° E. to a point on the prime meridian in latitude 40° N. ?

$$\begin{aligned}\cos c &= \cos a \cos b \\ &= \cos 40^\circ \cos 40^\circ.\end{aligned}$$

$$\log \cos a = 9.88425$$

$$\log \cos b = \underline{9.88425}$$

$$\log \cos c = 9.76850$$

$$c = 54^\circ 4' 7''.$$

22. Greenwich lies on the prime meridian $51^\circ 28' 38''$ N. The arc of a great circle drawn from Greenwich to a point on the equator in longitude 25° W. makes what angle with the equator ?

$$a = 25^\circ, b = 51^\circ 28' 38''.$$

B = the required angle.

$$\tan B = \tan b \csc a.$$

$$\log \tan b = 10.09904$$

$$\log \csc a = \underline{0.37405}$$

$$\log \tan B = 10.47309$$

$$B = 71^\circ 24' 17''.$$

23. The arc of a great circle drawn from Greenwich to a point on the equator in longitude 150° E. makes what angle with the prime meridian ?

$$a = 150^\circ, b = 51^\circ 28' 38''.$$

A = the required angle.

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 9.76144 \text{ (} n \text{)}$$

$$\log \csc b = \underline{0.10660}$$

$$\log \tan A = 9.86804 \text{ (} n \text{)}$$

$$A = 143^\circ 34' 25''.$$

24. How many degrees are there in the arc of a great circle drawn from a point on the equator in longitude 0° to a point in longitude 48° W., latitude 30° N. ?

$$a = 48^\circ, b = 30^\circ.$$

c = the required arc.

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.82551$$

$$\log \cos b = \underline{9.93753}$$

$$\log \cos c = 9.76304$$

$$c = 54^\circ 35' 10''$$

25. In a right spherical triangle on a sphere of radius 6 in. it is given that $a = 45^\circ$ and $b = 70^\circ$. Find the length of c in inches.

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.84949$$

$$\log \cos b = 9.53405$$

$$\log \cos c = 9.38354$$

$$c = 76^\circ 0' 17''.$$

$$c \text{ in inches} = \frac{2 \times 3\frac{1}{7} \times 6}{360} \times 76_{3600}^{17} = \frac{22 \times 6}{7 \times 180} \times 76_{3600}^{17} = 7.9624.$$

26. In a right spherical triangle on a sphere of diameter 2 ft. it is given that $a = 75^\circ$ and $b = 75^\circ$. Find the length of c in inches.

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.41300$$

$$\log \cos b = 9.41300$$

$$\log \cos c = 8.82600$$

$$c = 86^\circ 9' 32''.$$

$$c \text{ in inches} = \frac{2 \times 3\frac{1}{7} \times 12}{360} \times 86_{3600}^{57.2} = \frac{22 \times 12}{7 \times 180} \times 86_{3600}^{57.2} = 18.052.$$

27. Taking the radius of the earth as 4000 mi., how many miles is it, on a great circle, from a point on the equator in longitude 70° W. to a point on the prime meridian in latitude 60° N.?

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.53405$$

$$\log \cos b = 9.69897$$

$$\log \cos c = 9.23302$$

$$c = 80^\circ 9' 12''.$$

$$c \text{ in miles} = \frac{2 \times 3\frac{1}{7} \times 4000}{360} \times 80_{3600}^{55.2} = \frac{22 \times 4000}{7 \times 180} \times 80_{3600}^{55.2} = 5598_{180}^{\frac{2}{3}}.$$

28. The arc of a great circle drawn from a point on the prime meridian $60'$ N. to a point on the equator 60° W. makes what angle with the prime meridian and with the equator?

$$\tan A = \tan a \csc b.$$

$$\log \tan a = 10.23856$$

$$\log \csc b = 0.06247$$

$$\log \tan A = 10.30103$$

$$A = 63^\circ 26' 6''.$$

$$B = 63^\circ 26' 6''.$$

29. In Ex. 28, what is the length of the arc, taking the radius of the earth as 4000 mi.?

$$\cos c = \cos a \cos b.$$

$$\log \cos a = 9.69897$$

$$\log \cos b = \underline{9.69897}$$

$$\log \cos c = 9.39794$$

$$c = 75^\circ 31' 21''.$$

$$\begin{aligned} c \text{ in miles} &= \frac{2 \times 3\frac{1}{7} \times 4000}{360} \times 75\frac{1881}{3600} \\ &= \frac{22 \times 4000}{7 \times 180} \times 75\frac{1881}{3600} = 5274\frac{37}{63}. \end{aligned}$$

Exercise 88. Given the Hypotenuse and a Side
(Page 200)

Solve the right spherical triangles, given c and a as follows:

1. $c = 54^\circ 20'$, $a = 36^\circ 27'$.

$$\cos b = \cos c \sec a.$$

$$\log \cos c = 9.76572$$

$$\log \sec a = \underline{0.09454}$$

$$\log \cos b = 9.86026$$

$$b = 43^\circ 32' 30''$$

$$\sin A = \sin a \csc c.$$

$$\log \sin a = 9.77387$$

$$\log \csc c = \underline{0.09022}$$

$$\log \sin A = 9.86409$$

$$A = 46^\circ 59' 40''.$$

$$\cos B = \tan a \cot c.$$

$$\log \tan a = 9.86842$$

$$\log \cot c = \underline{9.85594}$$

$$\log \cos B = 9.72436$$

$$B = 57^\circ 59' 15''.$$

2. $c = 87^\circ 11' 40''$, $a = 86^\circ 40'$.

$$\cos b = \cos c \sec a.$$

$$\log \cos c = 8.68972$$

$$\log \sec a = 1.23549$$

$$\log \cos b = 9.92521$$

$$b = 32^\circ 40' 8''.$$

$$\tan^2(45^\circ - \frac{1}{2}A)$$

$$= \tan \frac{1}{2}(c - a) \cot \frac{1}{2}(c + a).$$

$$\log \tan \frac{1}{2}(c - a) = 7.66330$$

$$\log \cot \frac{1}{2}(c + a) = \underline{8.72935}$$

$$\log \tan^2(45^\circ - \frac{1}{2}A) = 16.39265$$

$$\log \tan(45^\circ - \frac{1}{2}A) = 8.19632.$$

$$45^\circ - \frac{1}{2}A = 0^\circ 54' 1''.$$

$$A = 88^\circ 11' 58''.$$

$$\cos B = \tan a \cot c.$$

$$\log \tan a = 11.23475$$

$$\log \cot c = \underline{8.69024}$$

$$\log \cos B = 9.92499$$

$$B = 32^\circ 42' 53''.$$

3. $c = 59^\circ 4' 26''$, $a = 50^\circ$.

$$\cos b = \cos c \sec a.$$

$$\log \cos c = 9.71091$$

$$\log \sec a = \underline{0.19193}$$

$$\log \cos b = \underline{9.90284}$$

$$b = 36^\circ 54' 48''.$$

$$\sin A = \sin a \csc c.$$

$$\log \sin a = 9.88425$$

$$\log \csc c = \underline{0.06660}$$

$$\log \sin A = \underline{9.95085}$$

$$A = 63^\circ 15' 10''.$$

$$\cos B = \tan a \cot c.$$

$$\log \tan a = 10.07619$$

$$\log \cot c = \underline{9.77750}$$

$$\log \cos B = \underline{9.85369}$$

$$B = 44^\circ 26' 23''.$$

4. $c = 63^\circ 55' 43''$, $a = 120^\circ 10'$.

$$\cos b = \cos c \sec a.$$

$$\log \cos c = 9.64295$$

$$\log \sec a = \underline{0.29885} \ (n)$$

$$\log \cos b = \underline{9.94180} \ (n)$$

$$b = 150^\circ 59' 43''.$$

$$\sin A = \sin a \csc c.$$

$$\log \sin a = 9.93680$$

$$\log \csc c = \underline{0.04661}$$

$$\log \sin A = \underline{9.98341}$$

$$A = 105^\circ 44' 15''.$$

$$\cos B = \tan a \cot c.$$

$$\log \tan a = 10.23565 \ (n)$$

$$\log \cot c = \underline{9.68955}$$

$$\log \cos B = \underline{9.92520} \ (n)$$

$$B = 147^\circ 19' 45''.$$

5. $c = 55^\circ 9' 32''$, $a = 22^\circ 15' 7''$.

$$\cos b = \cos c \sec a.$$

$$\log \cos c = 9.75686$$

$$\log \sec a = \underline{0.03361}$$

$$\log \cos b = \underline{9.79047}$$

$$b = 51^\circ 53'.$$

$$\sin A = \sin a \csc c.$$

$$\log \sin a = 9.57828$$

$$\log \csc c = \underline{0.08579}$$

$$\log \sin A = \underline{9.66407}$$

$$A = 27^\circ 28' 38''.$$

$$\cos B = \tan a \cot c.$$

$$\log \tan a = 9.61188$$

$$\log \cot c = \underline{9.84266}$$

$$\log \cos B = \underline{9.45454}$$

$$B = 73^\circ 27' 11''.$$

6. $c = 44^\circ 33' 17''$, $a = 32^\circ 9' 17''$.

$$\cos b = \cos c \sec a.$$

$$\log \cos c = 9.85283$$

$$\log \sec a = \underline{0.07231}$$

$$\log \cos b = \underline{9.92514}$$

$$b = 32^\circ 41'.$$

$$\sin A = \sin a \csc c.$$

$$\log \sin a = 9.72608$$

$$\log \csc c = \underline{0.15391}$$

$$\log \sin A = \underline{9.87999}$$

$$A = 49^\circ 20' 16''.$$

$$\cos B = \tan a \cot c.$$

$$\log \tan a = 9.79840$$

$$\log \cot c = \underline{10.00675}$$

$$\log \cos B = \underline{9.80515}$$

$$B = 50^\circ 19' 16''.$$

$$7. \ c = 97^{\circ} 13' 4'', \ a = 132^{\circ} 14' 12''.$$

$$\cos b = \cos c \sec a.$$

$$\log \cos c = 9.09914 \ (n)$$

$$\log \sec a = 0.17250 \ (n)$$

$$\log \cos b = 9.27164$$

$$b = 79^{\circ} 13' 38''.$$

$$\sin A = \sin a \csc c.$$

$$\log \sin a = 9.86945$$

$$\log \csc c = 0.00345$$

$$\log \sin A = 9.87290$$

$$A = 131^{\circ} 43' 50''.$$

$$\cos B = \tan a \cot c.$$

$$\log \tan a = 10.04196 \ (n)$$

$$\log \cot c = 9.10259 \ (n)$$

$$\log \cos B = 9.14455$$

$$B = 81^{\circ} 58' 53''$$

$$8. \ c = 69^{\circ} 25' 11'', \ a = 50^{\circ}.$$

$$\cos b = \cos c \sec a.$$

$$\log \cos c = 9.54595$$

$$\log \sec a = 0.19193$$

$$\log \cos b = 9.73788$$

$$b = 56^{\circ} 50' 51''$$

$$\sin A = \sin a \csc c.$$

$$\log \sin a = 9.88425$$

$$\log \csc c = 0.02864$$

$$\log \sin A = 9.91289$$

$$A = 54^{\circ} 54' 40''.$$

$$\cos B = \tan a \cot c.$$

$$\log \tan a = 10.07619$$

$$\log \cot c = 9.57459$$

$$\log \cos B = 9.65078$$

$$B = 63^{\circ} 25' 2''.$$

$$9. \ c = 2^{\circ} 3' 56'', \ a = 2^{\circ} 0' 55''.$$

$$\tan^2 \frac{1}{2} b = \tan \frac{1}{2} (c - a) \tan \frac{1}{2} (c + a).$$

$$\log \tan \frac{1}{2} (c - a) = 6.64222$$

$$\log \tan \frac{1}{2} (c + a) = 8.55181$$

$$\log \tan^2 \frac{1}{2} b = 15.19403$$

$$\log \tan \frac{1}{2} b = 7.59702.$$

$$b = 0^{\circ} 27' 7''.$$

$$\sin A = \sin a \csc c.$$

$$\log \sin a = 8.54612$$

$$\log \csc c = 1.44318$$

$$\log \sin A = 9.98930$$

$$A = 77^{\circ} 20'.$$

$$\cos B = \tan a \cot c.$$

$$\log \tan a = 8.54639$$

$$\log \cot c = 11.44289$$

$$\log \cos B = 9.98928$$

$$B = 12^{\circ} 40' 40''.$$

$$10. \ c = 90^{\circ}, \ a = 90^{\circ}.$$

$$\sin A = \sin a \csc c.$$

$$\sin A = 1 \times 1 = 1.$$

$$\therefore A = 90^{\circ}.$$

$$a = A = c = C = 90^{\circ}$$

$$\cos c = \cos a \cos b.$$

$$\therefore \cos b = \frac{\cos c}{\cos a} = \frac{\cos 90^{\circ}}{\cos 90^{\circ}} = \frac{0}{0},$$

which is indefinite.

Therefore b is indeterminate, and, by § 172, (2), $B = b$.

11. A point on the equator in longitude $62^{\circ} 30' \text{ W.}$ is 85° from a point A on the prime meridian. What is the latitude of A ?

$$c = 85^{\circ}, \ a = 62^{\circ} 30'.$$

$$\cos b = \cos c \sec a.$$

$$\log \cos c = 8.94030$$

$$\log \sec a = 0.33559$$

$$\log \cos b = 9.27589$$

$$b = 79^{\circ} 7' 12''$$

In a right spherical triangle show that :

$$12. \cos^2 A \sin^2 c = \sin(c + a) \sin(c - a).$$

$$\sin A = \frac{\sin a}{\sin c}.$$

$$\sin^2 A = \frac{\sin^2 a}{\sin^2 c}.$$

$$\cos^2 A = 1 - \frac{\sin^2 a}{\sin^2 c} = \frac{\sin^2 c - \sin^2 a}{\sin^2 c}.$$

$$\cos^2 A \sin^2 c = \sin^2 c - \sin^2 a.$$

$$\sin(c + a) = \sin c \cos a + \cos c \sin a.$$

$$\sin(c - a) = \sin c \cos a - \cos c \sin a.$$

$$\begin{aligned} \sin(c + a) \sin(c - a) &= \sin^2 c \cos^2 a - \cos^2 c \sin^2 a \\ &= \sin^2 c (1 - \sin^2 a) - (1 - \sin^2 c) \sin^2 a \\ &= \sin^2 c - \sin^2 c \sin^2 a - \sin^2 a + \sin^2 c \sin^2 a \\ &= \sin^2 c - \sin^2 a. \end{aligned}$$

$$\therefore \cos^2 A \sin^2 c = \sin(c + a) \sin(c - a).$$

$$13. \tan a \cos c = \sin b \cot B.$$

$$\sin b = \tan a \cot A.$$

$$\cot A = \frac{\sin b}{\tan a}.$$

$$\cos c = \cot A \cot B.$$

$$\cot A = \frac{\cos c}{\cot B}.$$

$$\cos c = \frac{\sin b}{\tan a}.$$

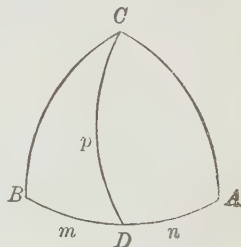
$$\therefore \tan a \cos c = \sin b \cot B.$$

14. If, in a right spherical triangle, p denotes the arc of the great circle passing through the vertex of the right angle and perpendicular to the hypotenuse, m and n the segments of the hypotenuse made by this arc adjacent to the sides a and b , show that $\tan^2 a = \tan c \tan m$, and that $\sin^2 p = \tan m \tan n$.

In the triangle BCA , by Napier's Rules,

$$\cos B = \tan a \cot c.$$

$$\tan a = \frac{\cos B}{\cot c}. \quad (1)$$



In the triangle CBD , by Napier's Rules,

$$\cos B = \tan BD \cot BC = \tan m \cot a.$$

$$\tan a = \frac{\tan m}{\cos B}. \quad (2)$$

Multiply (1) by (2), $\tan^2 a = \frac{\tan m}{\cos B} \times \frac{\cos B}{\cot c} = \tan m \tan c.$

In the triangle CBD , by Napier's Rules,

$$\sin p = \tan m \cot BCD. \quad (3)$$

In the triangle CAD , by Napier's Rules,

$$\sin p = \tan n \cot DCA. \quad (4)$$

Multiply (3) by (4), $\sin^2 p = \tan m \tan n \cot BCD \cot DCA$

But $BCD + DCA = 90^\circ.$

$$\therefore \cot BCD \times \cot DCA = 1.$$

$$\therefore \sin^2 p = \tan m \tan n.$$

Exercise 89. Given a Side and the Opposite Angle
(Page 201)

Solve the right spherical triangles, given a and A as follows:

1. $a = 50^\circ$, $A = 63^\circ 15' 13''.$

2. $a = 36^\circ 27'$, $A = 46^\circ 59' 43''.$

$$\sin c = \sin a \csc A.$$

$$\sin c = \sin a \csc A.$$

$$\log \sin a = 9.88425$$

$$\log \sin a = 9.77387$$

$$\log \csc A = 0.04915$$

$$\log \csc A = 0.13590$$

$$\log \sin c = 9.93340$$

$$\log \sin c = 9.90977$$

$$c = 59^\circ 4' 26''.$$

$$c = 54^\circ 19' 53''.$$

$$\sin b = \tan a \cot A.$$

$$\sin b = \tan a \cot A$$

$$\log \tan a = 10.07619$$

$$\log \tan a = 9.86842$$

$$\log \cot A = 9.70240$$

$$\log \cot A = 9.96973$$

$$\log \sin b = 9.77859$$

$$\log \sin b = 9.83815$$

$$b = 36^\circ 54' 49''.$$

$$b = 43^\circ 32' 32''$$

$$\sin B = \sec a \cos A.$$

$$\sin B = \sec a \cos A$$

$$\log \sec a = 0.19193$$

$$\log \sec a = 0.09454$$

$$\log \cos A = 9.65326$$

$$\log \cos A = 9.83382$$

$$\log \sin B = 9.84519$$

$$\log \sin B = 9.92836$$

$$B = 44^\circ 26' 18''$$

$$B = 57^\circ 59' 15''.$$

3. $a = 86^\circ 40'$, $A = 88^\circ 11' 58''$.

$$\sin c = \sin a \csc A.$$

$$\log \sin a = 9.99926$$

$$\log \csc A = 0.00021$$

$$\log \sin c = 9.99947$$

$$c = 87^\circ 10'.$$

$$\sin b = \tan a \cot A.$$

$$\log \tan a = 11.23475$$

$$\log \cot A = 8.49742$$

$$\log \sin b = 9.73217$$

$$b = 32^\circ 39' 54''.$$

$$\sin B = \sec a \cos A.$$

$$\log \sec a = 1.23549$$

$$\log \cos A = 8.49721$$

$$\log \sin B = 9.73270$$

$$B = 32^\circ 42' 35''.$$

4. $a = 120^\circ 10'$, $A = 105^\circ 44' 21''$.

$$\sin c = \sin a \csc A.$$

$$\log \sin a = 9.93680$$

$$\log \csc A = 0.01659$$

$$\log \sin c = 9.95339$$

$$c = 63^\circ 55' 40''.$$

$$\sin b = \tan a \cot A.$$

$$\log \tan a = 10.23565 (n)$$

$$\log \cot A = 9.44998 (n)$$

$$\log \sin b = 9.68563$$

$$b = 150^\circ 59' 44''.$$

$$\sin B = \sec a \cos A.$$

$$\log \sec a = 0.29885 (n)$$

$$\log \cos A = 9.43338 (n)$$

$$\log \sin B = 9.73223$$

$$B = 147^\circ 19' 48''.$$

5. $a = 115^\circ 30'$, $A = 110^\circ 10' 10''$.

$$\sin c = \sin a \csc A$$

$$\log \sin a = 9.95549$$

$$\log \csc A = 0.02749$$

$$\log \sin c = 9.98298$$

$$c = 74^\circ 3' 45''.$$

$$\sin b = \tan a \cot A.$$

$$\log \tan a = 10.32150 (n)$$

$$\log \cot A = 9.56504 (n)$$

$$\log \sin b = 9.88654$$

$$b = 129^\circ 38' 18''.$$

$$\sin B = \sec a \cos A.$$

$$\log \sec a = 0.36602 (n)$$

$$\log \cos A = 9.53757 (n)$$

$$\log \sin B = 9.90359$$

$$B = 126^\circ 46' 54''.$$

6. $a = 122^\circ 30'$, $A = 120^\circ 20' 20''$.

$$\sin c = \sin a \csc A.$$

$$\log \sin a = 9.92603$$

$$\log \csc A = 0.06396$$

$$\log \sin c = 9.98999$$

$$c = 77^\circ 44' 40''.$$

$$\sin b = \tan a \cot A.$$

$$\log \tan a = 10.19581 (n)$$

$$\log \cot A = 9.76735 (n)$$

$$\log \sin b = 9.96316$$

$$b = 113^\circ 16'.$$

$$\sin B = \sec a \cos A.$$

$$\log \sec a = 0.26978 (n)$$

$$\log \cos A = 9.70339 (n)$$

$$\log \sin B = 9.97317$$

$$B = 109^\circ 56'.$$

$$7. a = 22^\circ 15' 7'', A = 27^\circ 28' 38''.$$

$$\sin c = \sin a \csc A.$$

$$\log \sin a = 9.57828$$

$$\log \csc A = \underline{0.33593}$$

$$\log \sin c = \underline{9.91421}$$

$$c = 55^\circ 9' 33''.$$

$$\sin b = \tan a \cot A.$$

$$\log \tan a = 9.61188$$

$$\log \cot A = \underline{10.28394}$$

$$\log \sin b = \underline{9.89582}$$

$$b = 51^\circ 52' 48''.$$

$$\sin B = \sec a \cos A.$$

$$\log \sec a = 0.03361$$

$$\log \cos A = \underline{9.94802}$$

$$\log \sin B = \underline{9.98163}$$

$$B = 73^\circ 27' 15''.$$

$$8. a = 14^\circ 16' 35'', A = 37^\circ 36' 49''.$$

$$\sin c = \sin a \csc A.$$

$$\log \sin a = 9.39199$$

$$\log \csc A = \underline{0.21443}$$

$$\log \sin c = \underline{9.60642}$$

$$c = 23^\circ 49' 51''.$$

$$\sin b = \tan a \cot A.$$

$$\log \tan a = 9.40562$$

$$\log \cot A = \underline{10.11324}$$

$$\log \sin b = \underline{9.51886}$$

$$b = 19^\circ 17' 5''.$$

$$\sin B = \sec a \cos A.$$

$$\log \sec a = 0.01362$$

$$\log \cos A = \underline{9.89881}$$

$$\log \sin B = \underline{9.91243}$$

$$B = 54^\circ 49' 27''.$$

$$9. a = 32^\circ 9' 17'', A = 49^\circ 20' 16''.$$

$$\sin c = \sin a \csc A.$$

$$\log \sin a = 9.72608$$

$$\log \csc A = \underline{0.12001}$$

$$\log \sin c = \underline{9.84609}$$

$$c = 44^\circ 33' 18''.$$

$$\sin b = \tan a \cot A.$$

$$\log \tan a = 9.79840$$

$$\log \cot A = \underline{9.93399}$$

$$\log \sin b = \underline{9.73239}$$

$$b = 32^\circ 41'.$$

$$\sin B = \sec a \cos A.$$

$$\log \sec a = 0.07231$$

$$\log \cos A = \underline{9.81398}$$

$$\log \sin B = \underline{9.88629}$$

$$B = 50^\circ 19' 18''.$$

$$10. a = 77^\circ 21' 50'', A = 40^\circ 40' 40''$$

$$\sin c = \sin a \csc A.$$

$$\text{But } \sin A < \sin c.$$

$$\therefore \sin c > 1, \text{ which is impossible.}$$

$$11. a = 77^\circ 21' 50'', A = 83^\circ 56' 40''$$

$$\sin c = \sin a \csc A.$$

$$\log \sin a = 9.98935$$

$$\log \csc A = \underline{0.00243}$$

$$\log \sin c = \underline{9.99178}$$

$$c = 78^\circ 53' 20'' \text{ or } 101^\circ 6' 40''.$$

$$\sin b = \tan a \cot A.$$

$$\log \tan a = 10.64939$$

$$\log \cot A = \underline{9.02565}$$

$$\log \sin b = \underline{9.67504}$$

$$b = 28^\circ 14' 31'' \text{ or } 151^\circ 45' 29''.$$

$$\sin B = \sec a \cos A.$$

$$\log \sec a = 0.66004$$

$$\log \cos A = \underline{9.02323}$$

$$\log \sin B = \underline{9.68327}$$

$$B = 28^\circ 49' 57'' \text{ or } 151^\circ 10' 3''.$$

$$12. a = 132^\circ 14' 12'', A = 131^\circ 43' 50''.$$

$$\sin c = \sin a \csc A, \quad \sin b = \tan a \cot A, \quad \sin B = \sec a \cos A.$$

$$\log \sin a = 9.86945 \quad \log \tan a = 10.04196 (n) \quad \log \sec a = 0.17250 (n)$$

$$\log \csc A = 0.12710 \quad \log \cot A = 9.95033 (n) \quad \log \cos A = 9.82323 (n)$$

$$\log \sin c = 9.99655 \quad \log \sin b = 9.99229 \quad \log \sin B = 9.99573$$

$$c = 97^\circ 13'.$$

$$b = 79^\circ 14'.$$

$$B = 81^\circ 58' 30''.$$

In a right spherical triangle show that :

$$13. \sin^2 A = \cos^2 B + \sin^2 a \sin^2 B.$$

$$\cos B = \cos b \sin A.$$

$$\therefore \sin A = \frac{\cos B}{\cos b}.$$

$$\sin^2 A = \frac{\cos^2 B}{\cos^2 b}$$

$$= \cos^2 B \sec^2 b$$

$$= \cos^2 B (1 + \tan^2 b)$$

$$= \cos^2 B + \tan^2 b \cos^2 B. \quad (1)$$

$$\sin a = \tan b \cot B.$$

$$\tan b = \sin a \tan B.$$

$$\text{Substitute in (1),} \quad \sin^2 A = \cos^2 B + \sin^2 a \tan^2 B \cos^2 B.$$

$$\therefore \sin^2 A = \cos^2 B + \sin^2 a \sin^2 B.$$

$$14. \sin(b + c) = 2 \cos^2 \frac{1}{2} A \cos b \sin c.$$

$$\sin(b + c) = \sin b \cos c + \cos b \sin c$$

$$= \left(\frac{\sin b \cos c}{\cos b \sin c} + 1 \right) \cos b \sin c$$

$$= (\tan b \cot c + 1) \cos b \sin c. \quad (1)$$

$$\tan b \cot c = \cos A.$$

$$\tan b \cot c + 1 = \cos A + 1$$

$$= 2 \cos^2 \frac{1}{2} A.$$

$$\text{Substitute in (1),} \quad \sin(b + c) = 2 \cos^2 \frac{1}{2} A \cos b \sin c.$$

$$15. \sin(c - b) = 2 \sin^2 \frac{1}{2} A \cos b \sin c.$$

$$\sin(c - b) = \sin c \cos b - \cos c \sin b$$

$$= \sin c \cos b \left(1 - \frac{\cos c \sin b}{\sin c \cos b} \right)$$

$$= \sin c \cos b (1 - \cot c \tan b). \quad (1)$$

$$\cot c \tan b = \cos A.$$

$$1 - \cot c \tan b = 1 - \cos A$$

$$= 2 \sin^2 \frac{1}{2} A.$$

$$\text{Substitute in (1),} \quad \sin(c - b) = 2 \sin^2 \frac{1}{2} A \cos b \sin c.$$

Exercise 90. Given a Side and an Adjacent Angle
(Page 202)

Solve the right spherical triangles, given the following parts:

1. $a = 54^\circ 30'$, $B = 35^\circ 30'$.

$$\tan c = \tan a \sec B.$$

$$\log \tan a = 10.14673$$

$$\log \sec B = \underline{0.08931}$$

$$\log \tan c = 10.23604$$

$$c = 59^\circ 51' 21''.$$

$$\tan b = \sin a \tan B.$$

$$\log \sin a = 9.91069$$

$$\log \tan B = \underline{9.85327}$$

$$\log \tan b = \underline{9.76396}$$

$$b = 30^\circ 8' 39''.$$

$$\cos A = \cos a \sin B.$$

$$\log \cos a = 9.76395$$

$$\log \sin B = \underline{9.76395}$$

$$\log \cos A = \underline{9.52790}$$

$$A = 70^\circ 17' 35''.$$

2. $a = 92^\circ 47' 32''$, $B = 50^\circ 2' 1''$.

$$\tan c = \tan a \sec B.$$

$$\log \tan a = 11.31183 \text{ (}n\text{)}$$

$$\log \sec B = \underline{0.19223}$$

$$\log \tan c = 11.50406 \text{ (}n\text{)}$$

$$c = 91^\circ 47' 40''.$$

$$\tan b = \sin a \tan B.$$

$$\log \sin a = 9.99948$$

$$\log \tan B = \underline{10.07670}$$

$$\log \tan b = \underline{10.07618}$$

$$b = 49^\circ 59' 58''.$$

$$\cos A = \cos a \sin B.$$

$$\log \cos a = 8.68765 \text{ (}n\text{)}$$

$$\log \sin B = \underline{9.88447}$$

$$\log \cos A = 8.57212 \text{ (}n\text{)}$$

$$A = 92^\circ 8' 23''.$$

3. $a = 20^\circ 20' 20''$, $B = 38^\circ 10' 10''$.

$$\tan c = \tan a \sec B.$$

$$\log \tan a = 9.56900$$

$$\log \sec B = \underline{0.10448}$$

$$\log \tan c = \underline{9.67348}$$

$$c = 25^\circ 14' 38''.$$

$$\tan b = \sin a \tan B.$$

$$\log \sin a = 9.54104$$

$$\log \tan B = \underline{9.89545}$$

$$\log \tan b = \underline{9.43649}$$

$$b = 15^\circ 16' 50''.$$

$$\cos A = \cos a \sin B.$$

$$\log \cos a = 9.97204$$

$$\log \sin B = \underline{9.79098}$$

$$\log \cos A = \underline{9.76302}$$

$$A = 54^\circ 35' 17''.$$

4. $a = 50^\circ$, $B = 63^\circ 25' 4''$.

$$\tan c = \tan a \sec B.$$

$$\log \tan a = 10.07619$$

$$\log \sec B = \underline{0.34923}$$

$$\log \tan c = \underline{10.42542}$$

$$c = 69^\circ 25' 13''.$$

$$\tan b = \sin a \tan B.$$

$$\log \sin a = 9.88425$$

$$\log \tan B = \underline{10.30070}$$

$$\log \tan b = \underline{10.18495}$$

$$b = 56^\circ 50' 49''.$$

$$\cos A = \cos a \sin B.$$

$$\log \cos a = 9.80807$$

$$\log \sin B = \underline{9.95148}$$

$$\log \cos A = 9.75955$$

$$A = 54^\circ 54' 40''.$$

5. $a = 50^\circ$, $B = 120^\circ 3' 50''$.

$$\tan c = \tan a \sec B.$$

$$\log \tan a = 10.07619$$

$$\log \sec B = \underline{0.30019} \ (n)$$

$$\log \tan c = 10.37638 \ (n)$$

$$c = 112^\circ 48'.$$

$$\tan b = \sin a \tan B.$$

$$\log \sin a = 9.88425$$

$$\log \tan B = \underline{10.23744} \ (n)$$

$$\log \tan b = 10.12169 \ (n)$$

$$b = 127^\circ 4' 32''.$$

$$\cos A = \cos a \sin B.$$

$$\log \cos a = 9.80807$$

$$\log \sin B = \underline{9.93725}$$

$$\log \cos A = 9.74532$$

$$A = 56^\circ 11' 57''.$$

6. $c = 91^\circ 47' 40''$, $A = 92^\circ 8' 23''$.

$$\tan b = \tan c \cos A.$$

$$\log \tan c = 11.50406 \ (n)$$

$$\log \cos A = \underline{8.57213} \ (n)$$

$$\log \tan b = 10.07619$$

$$b = 50^\circ.$$

$$\sin b = \tan a \cot A.$$

$$\tan a = \sin b \tan A.$$

$$\log \sin b = 9.88425$$

$$\log \tan A = \underline{11.42756} \ (n)$$

$$\log \tan a = 11.31181 \ (n)$$

$$a = 92^\circ 47' 33''.$$

$$\cot B = \cos c \tan A.$$

$$\log \cos c = 8.49574 \ (n)$$

$$\log \tan A = \underline{11.42756} \ (n)$$

$$\log \cot B = \underline{9.92330}$$

$$B = 50^\circ 2'.$$

7. $c = 25^\circ 14' 38''$, $A = 54^\circ 35' 17''$.

$$\sin a = \sin c \sin A.$$

$$\log \sin c = 9.62989$$

$$\log \sin A = \underline{9.91117}$$

$$\log \sin a = 9.54106$$

$$a = 20^\circ 20' 23''.$$

$$\tan b = \tan c \cos A.$$

$$\log \tan c = 9.67348$$

$$\log \cos A = \underline{9.76302}$$

$$\log \tan b = 9.43650$$

$$b = 15^\circ 16' 52''.$$

$$\cot B = \cos c \tan A.$$

$$\log \cos c = 9.95641$$

$$\log \tan A = \underline{10.14815}$$

$$\log \cot B = 10.10456$$

$$B = 38^\circ 10' 7''.$$

8. $c = 59^\circ 51' 21''$, $A = 70^\circ 17' 35''$.

$$\sin a = \sin c \sin A.$$

$$\log \sin c = 9.93690$$

$$\log \sin A = \underline{9.97379}$$

$$\log \sin a = 9.91069$$

$$a = 54^\circ 30'.$$

$$\tan b = \tan c \cos A.$$

$$\log \tan c = 10.23604$$

$$\log \cos A = \underline{9.52790}$$

$$\log \tan b = 9.76394$$

$$b = 30^\circ 8' 35''.$$

$$\cot B = \cos c \tan A.$$

$$\log \cos c = 9.70086$$

$$\log \tan A = \underline{10.44589}$$

$$\log \cot B = \underline{10.14675}$$

$$B = 35^\circ 29' 56''.$$

9. $c = 112^\circ 48'$, $A = 56^\circ 11' 56''$.

10. $c = 2^\circ 3' 56''$, $A = 77^\circ 20' 28''$.

$$\sin a = \sin c \sin A.$$

$$\sin a = \sin c \sin A.$$

$$\log \sin c = 9.96467$$

$$\log \sin c = 8.55682$$

$$\log \sin A = 9.91958$$

$$\log \sin A = 9.98931$$

$$\log \sin a = 9.88425$$

$$\log \sin a = 8.54613$$

$$a = 50^\circ.$$

$$a = 2^\circ 0' 55''.$$

$$\tan b = \tan c \cos A.$$

$$\tan b = \tan c \cos A.$$

$$\log \tan c = 10.37638 (n)$$

$$\log \tan c = 8.55711$$

$$\log \cos A = 9.74532$$

$$\log \cos A = 9.34073$$

$$\log \tan b = 10.12170 (n)$$

$$\log \tan b = 7.89784$$

$$b = 127^\circ 4' 30''.$$

$$b = 0^\circ 27' 10''.$$

$$\cot B = \cos c \tan A.$$

$$\cot B = \cos c \tan A.$$

$$\log \cos c = 9.58829 (n)$$

$$\log \cos c = 9.99972$$

$$\log \tan A = 10.17427$$

$$\log \tan A = 10.64858$$

$$\log \cot B = 9.76256 (n)$$

$$\log \cot B = 10.64830$$

$$B = 120^\circ 3' 50''.$$

$$B = 12^\circ 40'.$$

11. Define a quadrantal triangle, and show how its solution may be reduced to that of the right triangle.

A quadrantal triangle is a triangle that has one or more of its sides equal to a quadrant.

Let $A'B'C'$ be a quadrantal triangle with side $A'B' = 90^\circ$, or a quadrant.

Let ABC be its polar triangle.

Then, since $A'B' + C = 180^\circ$, $C = 90^\circ$.

Hence ABC is a right triangle.

Therefore all parts of the polar triangle may be found by formulas for the right triangle.

The parts of $A'B'C'$ may then be found by subtracting proper parts of ABC from 180° .

12. Solve the quadrantal triangle the sides of which are $a = 174^\circ 12' 49''$, $b = 94^\circ 8' 20''$, $c = 90^\circ$.

Let A' , B' , C' , a' , b' , c' represent the corresponding angles and sides of the polar triangle.

Then $A' = 5^\circ 47' 11''$, $B' = 85^\circ 51' 40''$, $C' = 90^\circ$.

By 4, page 194, $\tan^2 \frac{1}{2} c' = -\cos(B' + A') \sec(B' - A')$.

By 5, page 194, $\tan^2 \frac{1}{2} b' = \tan[\frac{1}{2}(B' + A') - 45^\circ] \tan[45^\circ + \frac{1}{2}(B' - A')]$.

$$\tan^2 \frac{1}{2} a' = \tan[\frac{1}{2}(B' + A') - 45^\circ] \tan[45^\circ - \frac{1}{2}(B' - A')]$$

$$\begin{array}{rcl}
B' + A' = 91^\circ 38' 51'' & \log \tan [\tfrac{1}{2}(B' + A') - 45^\circ] = & 8.15770 \\
B' - A' = 80^\circ 4' 29'' & \log \tan [45^\circ + \tfrac{1}{2}(B' - A')] = & 11.06133 \\
\tfrac{1}{2}(B' + A') - 45^\circ = 0^\circ 49' 25.5'' & & 2) \underline{9.21903} \\
45^\circ + \tfrac{1}{2}(B' - A') = 85^\circ 2' 14.5'' & \log \tan \tfrac{1}{2}b' = & 9.60952 \\
45^\circ - \tfrac{1}{2}(B' - A') = 4^\circ 57' 45.5'' & \tfrac{1}{2}b' = & 22^\circ 8' 35'' \\
& b' = & 44^\circ 17' 10'' \\
& B = & 135^\circ 42' 50'' \\
\log \cos(B' + A') = & 8.45864 \\
\log \sec(B' - A') = & 0.76356 \\
& 2) \underline{9.22220} \\
\log \tan \tfrac{1}{2}c' = & 9.61110 \\
\tfrac{1}{2}c' = & 22^\circ 12' 56\frac{2}{3}'' \\
c' = & 44^\circ 25' 53'' \\
C = & 135^\circ 34' 7'' \\
& \log \tan [\tfrac{1}{2}(B' + A') - 45^\circ] = & 8.15770 \\
& \log \tan [45^\circ - \tfrac{1}{2}(B' - A')] = & 8.93867 \\
& & 2) \underline{7.09637} \\
& \log \tan \tfrac{1}{2}a' = & 8.54819 \\
& \tfrac{1}{2}a' = & 2^\circ 1' 25'' \\
& a' = & 4^\circ 2' 50'' \\
& A = & 175^\circ 57' 10''
\end{array}$$

Solve the right spherical triangles, given the following parts:

13. $c = 55^\circ, b = 45^\circ.$

14. $c = 65^\circ, A = 75^\circ.$

$$\begin{array}{rcl}
\cos c = \cos a \cos b. & \sin a = \sin c \sin A. \\
\cos a = \cos c \sec b. & \\
\log \cos c = 9.75859 & \log \sin c = 9.95728 \\
\log \sec b = 0.15051 & \log \sin A = 9.98494 \\
\log \cos a = 9.90910 & \log \sin a = 9.94222 \\
& a = 61^\circ 5' 43''. \\
a = 35^\circ 47' 33''. & \cos A = \tan b \cot c. \\
\cos A = \tan b \cot c. & \tan b = \tan c \cos A \\
\log \tan b = 10.00000 & \log \tan c = 10.33133 \\
\log \cot c = 9.84523 & \log \cos A = 9.41300 \\
\log \cos A = 9.84523 & \log \tan b = 9.74433 \\
A = 45^\circ 33' 23''. & b = 29^\circ 1' 56''. \\
\sin b = \sin c \sin B. & \cos c = \cot A \cot B \\
\sin B = \sin b \csc c. & \cot B = \cos c \tan A. \\
\log \sin b = 9.84949 & \log \cos c = 9.62595 \\
\log \csc c = 0.08664 & \log \tan A = 10.57195 \\
\log \sin B = 9.93613 & \log \cot B = 10.19790 \\
B = 59^\circ 40' 52''. & B = 32^\circ 22' 32''.
\end{array}$$

15. $a = 110^\circ$, $B = 45^\circ$.

$$\cos A = \cos a \sin B.$$

$$\log \cos a = 9.53405 \text{ (n)}$$

$$\log \sin B = \underline{9.84949}$$

$$\log \cos A = \underline{9.38354 \text{ (n)}}$$

$$A = 103^\circ 59' 44''.$$

$$\sin a = \tan b \cot B.$$

$$\tan b = \sin a \tan B.$$

$$\log \sin a = 9.97299$$

$$\log \tan B = \underline{10.00000}$$

$$\log \tan b = \underline{9.97299}$$

$$b = 43^\circ 13' 10''.$$

$$\cos B = \tan a \cot c.$$

$$\tan c = \tan a \sec B.$$

$$\log \tan a = 10.43893 \text{ (n)}$$

$$\log \sec B = \underline{0.15051}$$

$$\log \tan c = \underline{10.58944 \text{ (n)}}$$

$$c = 104^\circ 25' 59''.$$

16. $A = 78^\circ$, $c = 70^\circ$.

$$\sin a = \sin c \sin A.$$

$$\log \sin c = 9.97299$$

$$\log \sin A = \underline{9.99040}$$

$$\log \sin a = \underline{9.96339}$$

$$a = 66^\circ 48' 12''.$$

$$\tan b = \tan c \cos A.$$

$$\log \tan c = 10.43893$$

$$\log \cos A = \underline{9.31788}$$

$$\log \tan b = \underline{9.75681}$$

$$b = 29^\circ 44' 10''.$$

$$\cot B = \cos c \tan A.$$

$$\log \cos c = 9.53405$$

$$\log \tan A = \underline{10.67253}$$

$$\log \cot B = \underline{10.20658}$$

$$B = 31^\circ 51' 34''.$$

17. $c = 50^\circ$, $b = 44^\circ 18' 39''$.

$$\cos c = \cos a \cos b.$$

$$\cos a = \cos c \sec b.$$

$$\log \cos c = 9.80807$$

$$\log \sec b = \underline{0.14535}$$

$$\log \cos a = \underline{9.95342}$$

$$a = 26^\circ 3' 51''.$$

$$\cos A = \tan b \cot c.$$

$$\log \tan b = 9.98955$$

$$\log \cot c = \underline{9.92381}$$

$$\log \cos A = \underline{9.91336}$$

$$A = 35^\circ.$$

$$\sin b = \sin c \sin B.$$

$$\sin B = \sin b \csc c.$$

$$\log \sin b = 9.84419$$

$$\log \csc c = \underline{0.11575}$$

$$\log \sin B = \underline{9.95994}$$

$$B = 65^\circ 46'.$$

18. $A = 156^\circ 20' 30''$, $a = 65^\circ 15' 45''$.

The triangle is impossible, because a and A are unlike in kind.

19. $A = 74^\circ 12' 31''$, $c = 64^\circ 28' 47''$.

$$\sin a = \sin c \sin A.$$

$$\log \sin c = 9.95542$$

$$\log \sin A = \underline{9.98329}$$

$$\log \sin a = \underline{9.93871}$$

$$a = 60^\circ 16' 17''.$$

$$\cos A = \tan b \cot c.$$

$$\tan b = \tan c \cos A.$$

$$\log \tan c = 10.32111$$

$$\log \cos A = \underline{9.43479}$$

$$\log \tan b = \underline{9.75590}$$

$$b = 29^\circ 41' 4''.$$

$$\cos c = \cot A \cot B$$

$$\cot B = \cos c \tan A.$$

$$\log \cos c = 9.63431$$

$$\log \tan A = \underline{10.54851}$$

$$\log \cot B = \underline{10.18282}$$

$$B = 33^\circ 16' 54''$$

20. $a = 112^\circ 42' 38''$, $B = 44^\circ 28' 44''$.

$$\sin a = \tan b \cot B.$$

$$\tan b = \sin a \tan B.$$

$$\log \sin a = 9.96495$$

$$\log \tan B = 9.99210$$

$$\log \tan b = 9.95705$$

$$b = 42^\circ 10' 17''.$$

$$\cos B = \tan a \cot c.$$

$$\tan c = \tan a \sec B.$$

$$\log \tan a = 10.37828 \text{ (n)}$$

$$\log \sec B = 0.14660$$

$$\log \tan c = 10.52488 \text{ (n)}$$

$$c = 106^\circ 37' 37''.$$

$$\cos A = \cos a \sin B.$$

$$\log \cos a = 9.58667 \text{ (n)}$$

$$\log \sin B = 9.84550$$

$$\log \cos A = 9.43217 \text{ (n)}$$

$$A = 105^\circ 41' 39''.$$

Exercise 91. Given the Two Angles (Page 203)

Solve the right spherical triangles, given A and B as follows:

1. $A = 63^\circ 15' 12''$,

$B = 135^\circ 33' 39''$.

2. $A = 116^\circ 43' 12''$,

$B = 116^\circ 31' 25''$.

$$\cos a = \cos A \csc B.$$

$$\log \cos A = 9.65326$$

$$\log \csc B = 0.15480$$

$$\log \cos a = 9.80806$$

$$a = 50^\circ 0' 4''.$$

$$\cos a = \cos A \csc B.$$

$$\log \cos A = 9.65286 \text{ (n)}$$

$$\log \csc B = 0.04830$$

$$\log \cos a = 9.70116 \text{ (n)}$$

$$a = 120^\circ 10' 3''.$$

$$\cos b = \cos B \csc A.$$

$$\log \cos B = 9.85369 \text{ (n)}$$

$$\log \csc A = 0.04915$$

$$\log \cos b = 9.90284 \text{ (n)}$$

$$b = 143^\circ 5' 12''.$$

$$\cos b = \cos B \csc A.$$

$$\log \cos B = 9.64988 \text{ (n)}$$

$$\log \csc A = 0.04904$$

$$\log \cos b = 9.69892 \text{ (n)}$$

$$b = 119^\circ 59' 46''.$$

$$\cos c = \cot A \cot B.$$

$$\log \cot A = 9.70241$$

$$\log \cot B = 10.00850 \text{ (n)}$$

$$\log \cos c = 9.71091 \text{ (n)}$$

$$c = 120^\circ 55' 34''.$$

$$\cos c = \cot A \cot B.$$

$$\log \cot A = 9.70190 \text{ (n)}$$

$$\log \cot B = 9.69818 \text{ (n)}$$

$$\log \cos c = 9.40008$$

$$c = 75^\circ 26' 58''.$$

$$\begin{aligned} 3. \quad A &= 46^\circ 59' 43'', \\ B &= 57^\circ 59' 19''. \end{aligned}$$

$$\cos a = \cos A \csc B.$$

$$\log \cos A = 9.83382$$

$$\log \csc B = 0.07163$$

$$\log \cos a = 9.90545$$

$$a = 36^\circ 27' 7''.$$

$$\cos b = \cos B \csc A.$$

$$\log \cos B = 9.72435$$

$$\log \csc A = 0.13591$$

$$\log \cos b = 9.86026$$

$$b = 43^\circ 32' 30''.$$

$$\cos c = \cot A \cot B.$$

$$\log \cot A = 9.96973$$

$$\log \cot B = 9.79598$$

$$\log \cos c = 9.76571$$

$$c = 54^\circ 20' 3''.$$

$$4. \quad A = 90^\circ, \quad B = 88^\circ 24' 35''.$$

$$\cos a = \cos A \csc B.$$

$$\cos A = 0.$$

$$\cos a = 0.$$

$$a = 90^\circ.$$

$$\cos b = \cos B \csc A.$$

$$\csc A = 1.$$

$$b = B.$$

$$b = 88^\circ 24' 35''.$$

$$\cos c = \cot A \cot B.$$

$$\cot A = 0.$$

$$\cos c = 0.$$

$$c = 90^\circ.$$

$$5. \quad A = 92^\circ 8' 23'', \quad B = 50^\circ 2' 1''$$

$$\cos a = \cos A \csc B.$$

$$\log \cos A = 8.57213 \text{ (n)}$$

$$\log \csc B = 0.11553$$

$$\log \cos a = 8.68766 \text{ (n)}$$

$$a = 92^\circ 47' 32''.$$

$$\cos b = \cos B \csc A.$$

$$\log \cos B = 9.80777$$

$$\log \csc A = 0.00030$$

$$\log \cos b = 9.80807$$

$$b = 50^\circ.$$

$$\cos c = \cot A \cot B.$$

$$\log \cot A = 8.57244 \text{ (n)}$$

$$\log \cot B = 9.92330$$

$$\log \cos c = 8.49574 \text{ (n)}$$

$$c = 91^\circ 47' 40''.$$

$$6. \quad A = 77^\circ 20' 28'', \quad B = 12^\circ 40'.$$

$$\tan^2 \frac{1}{2} a = \tan \left[\frac{1}{2} (A + B) - 45^\circ \right] \tan \left[\frac{1}{2} (A - B) + 45^\circ \right].$$

$$\frac{1}{2} (A + B) - 45^\circ = 0^\circ 0' 14''.$$

$$\frac{1}{2} (A - B) + 45^\circ = 77^\circ 20' 14''.$$

$$\log \tan \left[\frac{1}{2} (A + B) - 45^\circ \right] = 5.83170$$

$$\log \tan \left[\frac{1}{2} (A - B) + 45^\circ \right] = 10.64844$$

$$2 \overline{) 16.48014}$$

$$\log \tan \frac{1}{2} a = 8.24007$$

$$\frac{1}{2} a = 0^\circ 59' 45''.$$

$$a = 1^\circ 59' 30''$$

$$\tan^2 \frac{1}{2} c = -\cos(A+B) \sec(A-B).$$

$$A+B = 90^\circ 0' 28''.$$

$$A-B = 64^\circ 40' 28''.$$

$$-\log \cos(A+B) = 6.13273$$

$$\log \sec(A-B) = 0.36880$$

$$2) 16.50153$$

$$\log \tan \frac{1}{2} c = 8.25077$$

$$\frac{1}{2} c = 1^\circ 1' 14''.$$

$$c = 2^\circ 2' 28''.$$

$$\tan^2 \frac{1}{2} b = \tan \frac{1}{2} (c+a) \tan \frac{1}{2} (c-a).$$

$$\log \tan \frac{1}{2} (c+a) = 8.54663$$

$$\log \tan \frac{1}{2} (c-a) = 6.63496$$

$$2) 15.18159$$

$$\log \tan \frac{1}{2} b = 7.59080$$

$$\frac{1}{2} b = 0^\circ 13' 24''.$$

$$b = 0^\circ 26' 48''.$$

7. $A = 54^\circ 35' 17''$, $B = 38^\circ 10' 10''$.

8. $A = 70^\circ 17' 35''$, $B = 35^\circ 30'$

$$\cos a = \cos A \csc B.$$

$$\cos a = \cos A \csc B.$$

$$\log \cos A = 9.76302$$

$$\log \cos A = 9.52790$$

$$\log \csc B = 0.20902$$

$$\log \csc B = 0.23605$$

$$\log \cos a = 9.97204$$

$$\log \cos a = 9.76395$$

$$a = 20^\circ 20' 24''.$$

$$a = 54^\circ 30'.$$

$$\cos b = \cos B \csc A.$$

$$\cos b = \cos B \csc A.$$

$$\log \cos B = 9.89552$$

$$\log \cos B = 9.91069$$

$$\log \csc A = 0.08883$$

$$\log \csc A = 0.02621$$

$$\log \cos b = 9.98435$$

$$\log \cos b = 9.93690$$

$$b = 15^\circ 17' 20''.$$

$$b = 30^\circ 8' 38''.$$

$$\cos c = \cot A \cot B.$$

$$\cos c = \cot A \cot B.$$

$$\log \cot A = 9.85185$$

$$\log \cot A = 9.55411$$

$$\log \cot B = 10.10455$$

$$\log \cot B = 10.14673$$

$$\log \cos c = 9.95640$$

$$\log \cos c = 9.70084$$

$$c = 25^\circ 14' 50''.$$

$$c = 59^\circ 51' 26''.$$

$$9. A = 54^\circ 54' 42'', B = 63^\circ 25' 4''. \quad 10. A = 56^\circ 11' 56'', B = 120^\circ 3' 50''$$

$$\cos a = \cos A \csc B.$$

$$\log \cos A = 9.75954$$

$$\log \csc B = 0.04852$$

$$\log \cos a = 9.80806$$

$$a = 50^\circ 0' 4''.$$

$$\cos b = \cos B \csc A.$$

$$\log \cos B = 9.65077$$

$$\log \csc A = 0.08711$$

$$\log \cos b = 9.73788$$

$$b = 56^\circ 50' 51''.$$

$$\cos c = \cot A \cot B.$$

$$\log \cot A = 9.84665$$

$$\log \cot B = 9.69930$$

$$\log \cos c = 9.54595$$

$$c = 69^\circ 25' 11''.$$

$$\cos a = \cos A \csc B.$$

$$\log \cos A = 9.74532$$

$$\log \csc B = 0.06275$$

$$\log \cos a = 9.80807$$

$$a = 50^\circ.$$

$$\cos b = \cos B \csc A.$$

$$\log \cos B = 9.69980 (n)$$

$$\log \csc A = 0.08042$$

$$\log \cos b = 9.78022 (n)$$

$$b = 127^\circ 4' 32''.$$

$$\cos c = \cot A \cot B.$$

$$\log \cot A = 9.82573$$

$$\log \cot B = 9.76256 (n)$$

$$\log \cos c = 9.58829 (n)$$

$$c = 112^\circ 48'.$$

Exercise 92. Isosceles Triangles (Page 204)

Solve the isosceles spherical triangles, given:

$$1. c = 50^\circ, a = 30^\circ.$$

Denote the elements of one of the right triangles by A' , B' , C' , and a' , b' , c' , where C' is the right angle.

$$a' = \frac{1}{2} \cdot 30^\circ = 15^\circ, c' = 50^\circ.$$

In the $\triangle A'B'C'$, given the hypotenuse c' and a side a' ,

$$\sin A' = \sin a' \csc c'.$$

$$\log \sin a' = 9.41300$$

$$\log \csc c' = 0.11575$$

$$\log \sin A' = 9.52875$$

$$A' = 19^\circ 44' 50''.$$

$$\cos B' = \tan a' \cot c'.$$

$$\log \tan a' = 9.42805$$

$$\log \cot c' = 9.92381$$

$$\log \cos B' = 9.35186$$

$$B' = 77^\circ 0' 25''.$$

$$B = C = B' = 77^\circ 0' 25''.$$

$$A = 2A' = 39^\circ 29' 40''.$$

$$b = c = 50^\circ.$$

$$2. c = 60^\circ, a = 40^\circ.$$

$$a' = \frac{1}{2} \cdot 40^\circ = 20^\circ, c' = 60^\circ.$$

$$\sin A' = \sin a' \csc c'.$$

$$\log \sin a' = 9.53405$$

$$\log \csc c' = 0.06247$$

$$\log \sin A' = 9.59652$$

$$A' = 23^\circ 15' 41''.$$

$$\cos B' = \tan a' \cot c'.$$

$$\log \tan a' = 9.56107$$

$$\log \cot c' = 9.76144$$

$$\log \cos B' = 9.32251$$

$$B' = 77^\circ 52' 10''.$$

$$B = C = B' = 77^\circ 52' 10''.$$

$$A = 2A' = 46^\circ 31' 22''$$

$$b = c = 60^\circ.$$

$$3. \ c = 62^\circ 37', \ a = 49^\circ 10'.$$

$$a' = \frac{1}{2} \cdot 49^\circ 10' = 24^\circ 35', \ c' = 62^\circ 37'.$$

$$\sin A' = \sin a' \csc c'.$$

$$\log \sin a' = 9.61911$$

$$\log \csc c' = \underline{0.05161}$$

$$\log \sin A' = \underline{9.67072}$$

$$A' = 27^\circ 56' 15''.$$

$$\cos B' = \tan a' \cot c'.$$

$$\log \tan a' = 9.66038$$

$$\log \cot c' = \underline{9.71431}$$

$$\log \cos B' = \underline{9.37469}$$

$$B' = 76^\circ 17' 32''.$$

$$B = C = B' = 76^\circ 17' 32''.$$

$$A = 2 A' = 55^\circ 52' 30''.$$

$$b = c = 62^\circ 37'.$$

$$4. \ c = 29^\circ 35', \ B = 15^\circ.$$

In the $\triangle A'B'C'$, given the hypotenuse c' and the adjacent angle,

$$B' = 15^\circ, \ c' = 29^\circ 35'.$$

$$\tan a' = \tan c' \cos B'.$$

$$\log \tan c' = 9.75411$$

$$\log \cos B' = \underline{9.98494}$$

$$\log \tan a' = \underline{9.73905}$$

$$a' = 28^\circ 44' 16''.$$

$$\tan A' = \cot B' \sec c'.$$

$$\log \cot B' = 10.57195$$

$$\log \sec c' = \underline{0.06066}$$

$$\log \tan A' = \underline{10.63261}$$

$$A' = 76^\circ 52' 59''.$$

$$A = 2 A' = 153^\circ 45' 58''.$$

$$B = C = B' = 15^\circ.$$

$$a = 2 a' = 57^\circ 28' 32''.$$

$$b = c = 29^\circ 35'.$$

$$5. \ c = 68^\circ 47', \ B = 42^\circ 30'.$$

$$B' = 42^\circ 30', \ c' = 68^\circ 47'.$$

$$\tan a' = \tan c' \cos B'$$

$$\log \tan c' = 10.41093$$

$$\log \cos B' = \underline{9.86763}$$

$$\log \tan a' = \underline{10.27856}$$

$$a' = 62^\circ 13' 52''.$$

$$\tan A' = \cot B' \sec c'.$$

$$\log \cot B' = 10.03795$$

$$\log \sec c' = \underline{0.44142}$$

$$\log \tan A' = \underline{10.47937}$$

$$A' = 71^\circ 39' 14''.$$

$$A = 2 A' = 143^\circ 18' 28''.$$

$$B = C = B' = 42^\circ 30'.$$

$$a = 2 a' = 124^\circ 27' 44''.$$

$$b = c = 68^\circ 47'.$$

$$6. \ c = 79^\circ 49', \ B = 49^\circ 37'.$$

$$B' = 49^\circ 37', \ c' = 79^\circ 49'.$$

$$\tan a' = \tan c' \cos B'.$$

$$\log \tan c' = 10.74563$$

$$\log \cos B' = \underline{9.81151}$$

$$\log \tan a' = \underline{10.55714}$$

$$a' = 74^\circ 30' 16''.$$

$$\tan A' = \cot B' \sec c'.$$

$$\log \cot B' = 9.92971$$

$$\log \sec c' = \underline{0.75252}$$

$$\log \tan A' = \underline{10.68223}$$

$$A' = 78^\circ 15' 28''.$$

$$A = 2 A' = 156^\circ 30' 56''.$$

$$B = C = B' = 49^\circ 37'.$$

$$a = 2 a' = 149^\circ 0' 32''.$$

$$b = c = 79^\circ 49'.$$

7. In an isosceles spherical triangle, given the base a and the side b , find B , A , and AD , as shown in the figure.

Let ABC be an isosceles triangle, B and C the equal angles, c and b the equal sides, and AD the arc of a great circle, drawn through $A \perp$ to BC , meeting BC in D .

Then, in the right triangle ACD ,

$$CD = \frac{1}{2}a \text{ in triangle } ABC,$$

$$AC = b \text{ in triangle } ABC,$$

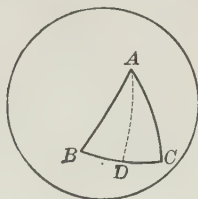
$$A' = \frac{1}{2}A \text{ in triangle } ABC.$$

In the rt. $\triangle ABD$ there is given the hypotenuse and a side.

$$\cos B = \cot b \tan \frac{1}{2}a.$$

$$\sin \frac{1}{2}A = \csc b \sin \frac{1}{2}a.$$

$$\cos AD = \cos b \sec \frac{1}{2}a.$$



Exercise 93. Law of Sines (Page 205)

Consider the Law of Sines when:

1. $A = 90^\circ$.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

$$\sin A = 1.$$

$$\therefore \sin a = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

$$\sin a \sin B = \sin b.$$

$$\sin a \sin C = \sin c.$$

2. $B = 90^\circ$.

$$\sin B = 1.$$

$$\therefore \sin b = \frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}.$$

$$\sin a = \sin b \sin A.$$

$$\sin c = \sin b \sin C.$$

3. $C = 90^\circ$.

$$\sin C = 1.$$

$$\therefore \sin c = \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}.$$

$$\sin a = \sin c \sin A.$$

$$\sin b = \sin c \sin B.$$

4. $a = 90^\circ$.

$$\sin a = 1.$$

$$\therefore \frac{1}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

$$\sin B = \sin b \sin A.$$

$$\sin C = \sin c \sin A.$$

5. $A = B = 90^\circ$.

$$\sin A = \sin B = 1.$$

$$\sin a = \sin b.$$

$$\sin c = \sin a \sin C$$

$$= \sin b \sin C.$$

6. $a = b = 90^\circ$.

$$\frac{1}{\sin A} = \frac{1}{\sin B}.$$

$$\sin B = \sin A.$$

$$\sin C = \sin c \sin A$$

$$= \sin c \sin B$$

7. $a = A = 90^\circ$.

$$1 = \frac{\sin b}{\sin B}.$$

$$\sin B = \sin b.$$

$$\sin C = \sin c.$$

8. $a = b = A = B = 90^\circ$.

$$1 = 1 = \frac{\sin c}{\sin C}.$$

$$\sin C = \sin c.$$

9. $A = B = C = 90^\circ$.

$$\sin a = \sin b = \sin c.$$

Exercise 94. Law of Cosines of Sides (Page 206)

Consider the Law of Cosines of Sides when:

1. $A = 90^\circ$.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

$$\cos A = 0.$$

$$\therefore \cos a = \cos b \cos c.$$

3. $A = B = 90^\circ$.

$$\cos A = \cos B = 0.$$

$$\therefore \cos a = \cos b \cos c,$$

$$\text{and} \quad \cos b = \cos a \cos c.$$

2. $B = 90^\circ$.

$$\cos b = \cos c \cos a + \sin c \sin a \cos B.$$

$$\cos B = 0.$$

$$\therefore \cos b = \cos a \cos c.$$

4. $A = B = C = 90^\circ$.

$$\cos A = \cos B = \cos C = 0.$$

$$\therefore \cos a = \cos b \cos c,$$

$$\cos b = \cos a \cos c,$$

$$\cos c = \cos a \cos b.$$

Prove the following formulas:

5. $1 - \cos a = 1 - \cos(b - c) + \sin b \sin c \operatorname{versin} A.$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

$$\cos A = 1 - \operatorname{versin} A.$$

Substitute the value of $\cos A$,

$$\cos a = \cos b \cos c + \sin b \sin c - \sin b \sin c \operatorname{versin} A$$

$$= \cos(b - c) - \sin b \sin c \operatorname{versin} A.$$

$$1 - \cos a = 1 - \cos(b - c) + \sin b \sin c \operatorname{versin} A.$$

6. $\operatorname{versin} a = \operatorname{versin}(b - c) \left[1 + \frac{\sin b \sin c \operatorname{versin} A}{\operatorname{versin}(b - c)} \right].$

$$\operatorname{versin} a = 1 - \cos a.$$

By Problem 5, $1 - \cos a = 1 - \cos(b - c) + \sin b \sin c \operatorname{versin} A.$

$$1 - \cos(b - c) = \operatorname{versin}(b - c).$$

$$\operatorname{versin} a = \operatorname{versin}(b - c) + \sin b \sin c \operatorname{versin} A$$

$$= \operatorname{versin}(b - c) \left[1 + \frac{\sin b \sin c \operatorname{versin} A}{\operatorname{versin}(b - c)} \right].$$

7. From the Law of Cosines find formulas for $\cos A$, $\cos B$, and $\cos C$ in terms of functions of a , b , and c .

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

$$\therefore \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

and, similarly,

$$\cos B = \frac{\cos b - \cos c \cos a}{\sin c \sin a},$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

8. Prove that $\cos c = \frac{\cos a - \sin b \sin c \cos A}{\cos b}$.

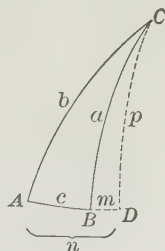
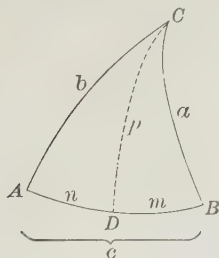
$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

$$\cos b \cos c = \cos a - \sin b \sin c \cos A.$$

$$\cos c = \frac{\cos a - \sin b \sin c \cos A}{\cos b}.$$

9. In the figures given prove that $\cos p = \cos a \sec m$.

In the right spherical triangle BDC , from § 172, (1),



$$\cos a = \cos p \cos m.$$

$$\cos p = \frac{\cos a}{\cos m} = \cos a \sec m.$$

Exercise 95. Law of Cosines of Angles (Page 207)

Consider the Law of Cosines of Angles when :

1. $A = 0^\circ$. $\cos A = -\cos B \cos C + \sin B \sin C \cos a.$

$$\cos A = 1, \sin A = 0.$$

$$\therefore 1 + \cos B \cos C = \sin B \sin C \cos a.$$

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b.$$

$$\therefore \cos B = -\cos C.$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

$$\therefore \cos C = -\cos B.$$

2. $A = 180^\circ$. $\cos A = -\cos B \cos C + \sin B \sin C \cos a.$

$$\cos A = -1, \sin A = 0.$$

$$\therefore \cos B \cos C - 1 = \sin B \sin C \cos a.$$

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b$$

$$\therefore \cos B = \cos C.$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

$$\therefore \cos C = \cos B.$$

$$\begin{aligned}
3. \quad A = 90^\circ, \quad \cos A &= -\cos B \cos C + \sin B \sin C \cos a, \\
&\cos A = 0, \sin A = 1. \\
\therefore \cos B \cos C &= \sin B \sin C \cos a, \\
&\cos B = -\cos A \cos C + \sin A \sin C \cos b, \\
\therefore \cos B &= \sin C \cos b, \\
&\cos C = -\cos A \cos B + \sin A \sin B \cos c, \\
\therefore \cos C &= \sin B \cos c.
\end{aligned}$$

$$\begin{aligned}
4. \quad A = B = 90^\circ, \quad \cos A &= -\cos B \cos C + \sin B \sin C \cos a, \\
&\cos A = \cos B = 0, \sin A = \sin B = 1. \\
\therefore \sin C \cos a &= 0, \\
&\cos B = -\cos A \cos C + \sin A \sin C \cos b, \\
\therefore \sin C \cos b &= 0, \\
&\cos C = -\cos A \cos B + \sin A \sin B \cos c, \\
\therefore \cos C &= \cos c.
\end{aligned}$$

5. Deduce the formulas of § 191 from those of § 190 by means of the relations between polar triangles (§ 171).

Substituting their equals, $180^\circ - A'$, $180^\circ - B'$, $180^\circ - C'$, and $180^\circ - a'$, for a , b , c , and A in the formulas of § 190, we obtain

$$\begin{aligned}
\cos(180^\circ - A') &= \cos(180^\circ - B') \cos(180^\circ - C') \\
&\quad + \sin(180^\circ - B') \sin(180^\circ - C') \cos(180^\circ - a') \\
\therefore -\cos A' &= \cos B' \cos C' - \sin B' \sin C' \cos a'.
\end{aligned}$$

$$\begin{aligned}
&\cos A' = -\cos B' \cos C' + \sin B' \sin C' \cos a'. \\
\text{Similarly,} \quad \cos B' &= -\cos A' \cos C' + \sin A' \sin C' \cos b', \\
\text{and} \quad \cos C' &= -\cos A' \cos B' + \sin A' \sin B' \cos c'.
\end{aligned}$$

Prove the following formulas:

$$\begin{aligned}
6. \quad 1 - \cos A &= 1 + \cos(B + C) + \sin B \sin C \operatorname{versin} a. \\
1 - \cos A &= 1 + \cos B \cos C - \sin B \sin C \cos a. \\
\cos a &= 1 - \operatorname{versin} a. \\
1 - \cos A &= 1 + \cos B \cos C - \sin B \sin C (1 - \operatorname{versin} a) \\
&= 1 + \cos B \cos C - \sin B \sin C + \sin B \sin C \operatorname{versin} a \\
&= 1 + \cos(B + C) + \sin B \sin C \operatorname{versin} a.
\end{aligned}$$

$$7. \quad \operatorname{versin} A = [1 + \cos(B + C)] \left[1 + \frac{\sin B \sin C \operatorname{versin} a}{1 + \cos(B + C)} \right].$$

$$\operatorname{versin} A = 1 - \cos A.$$

$$\begin{aligned}
\text{By Ex. 6, } 1 - \cos A &= 1 + \cos(B + C) + \sin B \sin C \operatorname{versin} a \\
&= [1 + \cos(B + C)] \left[1 + \frac{\sin B \sin C \operatorname{versin} a}{1 + \cos(B + C)} \right]. \\
\therefore, \operatorname{versin} A &= [1 + \cos(B + C)] \left[1 + \frac{\sin B \sin C \operatorname{versin} a}{1 + \cos(B + C)} \right].
\end{aligned}$$

From the Law of Cosines find formulas for the following in terms of functions of A , B , and C :

$$8. \cos a. \quad \cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

$$\sin B \sin C \cos a = \cos A + \cos B \cos C.$$

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

$$9. \cos b. \quad \cos B = -\cos A \cos C + \sin A \sin C \cos b.$$

$$\sin A \sin C \cos b = \cos B + \cos A \cos C.$$

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C}.$$

$$10. \cos c. \quad \cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

$$\sin A \sin B \cos c = \cos C + \cos A \cos B.$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

11. Investigate the dual of Ex. 8 in Exercise 94.

$$\cos c = \frac{\cos a - \sin b \sin c \cos A}{\cos b}. \quad \text{Required the value of } \cos C.$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

$$\cos B \cos C = \sin B \sin C \cos a - \cos A.$$

$$\cos C = \frac{\sin B \sin C \cos a - \cos A}{\cos B}.$$

$$= \frac{-\cos A + \sin B \sin C \cos a}{\cos B}.$$

Exercise 96. Formulas for Half Angles (Page 209)

Show that the following formulas are true:

$$1. \sin \frac{1}{2} A = \sqrt{\sin(s-b) \sin(s-c) \csc b \csc c}.$$

$$\sin \frac{1}{2} A = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}}.$$

$$\sin b \sin c = \frac{1}{\csc b \csc c}.$$

$$\therefore \sin \frac{1}{2} A = \sqrt{\sin(s-b) \sin(s-c) \csc b \csc c}.$$

$$2. \cos \frac{1}{2} A = \sqrt{\sin s \sin(s-a) \csc b \csc c}.$$

$$\cos \frac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$$

$$\sin b \sin c = \frac{1}{\csc b \csc c}.$$

$$\therefore \cos \frac{1}{2} A = \sqrt{\sin s \sin(s-a) \csc b \csc c}.$$

Find the value of A in each case, given:

3. $a = 95^\circ$, $b = 58^\circ$, $c = 42^\circ$.

$$\cos \frac{1}{2} A = \sqrt{\sin s \sin (s - a) \csc b \csc c},$$

$$s = 97^\circ 30'.$$

$$\begin{array}{rcl} \log \sin s & = & 9.99627 \\ \log \sin (s - a) & = & 8.63968 \\ \log \csc b & = & 0.07158 \\ \log \csc c & = & 0.17449 \\ & & 2) 18.88202 \\ \log \cos \frac{1}{2} A & = & 9.44101 \\ & & \frac{1}{2} A = 73^\circ 58' 29''. \\ & & A = 147^\circ 56' 58''. \end{array}$$

4. $a = 92^\circ$, $b = 61^\circ$, $c = 43^\circ$.

$$\cos \frac{1}{2} A = \sqrt{\sin s \sin (s - a) \csc b \csc c},$$

$$s = 98^\circ.$$

$$\begin{array}{rcl} \log \sin s & = & 9.99575 \\ \log \sin (s - a) & = & 9.01923 \\ \log \csc b & = & 0.05818 \\ \log \csc c & = & 0.16622 \\ & & 2) 19.23938 \\ \log \cos \frac{1}{2} A & = & 9.61969 \\ & & \frac{1}{2} A = 65^\circ 22' 53\frac{1}{2}''. \\ & & A = 130^\circ 45' 47''. \end{array}$$

5. $a = 96^\circ$, $b = 64^\circ$, $c = 48^\circ$.

$$\cos \frac{1}{2} A = \sqrt{\sin s \sin (s - a) \csc b \csc c},$$

$$s = 104^\circ.$$

$$\begin{array}{rcl} \log \sin s & = & 9.98690 \\ \log \sin (s - a) & = & 9.14356 \\ \log \csc b & = & 0.04634 \\ \log \csc c & = & 0.12893 \\ & & 2) 19.30573 \\ \log \cos \frac{1}{2} A & = & 9.65287 \\ & & \frac{1}{2} A = 63^\circ 16' 46''. \\ & & A = 126^\circ 33' 32''. \end{array}$$

6. $a = 98^\circ$, $b = 78^\circ$, $c = 60^\circ$.

$$\cos \frac{1}{2} A = \sqrt{\sin s \sin (s - a) \csc b \csc c},$$

$$s = 118^\circ.$$

$$\begin{aligned}\log \sin s &= 9.94593 \\ \log \sin (s - a) &= 9.53405 \\ \log \csc b &= 0.00960 \\ \log \csc c &= 0.06247 \\ 2 \overline{) 19.55205} \\ \log \cos \frac{1}{2} A &= 9.77603\end{aligned}$$

$$\begin{aligned}\frac{1}{2} A &= 53^\circ 20' 21''. \\ A &= 106^\circ 40' 42''.\end{aligned}$$

Find the value of B in each case, given :

7. $a = 95^\circ$, $b = 60^\circ$, $c = 40^\circ$.

$$\cos \frac{1}{2} B = \sqrt{\sin s \sin (s - b) \csc a \csc c},$$

$$s = 97^\circ 30'.$$

$$\begin{aligned}\log \sin s &= 9.99627 \\ \log \sin (s - b) &= 9.78445 \\ \log \csc a &= 0.00166 \\ \log \csc c &= 0.19193 \\ 2 \overline{) 19.97431} \\ \log \cos \frac{1}{2} B &= 9.98716\end{aligned}$$

$$\begin{aligned}\frac{1}{2} B &= 13^\circ 51' 45''. \\ B &= 27^\circ 43' 30''.\end{aligned}$$

8. $a = 97^\circ$, $b = 62^\circ$, $c = 38^\circ$

$$\cos \frac{1}{2} B = \sqrt{\sin s \sin (s - b) \csc a \csc c}$$

$$s = 98^\circ 30'.$$

$$\begin{aligned}\log \sin s &= 9.99520 \\ \log \sin (s - b) &= 9.77439 \\ \log \csc a &= 0.00325 \\ \log \csc c &= 0.21066 \\ 2 \overline{) 19.98350} \\ \log \cos \frac{1}{2} B &= 9.99175\end{aligned}$$

$$\begin{aligned}\frac{1}{2} B &= 11^\circ 8'. \\ B &= 22^\circ 16' .\end{aligned}$$

Find the value of C in each case, given :

9. $a = 92^\circ$, $b = 59^\circ$, $c = 37^\circ$

$$\cos \frac{1}{2} C = \sqrt{\sin s \sin (s - c) \csc a \csc b},$$

$$s = 94^\circ.$$

$$\begin{aligned} \log \sin s &= 9.99894 \\ \log \sin (s - c) &= 9.92359 \\ \log \csc a &= 0.00026 \\ \log \csc b &= 0.06693 \\ 2 \overline{) 19.98972} \\ \log \cos \frac{1}{2} C &= 9.99486 \\ \frac{1}{2} C &= 8^\circ 48'. \\ C &= 17^\circ 36'. \end{aligned}$$

10. $a = 96^\circ$, $b = 64^\circ$, $c = 39^\circ$.

$$\cos \frac{1}{2} C = \sqrt{\sin s \sin (s - c) \csc a \csc b},$$

$$s = 99^\circ 30'.$$

$$\begin{aligned} \log \sin s &= 9.99400 \\ \log \sin (s - c) &= 9.93970 \\ \log \csc a &= 0.00239 \\ \log \csc b &= 0.04634 \\ 2 \overline{) 19.982+3} \\ \cos \frac{1}{2} C &= 9.99122 \\ \frac{1}{2} C &= 11^\circ 29'. \\ C &= 22^\circ 58'. \end{aligned}$$

Prove the following formulas :

11. $\sin \frac{1}{2} (180^\circ - A) = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}.$

$$\begin{aligned} \sin \frac{1}{2} (180^\circ - A) &= \sin (90^\circ - \frac{1}{2} A) \\ &= \cos \frac{1}{2} A \\ &= \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}. \end{aligned}$$

12. $\cos \frac{1}{2} (180^\circ - A) = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}.$

$$\begin{aligned} \cos \frac{1}{2} (180^\circ - A) &= \cos (90^\circ - \frac{1}{2} A) \\ &= \sin \frac{1}{2} A \\ &= \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}. \end{aligned}$$

$$13. \tan \frac{1}{2}(180^\circ - A) = \sqrt{\frac{\sin s \sin(s-a)}{\sin(s-b) \sin(s-c)}}.$$

$$\begin{aligned} \tan \frac{1}{2}(180^\circ - A) &= \tan(90^\circ - \frac{1}{2}A) \\ &= \cot \frac{1}{2}A \\ &= \sqrt{\frac{\sin s \sin(s-a)}{\sin(s-b) \sin(s-c)}}. \end{aligned}$$

Exercise 97. Formulas for Half Sides (Page 211)

Consider the formula for $\sin \frac{1}{2}a$ when:

1. $B = 90^\circ$.

$$\begin{aligned} \sin \frac{1}{2}a &= \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}} \\ &= \sqrt{\frac{-\cos S \cos(S-A)}{\sin C}}. \end{aligned}$$

2. $C = 90^\circ$.

$$\sin \frac{1}{2}a = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B}}.$$

3. $B = C = 90^\circ$.

$$\sin \frac{1}{2}a = \sqrt{-\cos S \cos(S-A)}.$$

Consider the formula for $\sin \frac{1}{2}b$ when:

4. $A = 45^\circ$.

$$\begin{aligned} \sin \frac{1}{2}b &= \sqrt{\frac{-\cos S \cos(S-B)}{\sin A \sin C}} \\ &= \sqrt{\frac{-\cos S \cos(S-B)}{\frac{1}{2}\sqrt{2} \sin C}} \\ &= \sqrt{\frac{-2 \cos S \cos(S-B)}{\sqrt{2} \sin C}} \end{aligned}$$

5. $C = 45^\circ$.

$$\begin{aligned} \sin \frac{1}{2}b &= \sqrt{\frac{-\cos S \cos(S-B)}{\frac{1}{2}\sqrt{2} \sin A}} \\ &= \sqrt{\frac{-2 \cos S \cos(S-B)}{\sqrt{2} \sin A}} \end{aligned}$$

6. $A = C = 45^\circ$.

$$\sin \frac{1}{2}b = \sqrt{-2 \cos S \cos(S-B)}.$$

Consider the formula for $\sin \frac{1}{2} c$ when :

7. $A = 200^\circ$, $B = 100^\circ$, $C = 135^\circ$.

$$\sin \frac{1}{2} c = \sqrt{\frac{-\cos S \cos (S - C)}{\sin A \sin B}}.$$

$$S = 217^\circ 30'.$$

$$\log \cos S = 9.89947 \text{ (n)}$$

$$\log \cos (S - C) = 9.11570$$

$$\text{colog } \sin A = 0.46595 \text{ (n)}$$

$$\text{colog } \sin B = 0.00665$$

$$2 \overline{)19.48777} \text{ (n)}$$

$$\log \sin \frac{1}{2} c = 9.74389 \text{ (n)}$$

$$\frac{1}{2} c = -(33^\circ 40' 32'').$$

$$c = -(67^\circ 21' 4'').$$

8. $A = B = C = 90^\circ$.

$$\sin \frac{1}{2} c = \sqrt{\frac{-\cos S \cos (S - C)}{\sin A \sin B}}.$$

$$S = 135^\circ.$$

$$\sin \frac{1}{2} c = \sqrt{\frac{-\cos 135^\circ \cos 45^\circ}{\sin 90^\circ \sin 90^\circ}}.$$

$$\sin \frac{1}{2} c = \sqrt{\frac{-(-\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2})}{1 \cdot 1}}$$

$$= \frac{1}{2}\sqrt{2}.$$

$$\frac{1}{2} c = 45^\circ.$$

$$c = 90^\circ.$$

Show that the following formulas are true :

9. $\sin \frac{1}{2} a = \sqrt{-\cos S \cos (S - A) \csc B \csc C}.$

$$\sin \frac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}}.$$

$$\sin B \sin C = \frac{1}{\csc B \csc C}.$$

$$\therefore \sin \frac{1}{2} a = \sqrt{-\cos S \cos (S - A) \csc B \csc C}.$$

10. $\cos \frac{1}{2} a = \sqrt{\cos (S - B) \cos (S - C) \csc B \csc C}.$

$$\cos \frac{1}{2} a = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}}.$$

$$\sin B \sin C = \frac{1}{\csc B \csc C}.$$

$$\therefore \cos \frac{1}{2} a = \sqrt{\cos (S - B) \cos (S - C) \csc B \csc C}.$$

$$11. \tan \frac{1}{2} a = \sqrt{-\cos S \cos(S-A) \sec(S-B) \sec(S-C)}.$$

$$\tan \frac{1}{2} a = \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B) \cos(S-C)}}.$$

$$\cos(S-B) \cos(S-C) = \frac{1}{\sec(S-B) \sec(S-C)}.$$

$$\therefore \tan \frac{1}{2} a = \sqrt{-\cos S \cos(S-A) \sec(S-B) \sec(S-C)}.$$

$$12. \sin \frac{1}{2} b = \sqrt{-\cos S \cos(S-B) \csc A \csc C}.$$

$$\sin \frac{1}{2} b = \sqrt{\frac{-\cos S \cos(S-B)}{\sin A \sin C}}.$$

$$\sin A \sin C = \frac{1}{\csc A \csc C}.$$

$$\therefore \sin \frac{1}{2} b = \sqrt{-\cos S \cos(S-B) \csc A \csc C}.$$

$$13. \tan \frac{1}{2} c = \sqrt{-\cos S \cos(S-C) \sec(S-A) \sec(S-B)}.$$

$$\tan \frac{1}{2} c = \sqrt{\frac{-\cos S \cos(S-C)}{\cos(S-A) \cos(S-B)}}.$$

$$\cos(S-A) \cos(S-B) = \frac{1}{\sec(S-A) \sec(S-B)}.$$

$$\therefore \tan \frac{1}{2} c = \sqrt{-\cos S \cos(S-C) \sec(S-A) \sec(S-B)}.$$

14. From the formula for $\tan \frac{1}{2} b$ deduce another formula similar to that of Ex. 13.

$$\tan \frac{1}{2} b = \sqrt{\frac{-\cos S \cos(S-B)}{\cos(S-A) \cos(S-C)}}.$$

$$\cos(S-A) \cos(S-C) = \frac{1}{\sec(S-A) \sec(S-C)}.$$

$$\therefore \tan \frac{1}{2} b = \sqrt{-\cos S \cos(S-B) \sec(S-A) \sec(S-C)}.$$

15. From the formula for $\cos \frac{1}{2} b$ deduce another formula similar to that of Ex. 10.

$$\cos \frac{1}{2} b = \sqrt{\frac{\cos(S-A) \cos(S-C)}{\sin A \sin C}}.$$

$$\sin A \sin C = \frac{1}{\csc A \csc C}.$$

$$\therefore \cos \frac{1}{2} b = \sqrt{\cos(S-A) \cos(S-C) \csc A \csc C}.$$

16. From the formula for $\sin \frac{1}{2}c$ deduce another formula similar to that of Ex. 9.

$$\sin \frac{1}{2}c = \sqrt{\frac{-\cos S \cos(S-C)}{\sin A \sin B}}.$$

$$\sin A \sin B = \frac{1}{\csc A \csc B}.$$

$$\therefore \sin \frac{1}{2}c = \sqrt{-\cos S \cos(S-C) \csc A \csc B}.$$

Exercise 98. Given Two Sides and the Included Angle
(Page 214)

Solve the triangles, given the following parts:

1. $a = 88^\circ 12' 20''$,

$b = 124^\circ 7' 17''$,

$C = 50^\circ 2' 1''$.

2. $a = 120^\circ 55' 35''$,

$b = 88^\circ 12' 20''$,

$C = 47^\circ 42' 1''$.

$\frac{1}{2}(b-a) = 17^\circ 57' 28.5''$.

$\frac{1}{2}(b+a) = 106^\circ 9' 48.5''$.

$\frac{1}{2}C = 25^\circ 1' 0.5''$.

$\frac{1}{2}(a-b) = 16^\circ 21' 37.5''$.

$\frac{1}{2}(a+b) = 104^\circ 38' 57.5''$.

$\frac{1}{2}C = 23^\circ 51' 0.5''$.

$\log \cos \frac{1}{2}(b-a) = 9.97831$

$\log \sec \frac{1}{2}(b+a) = 0.55536 (n)$

$\log \cot \frac{1}{2}C = 10.33100$

$\log \tan \frac{1}{2}(B+A) = 10.86467 (n)$

$\frac{1}{2}(B+A) = 97^\circ 46' 34.7''$.

$\log \cos \frac{1}{2}(a-b) = 9.98205$

$\log \sec \frac{1}{2}(a+b) = 0.59947 (n)$

$\log \cot \frac{1}{2}C = 10.35448$

$\log \tan \frac{1}{2}(A+B) = 10.93600 (n)$

$\frac{1}{2}(A+B) = 96^\circ 36' 35.5''$.

$\log \sin \frac{1}{2}(b-a) = 9.48900$

$\log \csc \frac{1}{2}(b+a) = 0.01751$

$\log \cot \frac{1}{2}C = 0.33100$

$\log \tan \frac{1}{2}(B-A) = 9.83751$

$\frac{1}{2}(B-A) = 34^\circ 31' 23.6''$.

$\frac{1}{2}(B+A) = 97^\circ 46' 34.7''$.

$A = 63^\circ 15' 11''$.

$B = 132^\circ 17' 58''$.

$c = 59^\circ 4' 17''$.

$\log \sin \frac{1}{2}(a-b) = 9.44976$

$\log \csc \frac{1}{2}(a+b) = 0.01419$

$\log \cot \frac{1}{2}C = 10.35448$

$\log \tan \frac{1}{2}(A-B) = 9.81843$

$\frac{1}{2}(A-B) = 33^\circ 21' 26.7''$.

$\frac{1}{2}(A+B) = 96^\circ 36' 35.5''$.

$A = 129^\circ 58' 2''$.

$B = 63^\circ 15' 8''$.

$\log \sec \frac{1}{2}(A+B) = 0.93890 (n)$

$\log \cos \frac{1}{2}(a+b) = 9.40053 (n)$

$\log \sin \frac{1}{2}C = 9.60675$

$\log \cos \frac{1}{2}c = 9.94618$

$\frac{1}{2}c = 27^\circ 56' 20''$.

$c = 55^\circ 52' 40''$.

$\log \sec \frac{1}{2}(B+A) = 0.86868 (n)$

$\log \cos \frac{1}{2}(b+a) = 9.44464 (n)$

$\log \sin \frac{1}{2}C = 9.62622$

$\log \cos \frac{1}{2}c = 9.93954$

$\frac{1}{2}c = 29^\circ 32' 8.6''$.

$$\begin{aligned} 3. \quad b &= 63^\circ 15' 12'', \\ c &= 47^\circ 42' 1'', \\ A &= 59^\circ 4' 25''. \end{aligned}$$

$$\frac{1}{2}(b+c) = 55^\circ 28' 36.5''.$$

$$\frac{1}{2}(b-c) = 7^\circ 46' 35.5''.$$

$$\frac{1}{2}A = 29^\circ 32' 12.5''.$$

$$\log \cos \frac{1}{2}(b-c) = 9.99599$$

$$\log \sec \frac{1}{2}(b+c) = 0.24662$$

$$\log \cot \frac{1}{2}A = 10.24671$$

$$\log \tan \frac{1}{2}(B+C) = 10.48932$$

$$\frac{1}{2}(B+C) = 72^\circ 2' 32.7''.$$

$$\log \sin \frac{1}{2}(b-c) = 9.13133$$

$$\log \csc \frac{1}{2}(b+c) = 0.08413$$

$$\log \cot \frac{1}{2}A = 10.24671$$

$$\log \tan \frac{1}{2}(B-C) = 9.46217$$

$$\frac{1}{2}(B-C) = 16^\circ 9' 51.1''.$$

$$\frac{1}{2}(B+C) = 72^\circ 2' 32.7''.$$

$$B = 88^\circ 12' 24''.$$

$$C = 55^\circ 52' 42''.$$

$$\begin{aligned} 4. \quad b &= 69^\circ 25' 11'', \\ c &= 109^\circ 46' 19'', \\ A &= 54^\circ 54' 42''. \end{aligned}$$

$$\frac{1}{2}(c-b) = 20^\circ 10' 34''.$$

$$\frac{1}{2}(c+b) = 89^\circ 35' 45''.$$

$$\frac{1}{2}A = 27^\circ 27' 21''.$$

$$\log \cos \frac{1}{2}(c-b) = 9.97250$$

$$\log \sec \frac{1}{2}(c+b) = 2.15157$$

$$\log \cot \frac{1}{2}A = 10.28434$$

$$\log \tan \frac{1}{2}(C+B) = 12.40841$$

$$\frac{1}{2}(C+B) = 89^\circ 46' 34.6''.$$

$$\log \sin \frac{1}{2}(c-b) = 9.53770$$

$$\log \csc \frac{1}{2}(c+b) = 0.00001$$

$$\log \cot \frac{1}{2}A = 10.28434$$

$$\log \tan \frac{1}{2}(C-B) = 9.82205$$

$$\frac{1}{2}(C-B) = 33^\circ 34' 37.8''.$$

$$\frac{1}{2}(C+B) = 89^\circ 46' 34.6''.$$

$$C = 123^\circ 21' 12''.$$

$$B = 56^\circ 11' 57''.$$

$$\log \cos \frac{1}{2}(b+c) = 9.75338$$

$$\log \sec \frac{1}{2}(B+C) = 0.51101$$

$$\log \sin \frac{1}{2}A = 9.69284$$

$$\log \cos \frac{1}{2}a = 9.95723$$

$$\frac{1}{2}a = 25^\circ 0' 50''.$$

$$a = 50^\circ 1' 40''.$$

$$\log \cos \frac{1}{2}(c+b) = 7.84843$$

$$\log \sec \frac{1}{2}(C+B) = 2.40842$$

$$\log \sin \frac{1}{2}A = 9.66376$$

$$\log \cos \frac{1}{2}a = 9.92061$$

$$\frac{1}{2}a = 33^\circ 35' 53.3''.$$

$$a = 67^\circ 11' 47''.$$

5. Two sides of a triangle are 90° and 12° , and the included angle is 85° . Find the third side in degrees.

Let $a = 90^\circ$, $b = 12^\circ$, $C = 85^\circ$; required c .

$$\frac{1}{2}(a-b) = 39^\circ.$$

$$\frac{1}{2}(a+b) = 51^\circ.$$

$$\frac{1}{2}C = 42^\circ 30'.$$

$$\log \cos \frac{1}{2}(a-b) = 9.89050$$

$$\log \sec \frac{1}{2}(a+b) = 0.20113$$

$$\log \cot \frac{1}{2}C = 10.03795$$

$$\log \tan \frac{1}{2}(A+B) = 10.12958$$

$$\frac{1}{2}(A+B) = 53^\circ 25' 25''.$$

$$\log \cos \frac{1}{2}(a+b) = 9.79887$$

$$\log \sec \frac{1}{2}(A+B) = 0.22483$$

$$\log \sin \frac{1}{2}C = 9.82968$$

$$\log \cos \frac{1}{2}c = 9.85338$$

$$\frac{1}{2}c = 44^\circ 28' 55''.$$

$$c = 88^\circ 57' 50''.$$

Exercise 99. To find the Third Side (Page 215)*Find the value of c , given the following parts :*

$$\begin{aligned} 1. \quad a &= 88^\circ 30', \\ b &= 125^\circ 45', \\ C &= 49^\circ 15'. \end{aligned}$$

$$\begin{aligned} \log \tan a &= 11.58193 \\ \log \cos C &= \underline{9.81475} \\ \log \tan m &= 11.39668 \\ m &= 87^\circ 42' 10'' \\ b - m &= 38^\circ 2' 50''. \end{aligned}$$

$$\begin{aligned} \log \cos a &= 8.41792 \\ \log \sec m &= 1.39704 \\ \log \cos (b - m) &= \underline{9.89625} \\ \log \cos c &= 9.71121 \\ c &= 59^\circ 3'. \end{aligned}$$

$$\begin{aligned} 2. \quad a &= 121^\circ 45', \\ b &= 92^\circ 15', \\ C &= 48^\circ 30'. \end{aligned}$$

$$\begin{aligned} \log \tan a &= 10.20844 \ (n) \\ \log \cos C &= \underline{9.82126} \\ \log \tan m &= 10.02970 \ (n) \\ m &= 133^\circ 2' 32''. \\ b - m &= -40^\circ 47' 32''. \end{aligned}$$

$$\begin{aligned} \log \cos a &= 9.72116 \ (n) \\ \log \sec m &= 0.16588 \ (n) \\ \log \cos (b - m) &= \underline{9.87914} \\ \log \cos c &= 9.76618 \\ c &= 54^\circ 17' 23''. \end{aligned}$$

$$\begin{aligned} 3. \quad a &= 63.5^\circ, \\ b &= 89.25^\circ, \\ C &= 52.75^\circ. \end{aligned}$$

$$\begin{aligned} \log \tan a &= 10.30226 \\ \log \cos C &= \underline{9.78197} \\ \log \tan m &= 10.08423 \\ m &= 50^\circ 31' 18'' \\ b - m &= 38^\circ 43' 42''. \end{aligned}$$

$$\begin{aligned} \log \cos a &= 9.64953 \\ \log \sec m &= 0.19669 \\ \log \cos (b - m) &= \underline{9.89216} \\ \log \cos c &= 9.73838 \\ c &= 56^\circ 48' 16''. \end{aligned}$$

$$\begin{aligned} 4. \quad a &= 72.25^\circ, \\ b &= 93.75^\circ, \\ C &= 63.5^\circ. \end{aligned}$$

$$\begin{aligned} \log \tan a &= 10.49471 \\ \log \cos C &= \underline{9.64953} \\ \log \tan m &= 10.14424 \\ m &= 54^\circ 20' 40''. \\ b - m &= 39^\circ 24' 20''. \end{aligned}$$

$$\begin{aligned} \log \cos a &= 9.48411 \\ \log \sec m &= 0.23440 \\ \log \cos (b - m) &= \underline{9.88800} \\ \log \cos c &= 9.60651 \\ c &= 66^\circ 9' 50''. \end{aligned}$$

Exercise 100. Given Two Angles and the Included Side (Page 216)1. Write the formulas used in computing A , given B , C , and a .

$$\tan \frac{1}{2}(b - c) = \frac{\sin \frac{1}{2}(B - C)}{\sin \frac{1}{2}(B + C)} \tan \frac{1}{2}a.$$

$$\cos \frac{1}{2}A = \frac{\sin \frac{1}{2}(B + C)}{\cos \frac{1}{2}(b - c)} \cos \frac{1}{2}a.$$

2. Write the formulas used in computing B , given A , C , and b

$$\tan \frac{1}{2}(a - c) = \frac{\sin \frac{1}{2}(A - C)}{\sin \frac{1}{2}(A + C)} \tan \frac{1}{2}b,$$

$$\cos \frac{1}{2}B = \frac{\sin \frac{1}{2}(A + C)}{\cos \frac{1}{2}(a - c)} \cos \frac{1}{2}b.$$

3. Write the formulas used in computing b , given B , C , and a .

$$\tan \frac{1}{2}(b - c) = \frac{\sin \frac{1}{2}(B - C)}{\sin \frac{1}{2}(B + C)} \tan \frac{1}{2}a.$$

$$\tan \frac{1}{2}(b + c) = \frac{\cos \frac{1}{2}(B - C)}{\cos \frac{1}{2}(B + C)} \tan \frac{1}{2}a.$$

Solve the triangles, given the following parts:

4. $A = 28^\circ$, $B = 40^\circ$, $c = 90^\circ$.

$$\frac{1}{2}(B - A) = 6^\circ.$$

$$\frac{1}{2}(B + A) = 34^\circ.$$

$$\frac{1}{2}c = 45^\circ.$$

$$\log \cos \frac{1}{2}(B - A) = 9.99761$$

$$\log \sec \frac{1}{2}(B + A) = 0.08143$$

$$\log \tan \frac{1}{2}c = \underline{10.00000}$$

$$\log \tan \frac{1}{2}(b + a) = \underline{10.07904}$$

$$\frac{1}{2}(b + a) = 50^\circ 11' 7''.$$

$$\log \sin \frac{1}{2}(B - A) = 9.01923$$

$$\log \csc \frac{1}{2}(B + A) = 0.25244$$

$$\log \tan \frac{1}{2}c = \underline{10.00000}$$

$$\log \tan \frac{1}{2}(b - a) = \underline{9.27167}$$

$$\frac{1}{2}(b - a) = 10^\circ 35' 16''.$$

$$b = 60^\circ 46' 23''.$$

$$a = 39^\circ 35' 51''.$$

5. $A = 35^\circ$, $B = 56^\circ$, $c = 70^\circ$.

$$\frac{1}{2}(B - A) = 10^\circ 30'.$$

$$\frac{1}{2}(B + A) = 45^\circ 30'.$$

$$\frac{1}{2}c = 35^\circ.$$

$$\log \cos \frac{1}{2}(B - A) = 9.99267$$

$$\log \sec \frac{1}{2}(B + A) = 0.15434$$

$$\log \tan \frac{1}{2}c = \underline{9.84523}$$

$$\log \tan \frac{1}{2}(b + a) = \underline{9.99224}$$

$$\frac{1}{2}(b + a) = 44^\circ 29' 17''.$$

$$\log \sin \frac{1}{2}(B - A) = 9.26063$$

$$\log \csc \frac{1}{2}(B + A) = 0.14676$$

$$\log \tan \frac{1}{2}c = \underline{9.84523}$$

$$\log \tan \frac{1}{2}(b - a) = \underline{9.25262}$$

$$\frac{1}{2}(b - a) = 10^\circ 8' 35''.$$

$$b = 54^\circ 37' 52''.$$

$$a = 34^\circ 20' 42''.$$

$$\log \sin \frac{1}{2}(B + A) = 9.74756$$

$$\log \sec \frac{1}{2}(b - a) = 0.00746$$

$$\log \cos \frac{1}{2}c = \underline{9.84949}$$

$$\log \cos \frac{1}{2}C = \underline{9.60451}$$

$$\frac{1}{2}C = 66^\circ 16' 49''.$$

$$C = 132^\circ 33' 38''$$

$$\log \sin \frac{1}{2}(B + A) = 9.85324$$

$$\log \sec \frac{1}{2}(b - a) = 0.00684$$

$$\log \cos \frac{1}{2}c = \underline{9.91336}$$

$$\log \cos \frac{1}{2}C = \underline{0.77344}$$

$$\frac{1}{2}C = 53^\circ 35' 32''.$$

$$C = 107^\circ 11' 4''.$$

$$6. A = 46^\circ, B = 60^\circ, c = 80^\circ.$$

$$\frac{1}{2}(B - A) = 7^\circ.$$

$$\frac{1}{2}(B + A) = 53^\circ.$$

$$\frac{1}{2}c = 40^\circ.$$

$$\log \cos \frac{1}{2}(B - A) = 9.99675$$

$$\log \sec \frac{1}{2}(B + A) = 0.22054$$

$$\log \tan \frac{1}{2}c = 9.92381$$

$$\log \tan \frac{1}{2}(b + a) = 10.14110$$

$$\frac{1}{2}(b + a) = 54^\circ 8' 53''.$$

$$\log \sin \frac{1}{2}(B - A) = 9.08589$$

$$\log \csc \frac{1}{2}(B + A) = 0.09765$$

$$\log \tan \frac{1}{2}c = 9.92381$$

$$\log \tan \frac{1}{2}(b - a) = 9.10735$$

$$\frac{1}{2}(b - a) = 7^\circ 17' 47''.$$

$$b = 61^\circ 26' 40''.$$

$$a = 46^\circ 51' 6''$$

$$\log \sin \frac{1}{2}(B + A) = 9.90235$$

$$\log \sec \frac{1}{2}(b - a) = 0.00353$$

$$\log \cos \frac{1}{2}c = 9.88425$$

$$\log \cos \frac{1}{2}C = 9.79013$$

$$\frac{1}{2}C = 51^\circ 55' 8''.$$

$$C = 103^\circ 50' 16''.$$

$$7. A = 75^\circ, B = 30^\circ, c = 85^\circ.$$

$$\frac{1}{2}(A - B) = 22^\circ 30'.$$

$$\frac{1}{2}(A + B) = 52^\circ 30'.$$

$$\frac{1}{2}c = 42^\circ 30'.$$

$$\log \cos \frac{1}{2}(A - B) = 9.96562$$

$$\log \sec \frac{1}{2}(A + B) = 0.21555$$

$$\log \tan \frac{1}{2}c = 9.96205$$

$$\log \tan \frac{1}{2}(a + b) = 10.14322$$

$$\frac{1}{2}(a + b) = 54^\circ 16' 51''.$$

$$\log \sin \frac{1}{2}(A - B) = 9.58284$$

$$\log \csc \frac{1}{2}(A + B) = 0.10053$$

$$\log \tan \frac{1}{2}c = 9.96205$$

$$\log \tan \frac{1}{2}(a - b) = 9.64542$$

$$\frac{1}{2}(a - b) = 23^\circ 50' 43''.$$

$$a = 78^\circ 7' 34''.$$

$$b = 30^\circ 26' 8''.$$

$$\log \sin \frac{1}{2}(A + B) = 9.89947$$

$$\log \sec \frac{1}{2}(a - b) = 0.03875$$

$$\log \cos \frac{1}{2}c = 9.86763$$

$$\log \cos \frac{1}{2}C = 9.80585$$

$$\frac{1}{2}C = 50^\circ 14' 40''.$$

$$C = 100^\circ 29' 20''.$$

$$8. A = 60^\circ, B = 60^\circ, c = 40^\circ.$$

Solve as an isosceles triangle.

Let $A'B'C'$ represent one of the right spherical triangles of the isosceles triangle, with C' the right angle.

$$B' = 60^\circ.$$

$$a' = \frac{1}{2}c = 20^\circ.$$

$$\text{By (5), } \cos B' = \tan a' \cot c'.$$

$$\therefore \tan c' = \tan a' \sec B'.$$

$$\log \tan a' = 9.56107$$

$$\log \sec B' = 0.30103$$

$$\log \tan c' = 9.86210$$

$$c' = 36^\circ 3' 9''$$

$$= a = b.$$

$$\text{By (6), } \cos A' = \cos a' \sin B'.$$

$$\log \cos a' = 9.97299$$

$$\log \sin B' = 9.93753$$

$$\log \cos A' = 9.91052$$

$$A' = 35^\circ 31' 53''.$$

$$C = 2A' = 71^\circ 3' 46''.$$

$$9. A = 80^\circ, B = 80^\circ, c = 80^\circ.$$

Let $A'B'C'$ represent one right spherical triangle of the isosceles triangle, with C' the right angle.

$$B' = 80^\circ.$$

$$a' = \frac{1}{2}c = 40^\circ.$$

$$\text{By (5), } \tan c' = \tan a' \sec B'.$$

$$\log \tan a' = 9.92381$$

$$\log \sec B' = 0.76033$$

$$\log \tan c' = 10.68414$$

$$c' = 78^\circ 18' 28''$$

$$= a = b.$$

$$\text{By (6), } \cos A' = \cos a' \sin B'.$$

$$\log \cos A' = 9.88425$$

$$\log \sin B' = 9.99335$$

$$\log \cos A' = 9.87760$$

$$A' = 41^\circ 1' 38''.$$

$$C = 2 A' = 82^\circ 3' 16''.$$

$$10. A = 26^\circ, B = 39^\circ, c = 154''.$$

$$\frac{1}{2}(B - A) = 6^\circ 30'.$$

$$\frac{1}{2}(B + A) = 32^\circ 30'.$$

$$\frac{1}{2}c = 77''.$$

$$\log \cos \frac{1}{2}(B - A) = 9.99720$$

$$\log \sec \frac{1}{2}(B + A) = 0.07397$$

$$\log \tan \frac{1}{2}c = 10.63664$$

$$\log \tan \frac{1}{2}(b + a) = 10.70781$$

$$\frac{1}{2}(b + a) = 78^\circ 54' 44''.$$

$$\log \sin \frac{1}{2}(B - A) = 9.05386$$

$$\log \csc \frac{1}{2}(B + A) = 0.26978$$

$$\log \tan \frac{1}{2}c = 10.63664$$

$$\log \tan \frac{1}{2}(b - a) = 9.96028$$

$$\frac{1}{2}(b - a) = 42^\circ 23'.$$

$$b = 121^\circ 17' 44''.$$

$$a = 36^\circ 31' 44''.$$

$$\log \sin \frac{1}{2}(B + A) = 9.73022$$

$$\log \sec \frac{1}{2}(b - a) = 0.13156$$

$$\log \cos \frac{1}{2}c = 9.35209$$

$$\log \cos \frac{1}{2}C = 9.21387$$

$$\frac{1}{2}C = 80^\circ 34' 56''.$$

$$C = 161^\circ 9' 52''.$$

$$11. A = 128^\circ, B = 107^\circ, c = 124'.$$

$$\frac{1}{2}(A - B) = 10^\circ 30'.$$

$$\frac{1}{2}(A + B) = 117^\circ 30'.$$

$$\frac{1}{2}c = 62''.$$

$$\log \cos \frac{1}{2}(A - B) = 9.99267$$

$$\log \sec \frac{1}{2}(A + B) = 0.33559 (n)$$

$$\log \tan \frac{1}{2}c = 10.27433$$

$$\log \tan \frac{1}{2}(a + b) = 10.60259 (n)$$

$$\frac{1}{2}(a + b) = 104^\circ 1' 11''.$$

$$\log \sin \frac{1}{2}(A - B) = 9.26063$$

$$\log \csc \frac{1}{2}(A + B) = 0.05207$$

$$\log \tan \frac{1}{2}c = 10.27433$$

$$\log \tan \frac{1}{2}(a - b) = 9.58703$$

$$\frac{1}{2}(a - b) = 21^\circ 7' 35''.$$

$$a = 125^\circ 8' 46''.$$

$$b = 82^\circ 53' 36''.$$

$$\log \sin \frac{1}{2}(A + B) = 9.94793$$

$$\log \sec \frac{1}{2}(a - b) = 0.03022$$

$$\log \cos \frac{1}{2}c = 9.67161$$

$$\log \cos \frac{1}{2}C = 9.64976$$

$$\frac{1}{2}C = 63^\circ 29' 5''.$$

$$C = 126^\circ 58' 10''.$$

$$12. A = 153^\circ, C = 78^\circ, b = 86'.$$

$$\frac{1}{2}(A - C) = 37^\circ 30'.$$

$$\frac{1}{2}(A + C) = 115^\circ 30'.$$

$$\frac{1}{2}b = 43''$$

$$\log \sin \frac{1}{2}(A - C) = 9.78445$$

$$\log \csc \frac{1}{2}(A + C) = 0.04451$$

$$\log \tan \frac{1}{2}b = 9.96966$$

$$\log \tan \frac{1}{2}(a - c) = 9.79862$$

$$\frac{1}{2}(a - c) = 32^\circ 10' 4''.$$

$$\log \cos \frac{1}{2}(A - C) = 9.89947$$

$$\log \sec \frac{1}{2}(A + C) = 0.36602 (n)$$

$$\log \tan \frac{1}{2}b = 9.96966$$

$$\log \tan \frac{1}{2}(a + c) = 10.23515 (n)$$

$$\frac{1}{2}(a + c) = 120^\circ 11' 43''.$$

$$a = 152^\circ 21' 47''.$$

$$c = 88^\circ 1' 39''.$$

$$\log \sin \frac{1}{2}(A + C) = 9.95549$$

$$\log \sec \frac{1}{2}(a - c) = 0.07236$$

$$\log \cos \frac{1}{2}b = 9.86413$$

$$\log \cos \frac{1}{2}B = 9.89198$$

$$\frac{1}{2}B = 38^\circ 45' 30''.$$

$$B = 77^\circ 31'.$$

$$13. A = 125^\circ, C = 82^\circ, b = 52^\circ.$$

$$\cos B = \cos b \sin A.$$

$$\frac{1}{2}(A - C) = 21^\circ 30'.$$

$$\log \cos b = 9.48998$$

$$\frac{1}{2}(A + C) = 103^\circ 30'.$$

$$\log \sin A = 9.99335$$

$$\frac{1}{2}b = 26^\circ.$$

$$\log \cos B = 9.48333$$

$$B = 72^\circ 16' 59''.$$

$$\log \sin \frac{1}{2}(A - C) = 9.56408$$

$$\cos A = \tan b \cot c.$$

$$\log \csc \frac{1}{2}(A + C) = 0.01217$$

$$\cot c = \cos A \cot b.$$

$$\log \tan \frac{1}{2}b = 9.68818$$

$$\log \tan \frac{1}{2}(a - c) = 9.26443$$

$$\log \cos A = 9.23967 (n)$$

$$\log \cot b = 9.51178$$

$$\frac{1}{2}(a - c) = 10^\circ 25'.$$

$$\log \cot c = 8.75145 (n)$$

$$\log \cos \frac{1}{2}(A - C) = 9.96868$$

$$c = 93^\circ 13' 46''.$$

$$\log \sec \frac{1}{2}(A + C) = 0.63181 (n)$$

$$\log \tan \frac{1}{2}b = 9.68818$$

$$\log \tan \frac{1}{2}(a + c) = 10.28867 (n)$$

$$15. A = 120^\circ, C = 88^\circ, b = 75^\circ.$$

$$\frac{1}{2}(a + c) = 117^\circ 13' 22''.$$

$$\frac{1}{2}(A - C) = 16^\circ.$$

$$a = 127^\circ 38' 22''.$$

$$\frac{1}{2}(A + C) = 104^\circ.$$

$$c = 106^\circ 48' 22''.$$

$$\frac{1}{2}b = 37^\circ 30'.$$

$$\log \sin \frac{1}{2}(A + C) = 9.98783$$

$$\log \sin \frac{1}{2}(A - C) = 9.44034$$

$$\log \sec \frac{1}{2}(a - c) = 0.00722$$

$$\log \csc \frac{1}{2}(A + C) = 0.01310$$

$$\log \cos \frac{1}{2}b = 9.95366$$

$$\log \tan \frac{1}{2}b = 9.88498$$

$$\log \cos \frac{1}{2}B = 9.94871$$

$$\log \tan \frac{1}{2}(a - c) = 9.33842$$

$$\frac{1}{2}(a - c) = 12^\circ 17' 49''.$$

$$\frac{1}{2}B = 27^\circ 18'.$$

$$B = 54^\circ 36'.$$

$$\log \cos \frac{1}{2}(A - C) = 9.98284$$

$$\log \sec \frac{1}{2}(A + C) = 0.61632 (n)$$

$$\log \tan \frac{1}{2}b = 9.88498$$

$$\log \tan \frac{1}{2}(a + c) = 10.48414 (n)$$

$$\frac{1}{2}(a + c) = 108^\circ 9' 32''.$$

$$a = 120^\circ 27' 21''$$

$$c = 95^\circ 51' 43''.$$

$$14. A = 100^\circ, C = 90^\circ, b = 72^\circ.$$

$$\sin b = \tan a \cot A.$$

$$\tan a = \sin b \tan A.$$

$$\log \sin b = 9.97821$$

$$\log \tan A = 10.75368 (n)$$

$$\log \tan a = 10.73189 (n)$$

$$\log \sin \frac{1}{2}(A + C) = 9.98690$$

$$\log \sec \frac{1}{2}(a - c) = 0.01008$$

$$\log \cos \frac{1}{2}b = 9.89947$$

$$\log \cos \frac{1}{2}B = 9.89645$$

$$\frac{1}{2}B = 38^\circ 0' 48''.$$

$$B = 76^\circ 1' 36''.$$

$$a = 100^\circ 30' 12''.$$

Exercise 101. To find the Third Angle (Page 217)*Find the value of C , given the following parts:*

1. $A = 28^\circ$, $B = 40^\circ$, $c = 120^\circ$.

$$\cot x = \tan A \cos c.$$

$$\log \tan A = 9.72567$$

$$\log \cos c = \underline{9.69897} \text{ (n)}$$

$$\log \cot x = \underline{9.42464} \text{ (n)}$$

$$x = 104^\circ 53' 16''.$$

$$\cos C = \frac{\cos A \sin (B - x)}{\sin x}.$$

$$B - x = - (64^\circ 53' 16'').$$

$$\log \cos A = 9.94593$$

$$\log \sin (B - x) = 9.95688 \text{ (n)}$$

$$\text{colog } \sin x = \underline{0.01483}$$

$$\log \cos C = \underline{9.91764} \text{ (n)}$$

$$C = 145^\circ 49' 7''.$$

3. $A = 120^\circ$, $B = 100^\circ$, $c = 130^\circ$.

$$\cot x = \tan A \cos c.$$

$$\log \tan A = 10.23856 \text{ (n)}$$

$$\log \cos c = \underline{9.80807} \text{ (n)}$$

$$\log \cot x = \underline{10.04663}$$

$$x = 41^\circ 55' 48''.$$

$$\cos C = \frac{\cos A \sin (B - x)}{\sin x}.$$

$$B - x = 58^\circ 4' 12''.$$

$$\log \cos A = 9.69897 \text{ (n)}$$

$$\log \sin (B - x) = 9.92875$$

$$\text{colog } \sin x = \underline{0.17508}$$

$$\log \cos C = \underline{9.80280} \text{ (n)}$$

$$C = 129^\circ 25' 22''.$$

2. $A = 35^\circ$, $B = 45^\circ$, $c = 130^\circ$.

$$\cot x = \tan A \cos c.$$

$$\log \tan A = 9.84523$$

$$\log \cos c = \underline{9.80807} \text{ (n)}$$

$$\log \cot x = \underline{9.65330} \text{ (n)}$$

$$x = 114^\circ 13' 55''.$$

$$\cos C = \frac{\cos A \sin (B - x)}{\sin x}.$$

$$B - x = - (69^\circ 13' 55'').$$

$$\log \cos A = 9.91336$$

$$\log \sin (B - x) = 9.97083 \text{ (n)}$$

$$\text{colog } \sin x = \underline{0.04005}$$

$$\log \cos C = \underline{9.92424} \text{ (n)}$$

$$C = 147^\circ 7' 53''.$$

4. $A = 140^\circ$, $B = 75^\circ$, $c = 125^\circ$.

$$\cot x = \tan A \cos c.$$

$$\log \tan A = 9.92381 \text{ (n)}$$

$$\log \cos c = \underline{9.75859} \text{ (n)}$$

$$\log \cot x = \underline{9.68240}$$

$$x = 64^\circ 17' 58''.$$

$$\cos C = \frac{\cos A \sin (B - x)}{\sin x}.$$

$$B - x = 10^\circ 42' 2''.$$

$$\log \cos A = 9.88425 \text{ (n)}$$

$$\log \sin (B - x) = 9.26875$$

$$\text{colog } \sin x = \underline{0.04524}$$

$$\log \cos C = \underline{9.19824} \text{ (n)}$$

$$C = 99^\circ 4' 55''.$$

Find the value of the third angle, given the following parts :

$$\begin{aligned} 5. \quad A &= 26^\circ 58' 46'', \\ B &= 39^\circ 45' 10'', \\ c &= 154^\circ 46' 48''. \end{aligned}$$

$$\begin{aligned} \log \tan A &= 9.70678 \\ \log \cos c &= 9.95650 \text{ (n)} \\ \log \cot x &= 9.66328 \text{ (n)} \\ x &= 114^\circ 43' 44''. \end{aligned}$$

$$\begin{aligned} \log \cos A &= 9.94996 \\ \log \sin (B - x) &= 9.98490 \text{ (n)} \\ \text{colog} \sin x &= 0.04177 \\ \log \cos C &= 9.97663 \text{ (n)} \\ C &= 161^\circ 22' 15''. \end{aligned}$$

$$\begin{aligned} 6. \quad A &= 128^\circ 41' 49'', \\ B &= 107^\circ 33' 20'', \\ c &= 124^\circ 12' 31''. \end{aligned}$$

$$\begin{aligned} \log \tan A &= 10.09634 \text{ (n)} \\ \log \cos c &= 9.74990 \text{ (n)} \\ \log \cot x &= 9.84624 \\ x &= 54^\circ 56' 13''. \end{aligned}$$

$$\begin{aligned} \log \cos A &= 9.79602 \text{ (n)} \\ \log \sin (B - x) &= 9.90015 \\ \text{colog} \sin x &= 0.08697 \\ \log \cos C &= 9.78314 \text{ (n)} \\ C &= 127^\circ 22' 4''. \end{aligned}$$

Exercise 102. Given Two Sides and an Opposite Angle
(Page 219)

$$\begin{aligned} 1 \quad \text{Given } a &= 75^\circ, \\ b &= 110^\circ, \\ A &= 85^\circ, \text{ find } B. \end{aligned}$$

$$\sin B = \frac{\sin A \sin b}{\sin a}.$$

$$\begin{aligned} \log \sin A &= 9.99834 \\ \log \sin b &= 9.97299 \\ \log \sin a &= 0.01506 \\ \log \sin B &= 9.98639 \end{aligned}$$

$$B = 104^\circ 16' 15''$$

$$\begin{aligned} 2. \quad \text{Given } b &= 80^\circ, \\ c &= 115^\circ, \\ B &= 95^\circ, \text{ find } C. \end{aligned}$$

$$\sin C = \frac{\sin B \sin c}{\sin b}.$$

$$\begin{aligned} \log \sin B &= 9.99834 \\ \log \sin c &= 9.95728 \\ \text{colog} \sin b &= 0.00665 \\ \log \sin C &= 9.96227 \end{aligned}$$

$$C = 113^\circ 32' 20''$$

$$\begin{aligned} 3. \quad \text{Given } c &= 95^\circ, \\ a &= 120^\circ, \\ C &= 97^\circ, \text{ find } A. \end{aligned}$$

$$\sin A = \frac{\sin C \sin a}{\sin c}.$$

$$\begin{aligned} \log \sin C &= 9.99675 \\ \log \sin a &= 9.93753 \\ \text{colog} \sin c &= 0.00166 \\ \log \sin A &= 9.93594 \end{aligned}$$

$$A = 120^\circ 21' 37''.$$

Solve the triangles, given the following parts :

$$\begin{aligned} 4. \quad a &= 73^\circ 49' 38'', \\ b &= 120^\circ 53' 35'', \\ A &= 88^\circ 52' 42''. \end{aligned}$$

$$\begin{aligned} \log \sin A &= 9.99992 \\ \log \sin b &= 9.93355 \\ \text{colog} \sin a &= 0.01753 \\ \log \sin B &= 9.95100 \end{aligned}$$

$$\begin{aligned} B &= 180^\circ - (63^\circ 17' 30'') \\ &= 116^\circ 42' 30'', \end{aligned}$$

since the greater side is opposite the greater angle.

$$\begin{aligned} \frac{1}{2}(B + A) &= 102^\circ 47' 36'', \\ \frac{1}{2}(B - A) &= 13^\circ 54' 54'', \\ \frac{1}{2}(b + a) &= 97^\circ 21' 36.5'', \\ \frac{1}{2}(b - a) &= 23^\circ 31' 58.5''. \end{aligned}$$

$$\begin{aligned} \log \sin \frac{1}{2}(B + A) &= 9.98908 \\ \text{colog} \sin \frac{1}{2}(B - A) &= 0.61892 \\ \log \tan \frac{1}{2}(b - a) &= 9.63898 \\ \log \tan \frac{1}{2}c &= 10.24698 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}c &= 60^\circ 28' 43.4'', \\ c &= 120^\circ 57' 27'' \end{aligned}$$

$$\begin{aligned} \log \sin \frac{1}{2}(b + a) &= 9.99641 \\ \text{colog} \sin \frac{1}{2}(b - a) &= 0.39873 \\ \log \tan \frac{1}{2}(B - A) &= 9.39402 \\ \log \cot \frac{1}{2}C &= 9.78916 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}C &= 58^\circ 23' 30'' \\ C &= 116^\circ 47'. \end{aligned}$$

$$\begin{aligned} 5. \quad a &= 150^\circ 57' 5'', \\ b &= 134^\circ 15' 54'', \\ A &= 144^\circ 22' 42''. \end{aligned}$$

A and a are alike in kind and $\sin b > \sin a > \sin A \sin b$.
 $\therefore$ there are two solutions.

$$\begin{aligned} \log \sin A &= 9.76524 \\ \log \sin b &= 9.85498 \\ \text{colog} \sin a &= 0.31377 \\ \log \sin B &= 9.93399 \end{aligned}$$

$$\begin{aligned} B_1 &= 120^\circ 47' 45'', \\ B_2 &= 59^\circ 12' 15''. \end{aligned}$$

$$\begin{aligned} \frac{1}{2}(A + B_1) &= 132^\circ 35' 13.5'', \\ \frac{1}{2}(A + B_2) &= 101^\circ 47' 28.5'', \\ \frac{1}{2}(A - B_1) &= 11^\circ 47' 28.5'', \\ \frac{1}{2}(A - B_2) &= 42^\circ 35' 13.5'', \\ \frac{1}{2}(a - b) &= 8^\circ 20' 35.5'', \\ \frac{1}{2}(a + b) &= 142^\circ 36' 29.5''. \end{aligned}$$

$$\cot \frac{1}{2}C = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \frac{1}{2}(A + B)$$

$$\begin{aligned} \log \sin \frac{1}{2}(a + b) &= 9.78338 \\ \text{colog} \sin \frac{1}{2}(a - b) &= 0.83833 \\ \log \tan \frac{1}{2}(A - B_1) &= 9.31963 \end{aligned}$$

$$\begin{aligned} \log \cot \frac{1}{2}C_1 &= 9.94134 \\ \frac{1}{2}C_1 &= 48^\circ 51' 27.7'', \\ C_1 &= 97^\circ 42' 55''. \end{aligned}$$

$$\begin{aligned} \log \sin \frac{1}{2}(a + b) &= 9.78338 \\ \text{colog} \sin \frac{1}{2}(a - b) &= 0.83833 \\ \log \tan \frac{1}{2}(A - B_2) &= 9.96338 \end{aligned}$$

$$\begin{aligned} \log \cot \frac{1}{2}C_2 &= 10.58509 \\ \frac{1}{2}C_2 &= 14^\circ 34' 19.6'', \\ C_2 &= 29^\circ 8' 39''. \end{aligned}$$

$$\begin{aligned} \log \sin \frac{1}{2}(A + B_1) &= 9.86703 \\ \text{colog} \sin \frac{1}{2}(A - B_1) &= 0.68963 \\ \log \tan \frac{1}{2}(a - b) &= 9.16629 \\ \log \tan \frac{1}{2}c_1 &= 9.72295 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}c_1 &= 27^\circ 51' 4''. \\ c_1 &= 55^\circ 42' 8''. \end{aligned}$$

$$\begin{aligned} \log \sin \frac{1}{2}(A + B_2) &= 9.99074 \\ \text{colog} \sin \frac{1}{2}(A - B_2) &= 0.16960 \\ \log \tan \frac{1}{2}(a - b) &= 9.16629 \\ \log \tan \frac{1}{2}c_2 &= 9.32663 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}c_2 &= 11^\circ 58' 38.7'', \\ c_2 &= 23^\circ 57' 17''. \end{aligned}$$

6. $a = 79^\circ 0' 54''$, $b = 82^\circ 17' 4''$,
 $A = 82^\circ 9' 26''$.

$$\begin{aligned}\log \sin A &= 9.99592 \\ \log \sin b &= 9.99605 \\ \text{colog} \sin a &= \underline{0.00803} \\ \log \sin B &= \underline{0.00000}\end{aligned}$$

$$B = 90^\circ.$$

$$\begin{aligned}\tan c &= \cos A \tan b. \\ \cot C &= \tan A \cos b.\end{aligned}$$

$$\begin{aligned}\log \cos A &= 9.13499 \\ \log \tan b &= \underline{10.86812} \\ \log \tan c &= \underline{10.00311}\end{aligned}$$

$$c = 45^\circ 12' 19''.$$

$$\begin{aligned}\log \tan A &= 0.86093 \\ \log \cos b &= \underline{9.12793} \\ \log \cot C &= \underline{9.98886}\end{aligned}$$

$$C = 45^\circ 44' 5''.$$

7. Given $a = 30^\circ 52' 37''$, $b = 31^\circ 9' 16''$, and $A = 87^\circ 34' 12''$, show that the triangle is impossible.

$$\sin B = \sin A \sin b \csc a.$$

$$\begin{aligned}\log \sin A &= 9.99961 \\ \log \sin b &= 9.71378 \\ \text{colog} \sin a &= \underline{0.28972} \\ \log \sin B &= \underline{0.00311} \\ \sin B &= 1.0072.\end{aligned}$$

Since $\sin B > 1$, the triangle is impossible.

Reviewing preceding work, find the value of the third angle, given :

8. $A = 130^\circ 17'$, $B = 78^\circ 19'$,
 $c = 48^\circ 32'$.

$$\begin{aligned}\frac{1}{2}(A - B) &= 25^\circ 59'. \\ \frac{1}{2}(A + B) &= 104^\circ 18'. \\ \frac{1}{2}c &= 24^\circ 16'.$$

$$\begin{aligned}\log \sin \frac{1}{2}(A - B) &= 9.64158 \\ \text{colog} \sin \frac{1}{2}(A + B) &= 0.01367 \\ \log \tan \frac{1}{2}c &= \underline{9.65400} \\ \log \tan \frac{1}{2}(a - b) &= \underline{9.30925} \\ \frac{1}{2}(a - b) &= 11^\circ 31' 13''.\end{aligned}$$

$$\begin{aligned}\log \sin \frac{1}{2}(A + B) &= 9.98633 \\ \text{colog} \cos \frac{1}{2}(a - b) &= 0.00883 \\ \log \cos \frac{1}{2}c &= \underline{9.95982} \\ \log \cos \frac{1}{2}C &= \underline{9.95498} \\ \frac{1}{2}C &= 25^\circ 38' 20''. \\ C &= 51^\circ 16' 40''\end{aligned}$$

9. $B = 142^\circ 20'$, $C = 79^\circ 56'$,
 $a = 82^\circ 18'$.

$$\begin{aligned}\frac{1}{2}(B - C) &= 31^\circ 12'. \\ \frac{1}{2}(B + C) &= 111^\circ 8'. \\ \frac{1}{2}a &= 41^\circ 9'.$$

$$\begin{aligned}\log \sin \frac{1}{2}(B - C) &= 9.71435 \\ \text{colog} \sin \frac{1}{2}(B + C) &= 0.03024 \\ \log \tan \frac{1}{2}a &= \underline{9.94146} \\ \log \tan \frac{1}{2}(b - c) &= \underline{9.68605} \\ \frac{1}{2}(b - c) &= 25^\circ 53' 22''\end{aligned}$$

$$\begin{aligned}\log \sin \frac{1}{2}(B + C) &= 9.96976 \\ \text{colog} \cos \frac{1}{2}(b - c) &= 0.04593 \\ \log \cos \frac{1}{2}a &= \underline{9.87679} \\ \log \cos \frac{1}{2}A &= \underline{9.89248} \\ \frac{1}{2}A &= 38^\circ 40' 36''. \\ A &= 77^\circ 21' 12''.\end{aligned}$$

10. $B = 156^\circ 15'$, $C = 83^\circ 26'$,
 $a = 75^\circ 48'$.

$$\begin{aligned}\frac{1}{2}(B - C) &= 36^\circ 24' 30''. \\ \frac{1}{2}(B + C) &= 119^\circ 50' 30''. \\ \frac{1}{2}a &= 37^\circ 54'.$$

$$\begin{aligned}\log \sin \frac{1}{2}(B - C) &= 9.77345 \\ \text{colog} \sin \frac{1}{2}(B + C) &= 0.06177 \\ \log \tan \frac{1}{2}a &= \underline{9.89125} \\ \log \tan \frac{1}{2}(b - c) &= \underline{9.72647} \\ \frac{1}{2}(b - c) &= 28^\circ 2' 37''.\end{aligned}$$

$$\log \sin \frac{1}{2}(B + C) = 9.93823$$

$$\text{colog} \cos \frac{1}{2}(b - c) = 0.05424$$

$$\log \cos \frac{1}{2}a = 9.89712$$

$$\log \cos \frac{1}{2}A = 9.88959$$

$$\frac{1}{2}A = 39^\circ 8' 54''.$$

$$A = 78^\circ 17' 48''.$$

$$11. C = 75^\circ 48', A = 132^\circ 17',$$

$$b = 64^\circ 19'.$$

$$\frac{1}{2}(A - C) = 28^\circ 14' 30''.$$

$$\frac{1}{2}(A + C) = 104^\circ 2' 30''.$$

$$\frac{1}{2}b = 32^\circ 9' 30''.$$

$$\log \sin \frac{1}{2}(A - C) = 9.67504$$

$$\text{colog} \sin \frac{1}{2}(A + C) = 0.01317$$

$$\log \tan \frac{1}{2}b = 9.79846$$

$$\log \tan \frac{1}{2}(a - c) = 9.48677$$

$$\frac{1}{2}(a - c) = 17^\circ 3' 11''.$$

$$\log \sin \frac{1}{2}(A + C) = 9.98683$$

$$\text{colog} \cos \frac{1}{2}(a - c) = 0.01953$$

$$\log \cos \frac{1}{2}b = 9.92767$$

$$\log \cos \frac{1}{2}B = 9.93403$$

$$\frac{1}{2}B = 30^\circ 47' 15''.$$

$$B = 61^\circ 34' 30''.$$

$$12. C = 83^\circ 52', A = 127^\circ 48',$$

$$b = 72^\circ 50'.$$

$$\frac{1}{2}(A - C) = 21^\circ 58'.$$

$$\frac{1}{2}(A + C) = 105^\circ 50'.$$

$$\frac{1}{2}b = 36^\circ 25'.$$

$$\log \sin \frac{1}{2}(A - C) = 9.57295$$

$$\text{colog} \sin \frac{1}{2}(A + C) = 0.01680$$

$$\log \tan \frac{1}{2}b = 9.86789$$

$$\log \tan \frac{1}{2}(a - c) = 9.45764$$

$$\frac{1}{2}(a - c) = 16^\circ 0' 18''.$$

$$\log \sin \frac{1}{2}(A + C) = 9.98320$$

$$\text{colog} \cos \frac{1}{2}(a - c) = 0.01717$$

$$\log \cos \frac{1}{2}b = 9.90565$$

$$\log \cos \frac{1}{2}B = 9.90602$$

$$\frac{1}{2}B = 36^\circ 21'.$$

$$B = 72^\circ 42'.$$

$$13. A = 36.75^\circ,$$

$$B = 48.25^\circ,$$

$$c = 132.5^\circ.$$

$$\frac{1}{2}(B - A) = 5^\circ 45'.$$

$$\frac{1}{2}(B + A) = 42^\circ 30'.$$

$$\frac{1}{2}c = 66^\circ 15'.$$

$$\log \sin \frac{1}{2}(B - A) = 9.00082$$

$$\text{colog} \sin \frac{1}{2}(B + A) = 0.17032$$

$$\log \tan \frac{1}{2}c = \frac{10.35654}{9.52768}$$

$$\log \tan \frac{1}{2}(b - a) = 9.52768$$

$$\frac{1}{2}(b - a) = 18^\circ 37' 33''.$$

$$\log \sin \frac{1}{2}(B + A) = 9.82968$$

$$\text{colog} \cos \frac{1}{2}(b - a) = 0.02336$$

$$\log \cos \frac{1}{2}c = \frac{9.60503}{9.45807}$$

$$\log \cos \frac{1}{2}C = 9.45807$$

$$\frac{1}{2}C = 73^\circ 18' 51''.$$

$$C = 146^\circ 37' 42''.$$

$$14. A = 48.5^\circ,$$

$$B = 62.125^\circ,$$

$$c = 128.75^\circ.$$

$$\frac{1}{2}(B - A) = 6^\circ 48' 45''.$$

$$\frac{1}{2}(B + A) = 55^\circ 18' 45''.$$

$$\frac{1}{2}c = 64^\circ 22' 30''.$$

$$\log \sin \frac{1}{2}(B - A) = 9.07416$$

$$\text{colog} \sin \frac{1}{2}(B + A) = 0.08498$$

$$\log \tan \frac{1}{2}c = \frac{10.31907}{9.47821}$$

$$\log \tan \frac{1}{2}(b - a) = 9.47821$$

$$\frac{1}{2}(b - a) = 16^\circ 44' 20''.$$

$$\log \sin \frac{1}{2}(B + A) = 9.91502$$

$$\text{colog} \cos \frac{1}{2}(b - a) = 0.01880$$

$$\log \cos \frac{1}{2}c = \frac{9.63596}{9.56978}$$

$$\log \cos \frac{1}{2}C = 9.56978$$

$$\frac{1}{2}C = 68^\circ 12' 4''.$$

$$C = 136^\circ 24' 8''.$$

$$15. B = 156.6^\circ, A = 95.7^\circ, c = 117.8^\circ.$$

$$\begin{array}{ll} \frac{1}{2}(B - A) = 30^\circ 27'. & \log \sin \frac{1}{2}(B + A) = 9.90713 \\ \frac{1}{2}(B + A) = 126^\circ 9'. & \text{colog} \cos \frac{1}{2}(b - a) = 0.15928 \\ \frac{1}{2}c = 58^\circ 54'. & \log \cos \frac{1}{2}c = 9.71310 \\ \log \sin \frac{1}{2}(B - A) = 9.70482 & \log \cos \frac{1}{2}C = 9.77951 \\ \text{colog} \sin \frac{1}{2}(B + A) = 0.09287 & \frac{1}{2}C = 52^\circ 59' 42''. \\ \log \tan \frac{1}{2}c = 10.21951 & C = 105^\circ 59' 24''. \\ \log \tan \frac{1}{2}(b - a) = 10.01720 \\ \frac{1}{2}(b - a) = 46^\circ 8' 2''. \end{array}$$

Reviewing preceding work, solve the following triangles:

$$16. B = 153^\circ 17' 6'', \\ C = 78^\circ 43' 36'', \\ a = 86^\circ 15' 15''.$$

$$\begin{array}{l} \frac{1}{2}(B + C) = 116^\circ 0' 21''. \\ \frac{1}{2}(B - C) = 37^\circ 16' 45''. \\ \frac{1}{2}a = 43^\circ 7' 37.5''. \end{array}$$

$$\begin{array}{l} \log \cos \frac{1}{2}(B - C) = 9.90074 \\ \text{colog} \cos \frac{1}{2}(B + C) = 0.35807 (n) \\ \log \tan \frac{1}{2}a = 9.97159 \\ \log \tan \frac{1}{2}(b + c) = 10.23040 (n) \\ \frac{1}{2}(b + c) = 120^\circ 28' 6.2''. \end{array}$$

$$\begin{array}{l} \log \sin \frac{1}{2}(B - C) = 9.78226 \\ \text{colog} \sin \frac{1}{2}(B + C) = 0.04636 \\ \log \tan \frac{1}{2}a = 9.97159 \\ \log \tan \frac{1}{2}(b - c) = 9.80021 \\ \frac{1}{2}(b - c) = 32^\circ 15' 45''. \\ b = 152^\circ 43' 51''. \\ c = 88^\circ 12' 21''. \end{array}$$

$$\begin{array}{l} \log \sin \frac{1}{2}(B + C) = 9.95364 \\ \text{colog} \cos \frac{1}{2}(b - c) = 0.07283 \\ \log \cos \frac{1}{2}a = 9.86322 \\ \log \cos \frac{1}{2}A = 9.88969 \end{array}$$

$$\begin{array}{l} \frac{1}{2}A = 39^\circ 7' 54''. \\ A = 78^\circ 15' 48''. \end{array}$$

$$17. A = 125^\circ 41' 44'', \\ C = 82^\circ 47' 35'', \\ b = 52^\circ 37' 57''.$$

$$\begin{array}{l} \frac{1}{2}(A + C) = 104^\circ 14' 39.5''. \\ \frac{1}{2}(A - C) = 21^\circ 27' 4.5''. \\ \frac{1}{2}b = 26^\circ 18' 58.5''. \end{array}$$

$$\begin{array}{l} \log \cos \frac{1}{2}(A - C) = 9.96883 \\ \text{colog} \cos \frac{1}{2}(A + C) = 0.60896 (n) \\ \log \tan \frac{1}{2}b = 9.69424 \\ \log \tan \frac{1}{2}(a + c) = 10.27203 (n) \\ \frac{1}{2}(a + c) = 118^\circ 7' 32.9''. \end{array}$$

$$\begin{array}{l} \log \sin \frac{1}{2}(A - C) = 9.56313 \\ \text{colog} \sin \frac{1}{2}(A + C) = 0.01356 \\ \log \tan \frac{1}{2}b = 9.69424 \\ \log \tan \frac{1}{2}(a - c) = 9.27093 \\ \frac{1}{2}(a - c) = 10^\circ 34' 12.9''. \\ a = 128^\circ 41' 46''. \\ c = 107^\circ 33' 20''. \end{array}$$

$$\begin{array}{l} \log \sin \frac{1}{2}(A + C) = 9.98644 \\ \text{colog} \cos \frac{1}{2}(a - c) = 0.00743 \\ \log \cos \frac{1}{2}b = 9.95248 \\ \log \cos \frac{1}{2}B = 9.94635 \end{array}$$

$$\begin{array}{l} \frac{1}{2}B = 27^\circ 53' 50''. \\ B = 55^\circ 47' 40''. \end{array}$$

Exercise 103. Given Two Angles and an Opposite Side (Page 220)*Solve the triangles, given the following parts:*

1. $A = 110^\circ$, $B = 130^\circ$, $a = 150^\circ$.

$\log \sin a = 9.69897$

$\log \sin B = 9.88425$

$\log \sin A = 0.02701$

$\log \sin b = 9.61023$

$b = 155^\circ 56' 46''$.

$\frac{1}{2}(B + A) = 120^\circ$.

$\frac{1}{2}(B - A) = 10^\circ$.

$\frac{1}{2}a = 75^\circ$.

$\frac{1}{2}(b + a) = 152^\circ 58' 23''$.

$\frac{1}{2}(b - a) = 2^\circ 58' 23''$.

$\log \sin \frac{1}{2}(B + A) = 9.93753$

$\log \sin \frac{1}{2}(B - A) = 0.76033$

$\log \tan \frac{1}{2}(b - a) = 8.71546$

$\log \tan \frac{1}{2}c = 9.41332$

$\frac{1}{2}c = 14^\circ 31' 16''$.

$c = 29^\circ 2' 32''$.

$\log \sin \frac{1}{2}(b + a) = 9.65744$

$\log \sin \frac{1}{2}(b - a) = 1.28512$

$\log \tan \frac{1}{2}(B - A) = 9.24632$

$\log \cot \frac{1}{2}C = 10.18888$

$\frac{1}{2}C = 32^\circ 55' 58''$.

$C = 65^\circ 51' 56''$.

2. $A = 120^\circ$, $B = 115^\circ$, $a = 70^\circ$.

Since A and a are unlike in kind
and $\sin B > \sin A$,
there is no solution.

3. $A = 100^\circ$, $B = 100^\circ$, $a = 90^\circ$.

Since A and a are unlike in kind
and $\sin A = \sin B$,
there is no solution.

4. $A = 95^\circ$, $B = 96^\circ$, $a = 100^\circ$.

$\log \sin a = 9.99335$

$\log \sin B = 9.99761$

$\log \sin A = 0.00166$

$\log \sin b = 9.99262$

$b = 100^\circ 32'$.

$\frac{1}{2}(B + A) = 95^\circ 30'$.

$\frac{1}{2}(B - A) = 0^\circ 30'$.

$\frac{1}{2}(b + a) = 100^\circ 16'$.

$\frac{1}{2}(b - a) = 0^\circ 16'$.

$\log \sin \frac{1}{2}(B + A) = 9.99800$

$\log \sin \frac{1}{2}(B - A) = 2.05916$

$\log \tan \frac{1}{2}(b - a) = 7.66785$

$\log \tan \frac{1}{2}c = 9.72501$

$\frac{1}{2}c = 27^\circ 57' 50''$.

$c = 55^\circ 55' 40''$.

$\log \sin \frac{1}{2}(b + a) = 9.99299$

$\log \sin \frac{1}{2}(b - a) = 2.33216$

$\log \tan \frac{1}{2}(B - A) = 7.94086$

$\log \cot \frac{1}{2}C = 10.26601$

$\frac{1}{2}C = 28^\circ 27' 26''$.

$C = 56^\circ 54' 52''$.

5. $B = 98^\circ$, $C = 105^\circ$, $b = 80^\circ$.

$\log \sin b = 9.99335$

$\log \sin C = 9.98494$

$\log \sin B = 0.00425$

$\log \sin c = 9.98254$

$c = 106^\circ 8' 15''$.

$\frac{1}{2}(C + B) = 101^\circ 30'$.

$\frac{1}{2}(C - B) = 3^\circ 30'$.

$\frac{1}{2}(c + b) = 93^\circ 4' 7.5''$.

$\frac{1}{2}(c - b) = 13^\circ 4' 7.5''$.

$\log \sin \frac{1}{2}(C + B) = 9.99119$

$\log \sin \frac{1}{2}(C - B) = 1.21432$

$\log \tan \frac{1}{2}(c - b) = 9.36573$

$\log \tan \frac{1}{2}a = 10.57124$

$\frac{1}{2}a = 74^\circ 58' 36''$.

$a = 149^\circ 57' 12''$.

$\log \sin \frac{1}{2}(c + b) = 9.99938$

$\log \sin \frac{1}{2}(c - b) = 0.64566$

$\log \tan \frac{1}{2}(C - B) = 8.78649$

$\log \cot \frac{1}{2}A = 9.43153$

$\frac{1}{2}A = 74^\circ 58' 6''$.

$A = 149^\circ 46' 12''$.

$$6. C = 92^\circ, A = 115^\circ, c = 95^\circ$$

$$\log \sin c = 9.99834$$

$$\log \sin A = 9.95728$$

$$\text{colog} \sin C = \underline{0.00026}$$

$$\log \sin a = \underline{9.95588}$$

$$a = 115^\circ 23' 30''.$$

$$\frac{1}{2}(A + C) = 103^\circ 30'.$$

$$\frac{1}{2}(A - C) = 11^\circ 30'.$$

$$\frac{1}{2}(a + c) = 105^\circ 11' 45''.$$

$$\frac{1}{2}(a - c) = 10^\circ 11' 45''.$$

$$\log \sin \frac{1}{2}(A + C) = 9.98783$$

$$\text{colog} \sin \frac{1}{2}(A - C) = 0.70034$$

$$\log \tan \frac{1}{2}(a - c) = \underline{9.25492}$$

$$\log \tan \frac{1}{2}b = \underline{9.94309}$$

$$\frac{1}{2}b = 41^\circ 15' 24''.$$

$$b = 82^\circ 30' 48''.$$

$$\log \sin \frac{1}{2}(a + c) = 9.98454$$

$$\text{colog} \sin \frac{1}{2}(a - c) = 0.75199$$

$$\log \tan \frac{1}{2}(A - C) = \underline{9.30846}$$

$$\log \cot \frac{1}{2}B = \underline{10.04499}$$

$$\frac{1}{2}B = 42^\circ 2' 14''.$$

$$B = 84^\circ 4' 28''.$$

Find the side b , given the following parts :

$$7. A = 110^\circ 10', B = 133^\circ 18', a = 147^\circ 5' 32''.$$

$$\log \sin a = 9.73503$$

$$\log \sin B = 9.86200$$

$$\text{colog} \sin A = \underline{0.02748}$$

$$\log \sin b = \underline{9.62451}$$

$$b = 155^\circ 5' 18''.$$

$$8. A = 102^\circ 14' 12'',$$

$$C = 129^\circ 5' 46'',$$

$$a = 104^\circ 25' 9''.$$

$$9. C = 100^\circ 2' 11'',$$

$$A = 98^\circ 30' 28'',$$

$$c = 95^\circ 20' 39''.$$

$$\log \sin a = 9.98610$$

$$\log \sin C = 9.88991$$

$$\text{colog} \sin A = \underline{0.00998}$$

$$\log \sin c = 9.88599$$

$$c = 129^\circ 43' 33''.$$

$$\log \sin c = 9.99811$$

$$\log \sin A = 9.99519$$

$$\text{colog} \sin C = \underline{0.00670}$$

$$\log \sin a = \underline{10.00000}$$

$$a = 90^\circ.$$

$$\frac{1}{2}(C + A) = 115^\circ 39' 59''.$$

$$\frac{1}{2}(C - A) = 13^\circ 25' 47''.$$

$$\frac{1}{2}(c - a) = 12^\circ 39' 12''.$$

$$\log \sin \frac{1}{2}(C + A) = 9.95488$$

$$\text{colog} \sin \frac{1}{2}(C - A) = 0.63403$$

$$\log \tan \frac{1}{2}(c - a) = \underline{9.35123}$$

$$\log \tan \frac{1}{2}b = \underline{9.94014}$$

$$\frac{1}{2}b = 41^\circ 3' 50''.$$

$$b = 82^\circ 7' 40''.$$

$$\frac{1}{2}(C + A) = 99^\circ 16' 19.5''.$$

$$\frac{1}{2}(C - A) = 0^\circ 45' 51.5''.$$

$$\frac{1}{2}(c + a) = 92^\circ 40' 19.5''.$$

$$\frac{1}{2}(c - a) = 2^\circ 40' 19.5''.$$

$$\log \sin \frac{1}{2}(C + A) = 9.99428$$

$$\text{colog} \sin \frac{1}{2}(C - A) = 1.87487$$

$$\log \tan \frac{1}{2}(c - a) = \underline{8.66904}$$

$$\log \tan \frac{1}{2}b = \underline{10.53819}$$

$$\frac{1}{2}b = 73^\circ 50' 55''.$$

$$b = 147^\circ 41' 50''.$$

$$\begin{aligned} 10. \quad A &= 113^\circ 39' 21'', \\ B &= 123^\circ 40' 18'', \\ a &= 65^\circ 39' 46''. \end{aligned}$$

$$\begin{aligned} \log \sin a &= 9.95959 \\ \log \sin B &= 9.92024 \\ \text{colog} \sin A &= \underline{0.03812} \\ \log \sin b &= 9.91795 \end{aligned}$$

$$b = 124^\circ 7' 20''.$$

$$\begin{aligned} 11. \quad \text{Given } A &= 24^\circ 33' 9'', \\ B &= 38^\circ 0' 12'', \\ \text{and } a &= 65^\circ 20' 13'', \end{aligned}$$

show that the triangle is impossible.

$$\begin{aligned} \log \sin a &= 9.95845 \\ \log \sin B &= 9.78937 \\ \text{colog} \sin A &= \underline{0.38140} \\ \log \sin b &= 10.12922 \\ \sin b &= 1.3465. \end{aligned}$$

The triangle is impossible, since $\sin b > 1$.

Exercise 104. Given the Three Sides (Page 221)

Solve the triangles, given the following parts:

$$1. \quad a = 120^\circ, b = 60^\circ, c = 110^\circ.$$

$$\begin{aligned} a &= 120^\circ \\ b &= 60^\circ \\ c &= 110^\circ \\ 2s &= 290^\circ \\ s &= 145^\circ. \end{aligned}$$

$$\begin{aligned} s - a &= 25^\circ. \\ s - b &= 85^\circ. \\ s - c &= 35^\circ. \end{aligned}$$

$$\begin{aligned} \log \sin (s - b) &= 9.99834 \\ \log \sin (s - c) &= 9.75859 \\ \text{colog} \sin s &= 0.24141 \\ \text{colog} \sin (s - a) &= \underline{0.37405} \\ &2) 20.37239 \end{aligned}$$

$$\log \tan \frac{1}{2} A = 10.18619$$

$$\begin{aligned} \frac{1}{2} A &= 56^\circ 55' 19'', \\ A &= 113^\circ 50' 38''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2} B &= 9.81381. \\ \frac{1}{2} B &= 33^\circ 4' 41'', \\ B &= 66^\circ 9' 22''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2} C &= 10.05356. \\ \frac{1}{2} C &= 48^\circ 31' 26'', \\ C &= 97^\circ 2' 52''. \end{aligned}$$

$$2. \quad a = 50^\circ, b = 115^\circ, c = 130^\circ.$$

$$\begin{aligned} a &= 50^\circ \\ b &= 115^\circ \\ c &= 130^\circ \\ 2s &= 295^\circ \\ s &= 147^\circ 30'. \end{aligned}$$

$$\begin{aligned} s - a &= 97^\circ 30'. \\ s - b &= 32^\circ 30'. \\ s - c &= 17^\circ 30'. \end{aligned}$$

$$\begin{aligned} \log \sin (s - b) &= 9.73022 \\ \log \sin (s - c) &= 9.47814 \\ \text{colog} \sin s &= 0.26978 \\ \text{colog} \sin (s - a) &= \underline{0.00373} \\ &2) 19.48187 \end{aligned}$$

$$\log \tan \frac{1}{2} A = 9.74094$$

$$\begin{aligned} \frac{1}{2} A &= 28^\circ 50' 24'', \\ A &= 57^\circ 41' 8''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2} B &= 10.00699. \\ \frac{1}{2} B &= 45^\circ 27' 41'', \\ B &= 90^\circ 55' 22''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2} C &= 10.25907. \\ \frac{1}{2} C &= 61^\circ 9' 28'', \\ C &= 122^\circ 18' 56''. \end{aligned}$$

$$3. \ a = 130^\circ, \ b = 110^\circ, \ c = 85^\circ.$$

$$a = 130^\circ$$

$$b = 110^\circ$$

$$c = 85^\circ$$

$$2s = 325^\circ$$

$$s = 162^\circ 30'.$$

$$s - a = 32^\circ 30'.$$

$$s - b = 52^\circ 30'.$$

$$s - c = 77^\circ 30'.$$

$$\log \sin (s - b) = 9.89947$$

$$\log \sin (s - c) = 9.98958$$

$$\operatorname{colog} \sin s = 0.52186$$

$$\operatorname{colog} \sin (s - a) = 0.26978$$

$$2) 20.68069$$

$$\log \tan \frac{1}{2} A = 10.34035$$

$$\frac{1}{2} A = 65^\circ 27' 11''.$$

$$A = 130^\circ 54' 22''$$

$$\log \tan \frac{1}{2} B = 10.17110.$$

$$\frac{1}{2} B = 56^\circ 0' 19''.$$

$$B = 112^\circ 0' 38''.$$

$$\log \tan \frac{1}{2} C = 10.08099.$$

$$\frac{1}{2} C = 50^\circ 18' 42''.$$

$$C = 100^\circ 37' 24''.$$

$$4. \ a = 20^\circ, \ b = 60^\circ, \ c = 70^\circ.$$

$$a = 20^\circ$$

$$b = 60^\circ$$

$$c = 70^\circ$$

$$2s = 150^\circ$$

$$s = 75^\circ.$$

$$s - a = 55^\circ.$$

$$s - b = 15^\circ.$$

$$s - c = 5^\circ.$$

$$\log \sin (s - b) = 9.41300$$

$$\log \sin (s - c) = 8.94030$$

$$\operatorname{colog} \sin s = 0.01506$$

$$\operatorname{colog} \sin (s - a) = 0.08664$$

$$2) 18.45500$$

$$\log \tan \frac{1}{2} A = 9.22750$$

$$\frac{1}{2} A = 9^\circ 35' 2''.$$

$$A = 19^\circ 10' 4''.$$

$$\log \tan \frac{1}{2} B = 9.72786.$$

$$\frac{1}{2} B = 28^\circ 7' 11''.$$

$$B = 56^\circ 14' 22''.$$

$$\log \tan \frac{1}{2} C = 10.20056.$$

$$\frac{1}{2} C = 57^\circ 47'.$$

$$C = 115^\circ 34'.$$

$$5. \ a = 30^\circ, \ b = 50^\circ, \ c = 80^\circ.$$

The triangle is impossible, for

$$a + b = c.$$

$$6. \ a = 55^\circ, \ b = 100^\circ, \ c = 125^\circ.$$

$$a = 55^\circ$$

$$b = 100^\circ$$

$$c = 125^\circ$$

$$2s = 280^\circ$$

$$s = 140^\circ.$$

$$s - a = 85^\circ.$$

$$s - b = 40^\circ.$$

$$s - c = 15^\circ.$$

$$\log \sin (s - b) = 9.80807$$

$$\log \sin (s - c) = 9.41300$$

$$\operatorname{colog} \sin s = 0.19193$$

$$\operatorname{colog} \sin (s - a) = 0.00166$$

$$2) 19.41466$$

$$\log \tan \frac{1}{2} A = 9.70733$$

$$\frac{1}{2} A = 27^\circ 0' 31''.$$

$$A = 54^\circ 1' 2''.$$

$$\log \tan \frac{1}{2} B = 9.89760.$$

$$\frac{1}{2} B = 38^\circ 18' 25''.$$

$$B = 76^\circ 36' 50''.$$

$$\log \tan \frac{1}{2} C = 10.29267.$$

$$\frac{1}{2} C = 62^\circ 59' 29''.$$

$$C = 125^\circ 58' 58''.$$

Find the value of A , given the following parts:

7. $a = 120^\circ 55' 35''$, $b = 59^\circ 4' 25''$,
 $c = 106^\circ 10' 22''$.

$$\begin{aligned} a &= 120^\circ 55' 35'' \\ b &= 59^\circ 4' 25'' \\ c &= 106^\circ 10' 22'' \\ 2s &= 286^\circ 10' 22'' \\ s &= 143^\circ 5' 11''. \end{aligned}$$

$$\begin{aligned} s - a &= 22^\circ 9' 36''. \\ s - b &= 84^\circ 0' 46''. \\ s - c &= 36^\circ 54' 49''. \end{aligned}$$

$$\begin{aligned} \log \sin (s - b) &= 9.99763 \\ \log \sin (s - c) &= 9.77859 \\ \text{colog } \sin s &= 0.22141 \\ \text{colog } \sin (s - a) &= 0.42343 \end{aligned}$$

$$\begin{array}{r} 2 \overline{)20.42106} \\ \log \tan \frac{1}{2} A = 10.21053 \end{array}$$

$$\begin{aligned} \frac{1}{2} A &= 58^\circ 22' 24.8''. \\ A &= 116^\circ 44' 50''. \end{aligned}$$

9. $a = 131^\circ 35' 4''$, $b = 108^\circ 30' 14''$,
 $c = 84^\circ 46' 34''$.

$$\begin{aligned} a &= 131^\circ 35' 4'' \\ b &= 108^\circ 30' 14'' \\ c &= 84^\circ 46' 34'' \\ 2s &= 324^\circ 51' 52'' \\ s &= 162^\circ 25' 56''. \end{aligned}$$

$$\begin{aligned} s - a &= 30^\circ 50' 52''. \\ s - b &= 53^\circ 55' 42''. \\ s - c &= 77^\circ 39' 22''. \end{aligned}$$

$$\begin{aligned} \log \sin (s - b) &= 9.90756 \\ \log \sin (s - c) &= 9.98984 \\ \text{colog } \sin s &= 0.52023 \\ \text{colog } \sin (s - a) &= 0.29009 \end{aligned}$$

$$\begin{array}{r} 2 \overline{)20.70772} \\ \log \tan \frac{1}{2} A = 10.35386 \end{array}$$

$$\begin{aligned} \frac{1}{2} A &= 66^\circ 7' 11''. \\ A &= 132^\circ 14' 22''. \end{aligned}$$

8. $a = 50^\circ 12' 4''$, $b = 116^\circ 44' 48''$,
 $c = 129^\circ 11' 42''$.

$$\begin{aligned} a &= 50^\circ 12' 4'' \\ b &= 116^\circ 44' 48'' \\ c &= 129^\circ 11' 42'' \\ 2s &= 296^\circ 8' 34'' \\ s &= 148^\circ 4' 17''. \end{aligned}$$

$$\begin{aligned} s - a &= 97^\circ 52' 13''. \\ s - b &= 31^\circ 19' 29''. \\ s - c &= 18^\circ 52' 35''. \end{aligned}$$

$$\begin{aligned} \log \sin (s - b) &= 9.71591 \\ \log \sin (s - c) &= 9.50992 \\ \text{colog } \sin s &= 0.27666 \\ \text{colog } \sin (s - a) &= 0.00411 \end{aligned}$$

$$\begin{array}{r} 2 \overline{)19.50660} \\ \log \tan \frac{1}{2} A = 9.75330 \end{array}$$

$$\begin{aligned} \frac{1}{2} A &= 29^\circ 32' 14''. \\ A &= 59^\circ 4' 28''. \end{aligned}$$

10. $a = 20^\circ 16' 38''$, $b = 56^\circ 19' 40''$,
 $c = 66^\circ 20' 44''$.

$$\begin{aligned} a &= 20^\circ 16' 38'' \\ b &= 56^\circ 19' 40'' \\ c &= 66^\circ 20' 44'' \\ 2s &= 142^\circ 57' 2'' \\ s &= 71^\circ 28' 31''. \end{aligned}$$

$$\begin{aligned} s - a &= 51^\circ 11' 53''. \\ s - b &= 15^\circ 8' 51''. \\ s - c &= 5^\circ 7' 47''. \end{aligned}$$

$$\begin{aligned} \log \sin (s - b) &= 9.41715 \\ \log \sin (s - c) &= 8.95139 \\ \text{colog } \sin s &= 0.02311 \\ \text{colog } \sin (s - a) &= 0.10828 \end{aligned}$$

$$\begin{array}{r} 2 \overline{)18.49993} \\ \log \tan \frac{1}{2} A = 9.24997 \end{array}$$

$$\begin{aligned} \frac{1}{2} A &= 10^\circ 4' 58''. \\ A &= 20^\circ 9' 56''. \end{aligned}$$

Exercise 105. Given the Three Angles (Page 222)*Solve the triangles, given the following parts :*

1. $A = 120^\circ$, $B = 112^\circ$, $C = 85^\circ$.

$A = 120^\circ$

$B = 112^\circ$

$C = 85^\circ$

$2S = 317^\circ$

$S = 158^\circ 30'$

$S - A = 38^\circ 30'$

$S - B = 46^\circ 30'$

$S - C = 73^\circ 30'$

$\log \cos S = 9.96868 (n)$

$\log \cos(S - A) = 9.89354$

$\text{colog} \cos(S - B) = 0.16219$

$\text{colog} \cos(S - C) = 0.54666$

$2 \overline{20.57107}$

$\log \tan \frac{1}{2} a = 10.28554$

$\frac{1}{2} a = 62^\circ 36' 31''$

$a = 125^\circ 13' 2''$

$\log \tan \frac{1}{2} b = 10.22981$

$\frac{1}{2} b = 59^\circ 29' 52''$

$b = 118^\circ 59' 44''$

$\log \tan \frac{1}{2} c = 9.84534$

$\frac{1}{2} c = 35^\circ 0' 24''$

$c = 70^\circ 0' 48''$

2. $A = 60^\circ$, $B = 80^\circ$, $C = 60^\circ$.

$A = 60^\circ$

$B = 80^\circ$

$C = 60^\circ$

$2S = 200^\circ$

$S = 100^\circ$

$S - A = 40^\circ$

$S - B = 20^\circ$

$S - C = 40^\circ$

$\log \cos S = 9.23967 (n)$

$\log \cos(S - A) = 9.88425$

$\text{colog} \cos(S - B) = 0.02701$

$\text{colog} \cos(S - C) = 0.11575$

$2 \overline{19.26668}$

$\log \tan \frac{1}{2} a = 9.63334$

$\frac{1}{2} a = 23^\circ 15' 41''$

$a = 46^\circ 31' 22''$

c

$\log \tan \frac{1}{2} b = 9.72208$

$\frac{1}{2} b = 27^\circ 48' 14''$

$b = 55^\circ 36' 28''$

3. $A = 100^\circ$, $B = 55^\circ$, $C = 92^\circ$

$A = 100^\circ$

$B = 55^\circ$

$C = 92^\circ$

$2S = 247^\circ$

$S = 123^\circ 30'$

$S - A = 23^\circ 30'$

$S - B = 68^\circ 30'$

$S - C = 31^\circ 30'$

$\log \cos S = 9.74189 (n)$

$\log \cos(S - A) = 9.96240$

$\text{colog} \cos(S - B) = 0.43592$

$\text{colog} \cos(S - C) = 0.06923$

$2 \overline{20.20944}$

$\log \tan \frac{1}{2} a = 10.10472$

$\frac{1}{2} a = 51^\circ 50' 30''$

$a = 103^\circ 41''$

$\log \tan \frac{1}{2} b = 9.70640$

$\frac{1}{2} b = 26^\circ 57' 33''$

$b = 53^\circ 55' 6''$

$\log \tan \frac{1}{2} c = 10.07309$

$\frac{1}{2} c = 49^\circ 47' 55''$

$c = 99^\circ 35' 50''$

4. $A = 5^\circ$, $B = 39^\circ$, $C = 150^\circ$.

By § 193,

$$\cos \frac{1}{2} a = \sqrt{\frac{\cos(S-B) \cos(S-C)}{\sin B \sin C}}.$$

$$A = 5^\circ$$

$$B = 39^\circ$$

$$C = 150^\circ$$

$$2S = 194^\circ$$

$$S = 97^\circ.$$

$$S - A = 92^\circ.$$

$$S - B = 58^\circ.$$

$$S - C = -53^\circ.$$

$$\log \cos(S - B) = 9.72421$$

$$\log \cos(S - C) = 9.77946$$

$$\text{colog} \sin B = 0.20113$$

$$\text{colog} \sin C = 0.30103$$

$$2) 20.00583$$

$$\log \cos \frac{1}{2} a = 10.00583$$

The triangle is impossible, since $\cos \frac{1}{2} a > 1$.

5. $A = 75^\circ$, $B = 75^\circ$, $C = 75^\circ$.

$$A = 75^\circ$$

$$B = 75^\circ$$

$$C = 75^\circ$$

$$2S = 225^\circ$$

$$S = 112^\circ 30'.$$

$$S - A = 37^\circ 30'.$$

$$S - B = 37^\circ 30'.$$

$$S - C = 37^\circ 30'.$$

$$\log \cos S = 9.58284 (n)$$

$$\log \cos(S - A) = 9.89947$$

$$\text{colog} \cos(S - B) = 0.10053$$

$$\text{colog} \cos(S - C) = 0.10053$$

$$2) 19.68337$$

$$\log \tan \frac{1}{2} a = 9.84169$$

$$\frac{1}{2} a = 34^\circ 46' 51''.$$

$$a = 69^\circ 33' 42''$$

$$= b = c.$$

6. $A = 100^\circ$, $B = 105^\circ$, $C = 110^\circ$

$$A = 100^\circ$$

$$B = 105^\circ$$

$$C = 110^\circ$$

$$2S = 315^\circ$$

$$S = 157^\circ 30'.$$

$$S - A = 57^\circ 30'.$$

$$S - B = 52^\circ 30'.$$

$$S - C = 47^\circ 30'.$$

$$\log \cos S = 9.96562 (n)$$

$$\log \cos(S - A) = 9.73022$$

$$\text{colog} \cos(S - B) = 0.21555$$

$$\text{colog} \cos(S - C) = 0.17032$$

$$2) 20.08171$$

$$\log \tan \frac{1}{2} a = 10.04086$$

$$\frac{1}{2} a = 47^\circ 41' 29''.$$

$$a = 95^\circ 22' 58''.$$

$$\log \tan \frac{1}{2} b = 10.09509.$$

$$\frac{1}{2} b = 51^\circ 13' 23''.$$

$$b = 102^\circ 26' 46''.$$

$$\log \tan \frac{1}{2} c = 10.14032.$$

$$\frac{1}{2} c = 54^\circ 5' 58''.$$

$$c = 108^\circ 11' 56''.$$

Find a and b , given the following parts :

7. $A = 130^\circ$, $B = 110^\circ$, $C = 80^\circ$.

$$\begin{aligned} A &= 130^\circ \\ B &= 110^\circ \\ C &= 80^\circ \\ 2S &= 320^\circ \end{aligned}$$

$$S = 160^\circ.$$

$$S - A = 30^\circ.$$

$$S - B = 50^\circ.$$

$$S - C = 80^\circ.$$

$$\begin{aligned} \log \cos S &= 9.97299 \text{ (n)} \\ \log \cos (S - A) &= 9.93753 \\ \text{colog} \cos (S - B) &= 0.19193 \\ \text{colog} \cos (S - C) &= 0.76033 \\ 2) 20.86278 \end{aligned}$$

$$\log \tan \frac{1}{2} a = 10.43139$$

$$\begin{aligned} \frac{1}{2} a &= 69^\circ 40' 41'', \\ a &= 139^\circ 21' 22''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2} b &= 10.30193. \\ \frac{1}{2} b &= 63^\circ 28' 56'', \\ b &= 126^\circ 57' 52''. \end{aligned}$$

8. $A = 59^\circ 55' 10''$, $B = 85^\circ 36' 50''$,
 $C = 59^\circ 55' 10''$.

$$\begin{aligned} A &= 59^\circ 55' 10'' \\ B &= 85^\circ 36' 50'' \\ C &= 59^\circ 55' 10'' \\ 2S &= 205^\circ 27' 10'' \end{aligned}$$

$$S = 102^\circ 43' 35''.$$

$$S - A = 42^\circ 48' 25''.$$

$$S - B = 17^\circ 6' 45''.$$

$$S - C = 42^\circ 48' 25''.$$

$$\begin{aligned} \log \cos S &= 9.34301 \text{ (n)} \\ \log \cos (S - A) &= 9.86549 \\ \text{colog} \cos (S - B) &= 0.01967 \\ \text{colog} \cos (S - C) &= 0.13451 \\ 2) 19.36268 \\ \log \tan \frac{1}{2} a &= 9.68134 \end{aligned}$$

$$\begin{aligned} \frac{1}{2} a &= 25^\circ 38' 45.5'', \\ a &= 51^\circ 17' 31''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2} b &= 9.79618. \\ \frac{1}{2} b &= 32^\circ 1' 23.6'' \\ b &= 64^\circ 2' 47''. \end{aligned}$$

9. $A = 102^\circ 14' 12''$, $B = 54^\circ 32' 24''$,
 $C = 89^\circ 5' 46''$.

$$\begin{aligned} A &= 102^\circ 14' 12'' \\ B &= 54^\circ 32' 24'' \\ C &= 89^\circ 5' 46'' \\ 2S &= 245^\circ 52' 22'' \end{aligned}$$

$$S = 122^\circ 56' 11''.$$

$$\begin{aligned} S - A &= 20^\circ 41' 59'', \\ S - B &= 68^\circ 23' 47'', \\ S - C &= 33^\circ 50' 25''. \end{aligned}$$

$$\begin{aligned} \log S &= 9.73536 \text{ (n)} \\ \log \cos (S - A) &= 9.97102 \\ \text{colog} \cos (S - B) &= 0.43394 \\ \text{colog} \cos (S - C) &= 0.08061 \\ 2) 20.22093 \\ \log \tan \frac{1}{2} a &= 10.11047 \end{aligned}$$

$$\begin{aligned} \frac{1}{2} a &= 52^\circ 12' 34.6'', \\ a &= 104^\circ 25' 9''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2} b &= 9.70551. \\ \frac{1}{2} b &= 26^\circ 54' 42.6'' \\ b &= 53^\circ 49' 25''. \end{aligned}$$

$$\begin{aligned} 10. \quad A &= 4^\circ 23' 35'', \\ B &= 8^\circ 28' 20'', \\ C &= 172^\circ 17' 56''. \end{aligned}$$

$$\begin{aligned} A &= 4^\circ 23' 35'' \\ B &= 8^\circ 28' 20'' \\ C &= 172^\circ 17' 56'' \\ 2S &= 185^\circ 9' 51'' \\ S &= 92^\circ 34' 55.5''. \end{aligned}$$

$$\begin{aligned} S - A &= 88^\circ 11' 20.5'', \\ S - B &= 84^\circ 6' 35.5'', \\ S - C &= -(79^\circ 43' 0.5''). \end{aligned}$$

$$\begin{aligned} \log \cos S &= 8.65370 \text{ (n)} \\ \log \cos (S - A) &= 8.49971 \\ \text{colog } \cos (S - B) &= 0.98876 \\ \text{colog } \cos (S - C) &= 0.74833 \end{aligned}$$

$$\begin{array}{r} 2 \overline{)18.89050} \\ \log \tan \frac{1}{2} a = 9.44525 \end{array}$$

$$\begin{aligned} \frac{1}{2} a &= 15^\circ 34' 37'', \\ a &= 31^\circ 9' 14''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2} b &= 9.95678. \\ \frac{1}{2} b &= 42^\circ 9' 14''. \\ b &= 84^\circ 18' 28''. \end{aligned}$$

$$\begin{aligned} 11. \quad A &= 71^\circ 27' 30'', \\ B &= 16^\circ 29' 30'', \\ C &= 140^\circ 18' 50''. \end{aligned}$$

By § 193,

$$\cos \frac{1}{2} b = \sqrt{\frac{\cos (S - A) \cos (S - C)}{\sin A \sin C}}.$$

$$\begin{aligned} A &= 71^\circ 27' 30'' \\ B &= 16^\circ 29' 30'' \\ C &= 140^\circ 18' 50'' \\ 2S &= 228^\circ 15' 50'' \\ S &= 114^\circ 7' 55''. \end{aligned}$$

$$\begin{aligned} S - A &= 42^\circ 40' 25'', \\ S - B &= 97^\circ 38' 25'', \\ S - C &= -(26^\circ 10' 55'') \end{aligned}$$

$$\begin{aligned} \log \cos (S - A) &= 9.86642 \\ \log \cos (S - C) &= 9.95298 \\ \text{colog } \sin A &= 0.02315 \\ \text{colog } \sin C &= 0.19468 \\ 2 \overline{)20.03723} \\ \log \cos \frac{1}{2} b &= 10.01862 \end{aligned}$$

The triangle is impossible, since $\cos b > 1$.

$$\begin{aligned} 12. \quad A &= 42.75^\circ, \\ B &= 27.5^\circ, \\ C &= 150.3^\circ. \end{aligned}$$

$$\begin{aligned} A &= 42^\circ 45' \\ B &= 27^\circ 30' \\ C &= 150^\circ 18' \\ 2S &= 220^\circ 33' \end{aligned}$$

$$S = 110^\circ 16' 30''.$$

$$\begin{aligned} S - A &= 67^\circ 31' 30'', \\ S - B &= 82^\circ 46' 30'', \\ S - C &= -(40^\circ 1' 30''). \end{aligned}$$

$$\begin{aligned} \log \cos S &= 9.53974 \text{ (n)} \\ \log \cos (S - A) &= 9.58238 \\ \text{colog } \cos (S - B) &= 0.90044 \\ \text{colog } \cos (S - C) &= 0.11591 \\ 2 \overline{)20.13847} \end{aligned}$$

$$\log \tan \frac{1}{2} a = 10.06924$$

$$\begin{aligned} \frac{1}{2} a &= 49^\circ 32' 53'', \\ a &= 99^\circ 5' 46''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2} b &= 9.58642. \\ \frac{1}{2} b &= 21^\circ 5' 57'', \\ b &= 42^\circ 11' 54'' \end{aligned}$$

$$13. A = 72.51^\circ, B = 142.65^\circ, \\ C = 100.2^\circ.$$

$$\begin{aligned} A &= 72^\circ 30' 36'' \\ B &= 142^\circ 39' \\ C &= \underline{100^\circ 12'} \\ 2S &= \underline{315^\circ 21' 36''} \\ S &= 157^\circ 40' 48''. \\ S - A &= 85^\circ 10' 12''. \\ S - B &= 15^\circ 1' 48''. \\ S - C &= 57^\circ 28' 48''. \end{aligned}$$

$$\begin{aligned} \log \cos S &= 9.96618 \text{ (n)} \\ \log \cos (S - A) &= 8.92531 \\ \text{colog } \cos (S - B) &= 0.01511 \\ \text{colog } \cos (S - C) &= \underline{0.26955} \\ &2) 19.17615 \\ \log \tan \frac{1}{2} a &= 9.58808 \\ \frac{1}{2} a &= 21^\circ 10' 22''. \\ a &= 42^\circ 20' 44''. \\ \log \tan \frac{1}{2} b &= 10.64766. \\ \frac{1}{2} b &= 77^\circ 18' 55''. \\ b &= 154^\circ 37' 50''. \end{aligned}$$

$$14. A = 121^\circ 10' 10'', B = 68^\circ 42' 30'', \\ C = 21^\circ 17' 30''.$$

$$\begin{aligned} A &= 121^\circ 10' 10'' \\ B &= 68^\circ 42' 30'' \\ C &= \underline{21^\circ 17' 30''} \\ 2S &= \underline{211^\circ 10' 10''} \\ S &= 105^\circ 35' 5''. \\ S - A &= - (15^\circ 35' 5''). \\ S - B &= 36^\circ 52' 35''. \\ S - C &= 84^\circ 17' 35''. \end{aligned}$$

$$\begin{aligned} \log \cos S &= 9.42921 \text{ (n)} \\ \log \cos (S - A) &= 9.98373 \\ \text{colog } \cos (S - B) &= 0.09695 \\ \text{colog } \cos (S - C) &= \underline{1.00244} \\ &2) 20.51233 \\ \log \tan \frac{1}{2} a &= 10.25617 \\ \frac{1}{2} a &= 60^\circ 59' 44''. \\ a &= 121^\circ 59' 28''. \\ \log \tan \frac{1}{2} b &= 10.17548. \\ \frac{1}{2} b &= 56^\circ 16' 22''. \\ b &= 112^\circ 32' 44''. \end{aligned}$$

Exercise 106. Areas of Spherical Triangles (Page 223)

Find the areas of the following triangles :

$$1. A = 80^\circ, B = 35^\circ, \\ C = 70^\circ, r = 10.$$

$$\begin{aligned} A &= 80^\circ \\ B &= 35^\circ \\ C &= \underline{70^\circ} \\ &185^\circ \\ E = 5^\circ &= 18,000''. \\ 180^\circ &= 648,000''. \\ \log r^2 &= 2.00000 \\ \log E &= 4.25527 \\ \log \pi &= 0.49715 \\ \text{colog } 648,000 &= \underline{4.18842 - 10} \\ \log T &= 0.94084 \\ T &= 8.7265. \end{aligned}$$

$$2. A = 85^\circ 30', B = 29^\circ 45', \\ C = 72^\circ 15', r = 5.$$

$$\begin{aligned} A &= 85^\circ 30' \\ B &= 29^\circ 45' \\ C &= \underline{72^\circ 15'} \\ &187^\circ 30' \\ E = 7^\circ 30' &= 27,000''. \\ 180^\circ &= 648,000''. \\ \log r^2 &= 1.39794 \\ \log E &= 4.43126 \\ \log \pi &= 0.49715 \\ \text{colog } 648,000 &= \underline{4.18842 - 10} \\ \log T &= 0.51487 \\ T &= 3.2724. \end{aligned}$$

3. $A = 84^\circ 20' 19''$, $B = 27^\circ 22' 40''$, $C = 75^\circ 33'$, $r = 20$. 4. $A = 93^\circ 30' 10''$, $B = 32^\circ 35' 30''$, $C = 88^\circ 25'$, $r = 50$.

$$\begin{array}{r} A = 84^\circ 20' 19'' \\ B = 27^\circ 22' 40'' \\ C = 75^\circ 33' \\ \hline 187^\circ 15' 59'' \end{array}$$

$$\begin{aligned} E &= 7^\circ 15' 59'' = 26,159''. \\ 180^\circ &= 648,000''. \end{aligned}$$

$$\begin{aligned} \log r^2 &= 2.60206 \\ \log E &= 4.41762 \\ \log \pi &= 0.49715 \\ \text{colog } 648,000 &= 4.18842 - 10 \\ \log T &= 1.70525 \\ T &= 50.729. \end{aligned}$$

$$\begin{array}{r} A = 93^\circ 30' 10'' \\ B = 32^\circ 35' 30'' \\ C = 88^\circ 25' \\ \hline 214^\circ 30' 40'' \end{array}$$

$$\begin{aligned} E &= 34^\circ 30' 40'' = 124,240''. \\ 180^\circ &= 648,000''. \end{aligned}$$

$$\begin{aligned} \log r^2 &= 3.39794 \\ \log E &= 5.09426 \\ \log \pi &= 0.49715 \\ \text{colog } 648,000 &= 4.18842 - 10 \\ \log T &= 3.17777 \\ T &= 1505.8. \end{aligned}$$

Exercise 107. Finding Areas (Page 225)

Find the spherical excess, given:

1. $A = 80^\circ$, $B = 50^\circ$, $C = 75^\circ$.

$$\begin{array}{r} A = 80^\circ \\ B = 30^\circ \\ C = 75^\circ \\ \hline 185^\circ \end{array}$$

$$E = 185^\circ - 180^\circ = 5^\circ.$$

4. $A = 88^\circ$, $B = 95^\circ$, $C = 100^\circ$.

$$\begin{array}{r} A = 88^\circ \\ B = 95^\circ \\ C = 100^\circ \\ \hline 283^\circ \end{array}$$

$$E = 283^\circ - 180^\circ = 103^\circ.$$

2. $A = 70^\circ$, $B = 110^\circ$, $C = 80^\circ$.

$$\begin{array}{r} A = 70^\circ \\ B = 110^\circ \\ C = 80^\circ \\ \hline 260^\circ \end{array}$$

$$E = 260^\circ - 180^\circ = 80^\circ.$$

5. $A = 72^\circ$, $B = 98^\circ$, $C = 110^\circ$.

$$\begin{array}{r} A = 72^\circ \\ B = 98^\circ \\ C = 110^\circ \\ \hline 280^\circ \end{array}$$

$$E = 280^\circ - 180^\circ = 100^\circ.$$

3. $A = 95^\circ$, $B = 120^\circ$, $C = 85^\circ$.

$$\begin{array}{r} A = 95^\circ \\ B = 120^\circ \\ C = 85^\circ \\ \hline 300^\circ \end{array}$$

$$E = 300^\circ - 180^\circ = 120^\circ.$$

6. $A = 96^\circ$, $B = 97^\circ$, $C = 98^\circ$.

$$\begin{array}{r} A = 96^\circ \\ B = 97^\circ \\ C = 98^\circ \\ \hline 291^\circ \end{array}$$

$$E = 291^\circ - 180^\circ = 111^\circ.$$

Find the areas of the following triangles, given :

7. $a = 100^\circ, b = 75^\circ, c = 80^\circ.$

$$a = 100^\circ$$

$$b = 75^\circ$$

$$c = 80^\circ$$

$$2s = 255^\circ$$

$$s = 127^\circ 30'.$$

$$s - a = 27^\circ 30'.$$

$$s - b = 52^\circ 30'.$$

$$s - c = 47^\circ 30'.$$

$$\log \tan \frac{1}{2}s = 10.30702$$

$$\log \tan \frac{1}{2}(s - a) = 9.38863$$

$$\log \tan \frac{1}{2}(s - b) = 9.69298$$

$$\log \tan \frac{1}{2}(s - c) = 9.64346$$

$$\log \tan^2 \frac{1}{4}E = 19.03209$$

$$\log \tan \frac{1}{4}E = 9.51605.$$

$$\frac{1}{4}E = 18^\circ 9' 59''.$$

$$E = 72^\circ 39' 56''$$

$$= 261,596''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.41763$$

$$\log \pi = 0.49715$$

$$\text{colog } 648,000 = \frac{4.18842 - 10}{}$$

$$\log T = 0.10320 + \log r^2$$

$$T = 1.2682 r^2.$$

8. $a = 110^\circ, b = 85^\circ, c = 95^\circ.$

$$a = 110^\circ$$

$$b = 85^\circ$$

$$c = 95^\circ$$

$$2s = 290^\circ$$

$$s = 145^\circ$$

$$s - a = 35^\circ.$$

$$s - b = 60^\circ.$$

$$s - c = 50^\circ.$$

$$\log \tan \frac{1}{2}s = 10.50128$$

$$\log \tan \frac{1}{2}(s - a) = 9.49872$$

$$\log \tan \frac{1}{2}(s - b) = 9.76144$$

$$\log \tan \frac{1}{2}(s - c) = 9.66867$$

$$\log \tan^2 \frac{1}{4}E = 19.43011$$

$$\log \tan \frac{1}{4}E = 9.71506.$$

$$\frac{1}{4}E = 27^\circ 25' 25''.$$

$$E = 109^\circ 41' 40''$$

$$= 394,900''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.59649$$

$$\log \pi = 0.49715$$

$$\text{colog } 648,000 = \frac{4.18842 - 10}{}$$

$$\log T = 0.28206 + \log r^2$$

$$T = 1.9145 r^2.$$

9. $A = 120^\circ, B = 78^\circ, c = 115^\circ.$

By § 200, $\cot x = \tan A \cos c$

and $\cos C = \frac{\cos A \sin(B-x)}{\sin x}$

$$\log \tan A = 10.23856 (n)$$

$$\log \cos c = 9.62595 (n)$$

$$\log \cot x = 9.86451$$

$$x = 53^\circ 47' 46''.$$

$$B - x = 24^\circ 12' 14''.$$

$$\log \cos A = 9.69897 (n)$$

$$\log \sin(B - x) = 9.61277$$

$$\text{colog } \sin x = 0.09319$$

$$\log \cos C = 9.40493 (n)$$

$$C = 104^\circ 43' 4''.$$

$$A = 120^\circ$$

$$B = 78^\circ$$

$$C = 104^\circ 43' 4''$$

$$302^\circ 43' 4''$$

$$E = 122^\circ 43' 4''$$

$$= 441,784''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.64521$$

$$\log \pi = 0.49715$$

$$\text{colog } 648,000 = \frac{4.18842 - 10}{}$$

$$\log T = 0.33078 + \log r^2$$

$$T = 2.1418 r^2$$

$$10. A = 60^\circ, a = 75^\circ, b = 80^\circ.$$

There are two solutions, since a and A are alike in kind and $\sin b > \sin a > \sin A$ $\sin b$.

$$\sin B = \frac{\sin A \sin b}{\sin a}.$$

$$\log \sin A = 9.93753$$

$$\log \sin b = 9.99335$$

$$\text{colog } \sin a = 0.01506$$

$$\log \sin B = 9.94594$$

$$B_1 = 62^\circ 0' 9''.$$

$$B_2 = 117^\circ 59' 51''.$$

$$\frac{1}{2}(b + a) = 77^\circ 30'.$$

$$\frac{1}{2}(b - a) = 2^\circ 30'.$$

$$\frac{1}{2}(B_1 + A) = 61^\circ 0' 4.5''.$$

$$\frac{1}{2}(B_2 + A) = 88^\circ 59' 55.5''.$$

$$\frac{1}{2}(B_1 - A) = 1^\circ 0' 4.5''.$$

$$\frac{1}{2}(B_2 - A) = 28^\circ 59' 55.5''.$$

$$\log \sin \frac{1}{2}(B_1 + A) = 9.94183$$

$$\log \csc \frac{1}{2}(B_1 - A) = 1.75760$$

$$\log \tan \frac{1}{2}(b - a) = 8.64009$$

$$\log \tan \frac{1}{2}c_1 = 10.33952$$

$$\frac{1}{2}c_1 = 65^\circ 24' 41.8''$$

$$c_1 = 130^\circ 49' 24''.$$

$$\log \sin \frac{1}{2}(B_2 + A) = 9.99993$$

$$\log \csc \frac{1}{2}(B_2 - A) = 0.31445$$

$$\log \tan \frac{1}{2}(b - a) = 8.64009$$

$$\log \tan \frac{1}{2}c_2 = 8.95447$$

$$\frac{1}{2}c_2 = 5^\circ 8' 43.5''.$$

$$c_2 = 10^\circ 17' 27''.$$

$$a = 75^\circ$$

$$b = 80^\circ$$

$$c = 130^\circ 49' 24''$$

$$2s = 285^\circ 49' 24''$$

$$\frac{1}{2}s = 71^\circ 27' 21''$$

$$\frac{1}{2}(s - a) = 33^\circ 57' 21''$$

$$\frac{1}{2}(s - b) = 31^\circ 27' 21''$$

$$\frac{1}{2}(s - c) = 6^\circ 2' 39''$$

$$75^\circ$$

$$80^\circ$$

$$\text{or } 10^\circ 17' 27''$$

$$\text{or } 165^\circ 17' 27''$$

$$\text{or } 41^\circ 19' 22''.$$

$$\text{or } 3^\circ 49' 22''.$$

$$\text{or } 1^\circ 19' 22''.$$

$$\text{or } 36^\circ 10' 38''.$$

$$\log \tan \frac{1}{2}s = 10.47437$$

$$\log \tan \frac{1}{2}(s - a) = 9.82826$$

$$\log \tan \frac{1}{2}(s - b) = 9.78657$$

$$\log \tan \frac{1}{2}(s - c) = 9.02483$$

$$\log \tan^2 \frac{1}{4}E = 19.11403$$

$$\log \tan \frac{1}{4}E = 9.55702$$

$$\frac{1}{4}E = 19^\circ 49' 44.6''$$

$$E = 70^\circ 18' 58''$$

$$= 285,538''$$

$$\text{or } 9.94410$$

$$\text{or } 8.82490$$

$$\text{or } 8.36343$$

$$\text{or } 9.86408$$

$$\text{or } 16.99641$$

$$\text{or } 8.49821.$$

$$\text{or } 1^\circ 48' 13.7''.$$

$$\text{or } 7^\circ 12' 55''.$$

$$\text{or } 25,975''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.45567$$

$$\log \pi = 0.49715$$

$$\text{colog } 648,000 = 4.18842 - 10$$

$$\log T = 0.14124 + \log r^2$$

$$T = 1.3843 r^2$$

$$\log r^2$$

$$\text{or } 4.41455$$

$$0.49715$$

$$4.18842 - 10$$

$$\text{or } 1.10012 + \log r^2$$

$$\text{or } 0.12593 r^2.$$

$$11. A = 80^\circ, B = 75^\circ, a = 75^\circ.$$

$$\sin b = \frac{\sin a \sin B}{\sin A}.$$

$$\log \sin a = 9.98494$$

$$\log \sin B = 9.98494$$

$$\text{colog } \sin A = 0.00665$$

$$\log \sin b = 9.97653$$

$$b = 71^\circ 20'.$$

$$\frac{1}{2}(a + b) = 73^\circ 10'.$$

$$\frac{1}{2}(a - b) = 1^\circ 50'.$$

$$\frac{1}{2}(A - B) = 2^\circ 30'.$$

$$\log \sin \frac{1}{2}(a + b) = 9.98098$$

$$\log \csc \frac{1}{2}(a - b) = 1.49496$$

$$\log \tan \frac{1}{2}(A - B) = 8.64009$$

$$\log \cot \frac{1}{2}C = 10.11603$$

$$\frac{1}{2}C = 37^\circ 26' 9''.$$

$$C = 74^\circ 52' 18''.$$

$$A = 80^\circ$$

$$B = 75^\circ$$

$$C = 74^\circ 52' 18''$$

$$229^\circ 52' 18''$$

$$E = 49^\circ 52' 18''$$

$$= 179, 538''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.25416$$

$$\log \pi = 0.49715$$

$$\text{colog } 648,000 = 4.18842 - 10$$

$$\log T = 1.93973 + \log r^2$$

$$T = 0.87042 r^2.$$

$$12. A = 150^\circ, b = 45^\circ, c = 15^\circ.$$

$$\tan m = \tan c \cos A.$$

$$\cos a = \cos c \sec m \cos(b - m).$$

$$\log \tan c = 9.42805$$

$$\log \cos A = 9.93753 (n)$$

$$\log \tan m = 9.36558 (n)$$

$$m = 166^\circ 56' 8''.$$

$$b - m = -(121^\circ 56' 8'').$$

$$\log \cos c = 9.98494$$

$$\log \sec m = 0.01138 (n)$$

$$\log \cos(b - m) = 9.72343$$

$$\log \cos a = 9.71975 (n)$$

$$a = 121^\circ 38' 6''.$$

$$a = 121^\circ 38' 6''.$$

$$b = 45^\circ$$

$$c = 15^\circ$$

$$2s = 181^\circ 38' 6''$$

$$s = 90^\circ 49' 3''.$$

$$s - a = -(30^\circ 49' 3'').$$

$$s - b = 45^\circ 49' 3''.$$

$$s - c = 75^\circ 49' 3''.$$

$$\log \tan \frac{1}{2}s = 10.00620$$

$$\log \tan \frac{1}{2}(s - a) = 9.44030 (n)$$

$$\log \tan \frac{1}{2}(s - b) = 9.62592$$

$$\log \tan \frac{1}{2}(s - c) = 9.89139$$

$$\log \tan^2 \frac{1}{4}E = 18.96381 (n)$$

$$\log \tan \frac{1}{4}E = 9.48192 (n).$$

$$E = 112^\circ 30' 10''$$

$$= 405,010''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.60747$$

$$\log \pi = 0.49715$$

$$\text{colog } 648,000 = 4.18842 - 10$$

$$\log T = 0.29304 + \log r^2$$

$$T = 1.9635 r^2.$$

$$13 \quad A = 85^\circ, C = 95^\circ, b = 70^\circ.$$

$$\cot x = \tan A \cos b.$$

$$\log \tan A = 11.05805$$

$$\log \cos b = \underline{9.53405}$$

$$\log \cot x = 10.59210$$

$$x = 14^\circ 20' 54''.$$

$$C - x = 80^\circ 39' 6''.$$

$$\cos B = \frac{\cos A \sin(C-x)}{\sin x}.$$

$$\log \cos A = 8.94030$$

$$\log \sin(C-x) = 9.99419$$

$$\text{colog} \sin x = \underline{0.60587}$$

$$\log \cos B = 9.54036$$

$$B = 69^\circ 41' 41''.$$

$$A = 85^\circ$$

$$B = 69^\circ 41' 41''$$

$$C = 95^\circ$$

$$\underline{249^\circ 41' 41''}$$

$$E = 69^\circ 41' 41''$$

$$= 250,901''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.39950$$

$$\log \pi = 0.49715$$

$$\text{colog} 648,000 = \underline{4.18842 - 10}$$

$$\log T = 0.08507 + \log r^2 \theta$$

$$T = 1.2164 r^2.$$

$$14. \quad B = 75^\circ, b = 72^\circ, c = 55^\circ.$$

$$\sin C = \frac{\sin B \sin c}{\sin b}.$$

$$\log \sin B = 9.98494$$

$$\log \sin c = 9.91336$$

$$\text{colog} \sin b = \underline{0.02179}$$

$$\log \sin C = 9.92009$$

$$C = 56^\circ 17' 53''.$$

$$\frac{1}{2}(b+c) = 63^\circ 30'.$$

$$\frac{1}{2}(b-c) = 8^\circ 30'.$$

$$\frac{1}{2}(B+C) = 65^\circ 38' 56.5''.$$

$$\frac{1}{2}(B-C) = 9^\circ 21' 3.5''.$$

$$\log \sin \frac{1}{2}(B+C) = 9.95954$$

$$\text{colog} \sin \frac{1}{2}(B-C) = 0.78920$$

$$\log \tan \frac{1}{2}(b-c) = 9.17450$$

$$\log \tan \frac{1}{2}a = 9.92324$$

$$\frac{1}{2}a = 39^\circ 57' 46''.$$

$$a = 79^\circ 55' 32''.$$

$$a = 79^\circ 55' 32''$$

$$b = 72^\circ$$

$$c = 55^\circ$$

$$2s = \underline{206^\circ 55' 32''}$$

$$s = 103^\circ 27' 46''.$$

$$s - a = 23^\circ 32' 14''.$$

$$s - b = 31^\circ 27' 46''.$$

$$s - c = 48^\circ 27' 46''.$$

$$\log \tan \frac{1}{2}s = 10.10300$$

$$\log \tan \frac{1}{2}(s-a) = 9.31877$$

$$\log \tan \frac{1}{2}(s-b) = 9.44975$$

$$\log \tan \frac{1}{2}(s-c) = \underline{9.65329}$$

$$\log \tan^2 \frac{1}{4}E = 18.52481$$

$$\log \tan \frac{1}{4}E = 9.26241.$$

$$\frac{1}{4}E = 10^\circ 22' 10''.$$

$$E = 41^\circ 28' 40''$$

$$= 149,320''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.17412$$

$$\log \pi = 0.49715$$

$$\text{colog} 648,000 = \underline{4.18842 - 10}$$

$$\log T = 1.85969 + \log r^2$$

$$T = 0.72392 r^2.$$

Exercise 108. Miscellaneous Examples (Page 226)*Find the spherical excess, given :*

$$\begin{aligned} 1. \quad A &= 84^\circ 20' 19'', \\ B &= 27^\circ 22' 40'', \\ C &= 75^\circ 33'. \end{aligned}$$

$$\begin{aligned} A &= 84^\circ 20' 19'' \\ B &= 27^\circ 22' 40'' \\ C &= 75^\circ 33' \\ \hline &187^\circ 15' 59'' \\ E &= 187^\circ 15' 59'' - 180^\circ \\ &= 7^\circ 15' 59''. \end{aligned}$$

$$\begin{aligned} 2. \quad a &= 69^\circ 15' 6'', \\ b &= 120^\circ 42' 47'', \\ c &= 159^\circ 18' 33''. \end{aligned}$$

$$\begin{aligned} a &= 69^\circ 15' 6'' \\ b &= 120^\circ 42' 47'' \\ c &= 159^\circ 18' 33'' \\ 2s &= 349^\circ 16' 26'' \\ s &= 174^\circ 38' 13''. \end{aligned}$$

$$\begin{aligned} s - a &= 105^\circ 23' 7'', \\ s - b &= 53^\circ 55' 26'', \\ s - c &= 15^\circ 19' 40''. \end{aligned}$$

$$\begin{aligned} \frac{1}{2}s &= 87^\circ 19' 6.5'', \\ \frac{1}{2}(s - a) &= 52^\circ 41' 33.5'', \\ \frac{1}{2}(s - b) &= 26^\circ 57' 43'', \\ \frac{1}{2}(s - c) &= 7^\circ 39' 50''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2}s &= 11.32942 \\ \log \tan \frac{1}{2}(s - a) &= 10.11805 \\ \log \tan \frac{1}{2}(s - b) &= 9.70645 \\ \log \tan \frac{1}{2}(s - c) &= 9.12893 \\ \log \tan^2 \frac{1}{4}E &= 20.28285 \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{4}E &= 10.14142. \\ \frac{1}{4}E &= 54^\circ 10' 4.6''. \\ E &= 216^\circ 40' 18''. \end{aligned}$$

$$\begin{aligned} 3. \quad a &= 33^\circ 1' 45'', \\ b &= 155^\circ 5' 18'', \\ C &= 110^\circ 10'. \end{aligned}$$

$$\begin{aligned} \tan m &= \tan a \cos C. \\ \cos c &= \cos a \sec m \cos (b - m) \\ \log \tan a &= 9.81300 \\ \log \cos C &= 9.53751 \\ \log \tan m &= 9.35051 \\ m &= 167^\circ 22'. \\ b - m &= -(12^\circ 16' 42''). \end{aligned}$$

$$\begin{aligned} \log \cos a &= 9.92345 \\ \log \sec m &= 0.01064 (n) \\ \log \cos (b - m) &= 9.98995 \\ \log \cos c &= 9.92404 (n) \end{aligned}$$

$$c = 147^\circ 5' 30''.$$

$$\begin{aligned} a &= 33^\circ 1' 45'' \\ b &= 155^\circ 5' 18'' \\ c &= 147^\circ 5' 30'' \\ 2s &= 335^\circ 12' 33'' \end{aligned}$$

$$\begin{aligned} s &= 167^\circ 36' 16.5'', \\ s - a &= 134^\circ 34' 31.5'', \\ s - b &= 12^\circ 30' 58.5'', \\ s - c &= 20^\circ 30' 46.5''. \end{aligned}$$

$$\begin{aligned} \frac{1}{2}s &= 83^\circ 48' 8.25'', \\ \frac{1}{2}(s - a) &= 67^\circ 17' 15.75'', \\ \frac{1}{2}(s - b) &= 6^\circ 15' 29.25'', \\ \frac{1}{2}(s - c) &= 10^\circ 15' 23.25''. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2}s &= 10.96419 \\ \log \tan \frac{1}{2}(s - a) &= 10.37824 \\ \log \tan \frac{1}{2}(s - b) &= 9.04005 \\ \log \tan \frac{1}{2}(s - c) &= 9.25755 \\ \log \tan^2 \frac{1}{4}E &= 9.64003 \\ \log \tan \frac{1}{4}E &= 9.82002. \end{aligned}$$

$$\begin{aligned} \frac{1}{4}E &= 33^\circ 27' 13.3'', \\ E &= 133^\circ 48' 53''. \end{aligned}$$

Find the areas of the following triangles, given:

$$\begin{aligned} 4. \quad c &= 114^\circ 27' 57'', \\ A &= 78^\circ 42' 33'', \\ B &= 127^\circ 13' 7''. \end{aligned}$$

$$\begin{aligned} 5. \quad a &= 76^\circ 14' 47'', \\ b &= 82^\circ 40' 15'', \\ A &= 60^\circ 22' 44''. \end{aligned}$$

$$\begin{aligned} \frac{1}{2}(B - A) &= 24^\circ 15' 17'', \\ \frac{1}{2}(B + A) &= 102^\circ 57' 50'', \\ \frac{1}{2}c &= 57^\circ 13' 58.5''. \end{aligned}$$

$$\tan \frac{1}{2}(b - a) = \frac{\sin \frac{1}{2}(B - A)}{\sin \frac{1}{2}(B + A)} \tan \frac{1}{2}c.$$

$$\begin{aligned} \log \sin \frac{1}{2}(B - A) &= 9.61362 \\ \text{colog} \sin \frac{1}{2}(B + A) &= 0.01121 \\ \log \tan \frac{1}{2}c &= \underline{10.19128} \\ \log \tan \frac{1}{2}(b - a) &= 9.81611 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}(b - a) &= 33^\circ 13'. \\ \cos \frac{1}{2}C &= \frac{\sin \frac{1}{2}(B + A)}{\cos \frac{1}{2}(b - a)} \cos \frac{1}{2}c. \end{aligned}$$

$$\begin{aligned} \log \sin \frac{1}{2}(B + A) &= 9.98879 \\ \text{colog} \cos \frac{1}{2}(b - a) &= 0.07748 \\ \log \cos \frac{1}{2}c &= 9.73338 \\ \log \cos \frac{1}{2}C &= \underline{9.79965} \end{aligned}$$

$$\begin{aligned} \frac{1}{2}C &= 50^\circ 55'. \\ C &= 101^\circ 50'. \end{aligned}$$

$$\begin{aligned} A &= 78^\circ 42' 33'' \\ B &= 127^\circ 13' 7'' \\ C &= 101^\circ 50' \\ &= \underline{307^\circ 45' 40''} \end{aligned}$$

$$\begin{aligned} E &= 127^\circ 45' 40'' \\ &= 459,940''. \end{aligned}$$

$$\begin{aligned} \log r^2 &= \log r^2 \\ \log 459,940 &= 5.66270 \\ \log \pi &= 0.49715 \\ \text{colog } 648,000 &= 4.18842 - 10 \\ \log T &= 0.34827 + \log r^2 \\ T &= 2.2298 r^2. \end{aligned}$$

Since a and A are alike in kind and $\sin b > \sin a > \sin A$, $\sin b$, there are two solutions.

$$\begin{aligned} \log \sin A &= 9.93918 \\ \log \sin b &= 9.99643 \\ \text{colog} \sin a &= 0.01264 \\ \log \sin B &= 9.94825 \end{aligned}$$

$$\begin{aligned} B_1 &= 62^\circ 34' 51'' \\ B_2 &= 117^\circ 25' 9'' \end{aligned}$$

$$\begin{aligned} \frac{1}{2}(b + a) &= 79^\circ 27' 31'', \\ \frac{1}{2}(b - a) &= 3^\circ 12' 44'', \\ \frac{1}{2}(B_1 + A) &= 61^\circ 28' 47.5'', \\ \frac{1}{2}(B_1 - A) &= 1^\circ 6' 3.5''. \end{aligned}$$

$$\begin{aligned} \log \sin \frac{1}{2}(B_1 + A) &= 9.94382 \\ \text{colog} \sin \frac{1}{2}(B_1 - A) &= 1.71637 \\ \log \tan \frac{1}{2}(b - a) &= \underline{8.74914} \\ \log \tan \frac{1}{2}c_1 &= 10.40933 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}c_1 &= 68^\circ 42' 42.6'', \\ c_1 &= 137^\circ 25' 25''. \end{aligned}$$

$$\begin{aligned} \frac{1}{2}(B_2 + A) &= 88^\circ 53' 56.5'', \\ \frac{1}{2}(B_2 - A) &= 28^\circ 31' 12.5''. \end{aligned}$$

$$\begin{aligned} \log \sin \frac{1}{2}(B_2 + A) &= 9.99992 \\ \text{colog} \sin \frac{1}{2}(B_2 - A) &= 0.32105 \\ \log \tan \frac{1}{2}(b - a) &= 8.74914 \\ \log \tan \frac{1}{2}c_2 &= \underline{9.07011} \end{aligned}$$

$$\begin{aligned} \frac{1}{2}c_2 &= 6^\circ 42' 9.4'', \\ c_2 &= 13^\circ 24' 19''. \end{aligned}$$

$a = 76^{\circ} 14' 47''$	$76^{\circ} 14' 47''$
$b = 82^{\circ} 40' 15''$	$82^{\circ} 40' 15''$
$c = \underline{137^{\circ} 25' 25''}$	or $\underline{13^{\circ} 24' 19''}$
$2s = \underline{296^{\circ} 20' 27''}$	or $\underline{172^{\circ} 19' 21''}$
$\frac{1}{2}s = 74^{\circ} 5' 6.75''$	or $43^{\circ} 4' 50.25''$
$\frac{1}{2}(s - a) = 35^{\circ} 57' 43.25''$	or $4^{\circ} 57' 26.75''$
$\frac{1}{2}(s - b) = 32^{\circ} 44' 59.25''$	or $1^{\circ} 44' 42.75''$
$\frac{1}{2}(s - c) = 5^{\circ} 22' 24.25''$	or $36^{\circ} 22' 40.75''$
$\log \tan \frac{1}{2}s = 10.54494$	or 9.97088
$\log \tan \frac{1}{2}(s - a) = 9.86065$	or 8.93822
$\log \tan \frac{1}{2}(s - b) = 9.80836$	or 8.48386
$\log \tan \frac{1}{2}(s - c) = \underline{8.97340}$	or $\underline{9.86727}$
$\log \tan^2 \frac{1}{4}E = 9.18735$	or 7.26023
$\log \tan \frac{1}{4}E = 9.59368$	or 8.63012
$\frac{1}{4}E = 21^{\circ} 25' 22.7''$	or $2^{\circ} 26' 36.0''$
$E = 85^{\circ} 41' 31''$	or $9^{\circ} 46' 24''$
$= 308,491''$	or $35,184''$

$\log r^2 = \log r^2$	$\log r^2$
$\log E = 5.48925$	or 4.54635
$\log \pi = 0.49715$	0.49715
$\text{colog } 648,000 = \underline{4.18842 - 10}$	$\underline{4.18842 - 10}$
$\log T = 0.17482 + \log r^2$	or $9.23192 - 10 + \log r^2$
$T = 1.4956 r^2$	or $0.17085 r^2$

6. $A = 80^{\circ} 12' 35''$, $B = 77^{\circ} 38' 22''$, $a = 76^{\circ} 42' 28''$.

$\cot x = \cos a \tan B$	$A = 80^{\circ} 12' 35''$
$\sin(C - x) = \cos A \sec B \sin x$	$B = 77^{\circ} 38' 22''$
$\log \cos a = 9.36157$	$C = \underline{76^{\circ} 51' 34''}$
$\log \tan B = \underline{10.65927}$	$\underline{234^{\circ} 42' 31''}$
$\log \cot x = \underline{10.02084}$	$E = 54^{\circ} 42' 31''$
$x = 43^{\circ} 37' 34''$	$= 196,951''$

$\log \cos A = 9.23055$	$\log r^2 = \log r^2$
$\log \sec B = 0.66946$	$\log E = 5.29436$
$\log \sin x = \underline{9.83881}$	$\log \pi = 0.49715$
$\log \sin(C - x) = \underline{9.73882}$	$\text{colog } 648,000 = \underline{4.18842 - 10}$
$C - x = 33^{\circ} 14'$	$\log T = \underline{9.97993 - 10 + \log r^2}$
$C = 76^{\circ} 51' 34''$	$T = 0.95484 r^2$

$$7. \ b = 44^\circ 27' 40'', \ c = 15^\circ 22' 44'', \\ A = 167^\circ 42' 27''.$$

$$\frac{1}{2}(b - c) = 14^\circ 32' 28''.$$

$$\frac{1}{2}(b + c) = 29^\circ 55' 12''.$$

$$\frac{1}{2}A = 83^\circ 51' 13.5''.$$

$$\tan \frac{1}{2}(B + C) = \frac{\cos \frac{1}{2}(b - c)}{\cos \frac{1}{2}(b + c)} \cot \frac{1}{2}A.$$

$$\log \cos \frac{1}{2}(b - c) = 9.98586$$

$$\operatorname{colog} \cos \frac{1}{2}(b + c) = 0.06212$$

$$\log \cot \frac{1}{2}A = 9.03215$$

$$\log \tan \frac{1}{2}(B + C) = 9.08013$$

$$\frac{1}{2}(B + C) = 6^\circ 51' 27.5''.$$

$$B + C = 13^\circ 42' 55''.$$

$$A = 167^\circ 42' 27''$$

$$B + C = \frac{13^\circ 42' 55''}{181^\circ 25' 22''}$$

$$E = 1^\circ 25' 22''$$

$$= 5122''.$$

$$\log r^2 = \log r^2$$

$$\log E = 3.70944$$

$$\log \pi = 0.49715$$

$$\operatorname{colog} 648,000 = \frac{4.18842 - 10}{8.39501 - 10 + \log r^2}$$

$$\log T = 8.39501 - 10 + \log r^2$$

$$T = 0.024832 r^2.$$

$$8. \ b = 67^\circ 15' 42'', \ A = 84^\circ 55' 8'', \\ C = 96^\circ 18' 49''.$$

$$\cot x = \tan C \cos b.$$

$$\cos B = \cos C \csc x \sin (A - x).$$

$$\log \tan C = 10.95608 (n)$$

$$\log \cos b = 9.58718$$

$$\log \cot x = 10.54326 (n)$$

$$x = 164^\circ 1' 35''.$$

$$\log \cos C = 9.04128 (n)$$

$$\log \csc x = 0.56036$$

$$\log \sin (A - x) = 9.99210 (n)$$

$$\log \cos B = 9.59374$$

$$B = 66^\circ 53' 44''.$$

$$A = 84^\circ 55' 8''$$

$$B = 66^\circ 53' 44''$$

$$C = \frac{96^\circ 18' 49''}{248^\circ 7' 41''}$$

$$E = 68^\circ 7' 41''$$

$$= 245,261''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.38963$$

$$\log \pi = 0.49715$$

$$\operatorname{colog} 648,000 = \frac{4.18842 - 10}{\log T = 0.07520 + \log r^2}$$

$$\log T = 0.07520 + \log r^2$$

$$T = 1.1891 r^2.$$

$$9. \ b = 72^\circ 19' 38'', \ c = 54^\circ 58' 52'', \\ B = 77^\circ 15' 14''.$$

$$\tan m = \cos B \tan c.$$

$$\cos (a - m) = \cos b \sec c \cos m.$$

$$\log \cos B = 9.34367$$

$$\log \tan c = 10.15447$$

$$\log \tan m = 9.49814$$

$$m = 17^\circ 28' 41''.$$

$$\log \cos b = 9.48227$$

$$\log \sec c = 0.24121$$

$$\log \cos m = 9.97947$$

$$\log \cos (a - m) = 9.70295$$

$$a - m = 59^\circ 41' 41''.$$

$$a = 77^\circ 10' 22''.$$

$$a = 77^\circ 10' 22''$$

$$b = 72^\circ 19' 38''$$

$$c = 54^\circ 58' 52''$$

$$2s = 204^\circ 28' 52''$$

$$\frac{1}{2}s = 51^\circ 7' 13''.$$

$$\frac{1}{2}(s - a) = 12^\circ 32' 2''.$$

$$\frac{1}{2}(s - b) = 14^\circ 57' 24''.$$

$$\frac{1}{2}(s - c) = 23^\circ 37' 47''.$$

$$\log \tan \frac{1}{2} s = 10.09350$$

$$\log \tan \frac{1}{2} (s - a) = 9.34697$$

$$\log \tan \frac{1}{2} (s - b) = 9.42673$$

$$\log \tan \frac{1}{2} (s - c) = 9.64099$$

$$\log \tan^2 \frac{1}{4} E = 8.50819$$

$$\log \tan \frac{1}{4} E = 9.25410.$$

$$\frac{1}{4} E = 10^\circ 10' 37.5''.$$

$$E = 40^\circ 42' 30''$$

$$= 146,550''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.16599$$

$$\log \pi = 0.49715$$

$$\text{colog } 648,000 = 4.18842 - 10$$

$$\log T = 9.85156 - 10 + \log r^2$$

$$T = 0.7105 r^2.$$

$$10. B = 127^\circ 16' 4'',$$

$$C = 42^\circ 34' 19'',$$

$$b = 54^\circ 47' 55''.$$

$$\sin c = \sin b \sin C \csc B.$$

$$\log \sin b = 9.91229$$

$$\log \sin C = 9.83027$$

$$\log \csc B = 0.09919$$

$$\log \sin c = 9.84175$$

$$c = 43^\circ 59' 51''.$$

$$\frac{1}{2} (B + C) = 84^\circ 55' 11.5'',$$

$$\frac{1}{2} (B - C) = 42^\circ 20' 52.5'',$$

$$\frac{1}{2} (b + c) = 49^\circ 23' 53'',$$

$$\frac{1}{2} (b - c) = 5^\circ 24' 2''.$$

$$\log \sin \frac{1}{2} (b + c) = 9.88039$$

$$\log \csc \frac{1}{2} (b - c) = 1.02633$$

$$\log \tan \frac{1}{2} (B - C) = 9.95974$$

$$\log \cot \frac{1}{2} A = 10.86646$$

$$\frac{1}{2} A = 7^\circ 44' 41.1''.$$

$$A = 15^\circ 29' 22''.$$

$$A = 15^\circ 29' 22''$$

$$B = 127^\circ 16' 4''$$

$$C = 42^\circ 34' 19''$$

$$= 185^\circ 19' 45''$$

$$E = 5^\circ 19' 45''$$

$$= 19,185''.$$

$$\log r^2 = \log r^2$$

$$\log E = 4.28296$$

$$\log \pi = 0.49715$$

$$\text{colog } 648,000 = 4.18842 - 10$$

$$\log T = 8.96853 - 10 + \log r^2$$

$$T = 0.09301 r^2.$$

$$11. a = 128^\circ 42' 56'',$$

$$b = 107^\circ 13' 48'',$$

$$c = 88^\circ 37' 51''.$$

$$a = 128^\circ 42' 56''$$

$$b = 107^\circ 13' 48''$$

$$c = 88^\circ 37' 51''$$

$$2s = 324^\circ 34' 35''$$

$$s = 162^\circ 17' 17.5''.$$

$$s - a = 33^\circ 34' 21.5''.$$

$$s - b = 55^\circ 3' 29.5''.$$

$$s - c = 73^\circ 39' 26.5''.$$

$$\frac{1}{2} s = 81^\circ 8' 38.75''.$$

$$\frac{1}{2} (s - a) = 16^\circ 47' 10.75''.$$

$$\frac{1}{2} (s - b) = 27^\circ 31' 44.75''.$$

$$\frac{1}{2} (s - c) = 36^\circ 49' 43.25''.$$

$$\log \tan \frac{1}{2} s = 10.80742$$

$$\log \tan \frac{1}{2} (s - a) = 9.47951$$

$$\log \tan \frac{1}{2} (s - b) = 9.71701$$

$$\log \tan \frac{1}{2} (s - c) = 9.87441$$

$$\log \tan^2 \frac{1}{4} E = 9.87835$$

$$\log \tan \frac{1}{4} E = 9.93918.$$

$$\frac{1}{4} E = 41^\circ 0' 4.6''.$$

$$E = 164^\circ 0' 18''$$

$$= 590,418''.$$

$$\log r^2 = \log r^2$$

$$\log E = 5.77116$$

$$\log \pi = 0.49715$$

$$\text{colog } 648,000 = 4.18842 - 10$$

$$\log T = 0.45673 + \log r^2$$

$$T = 2.8624 r^2.$$

$$\begin{aligned} 12. \quad A &= 127^\circ 22' 28'', \\ B &= 131^\circ 45' 27'', \\ C &= 100^\circ 52' 16''. \end{aligned}$$

$$\begin{aligned} A &= 127^\circ 22' 28'' \\ B &= 131^\circ 45' 27'' \\ C &= 100^\circ 52' 16'' \\ \hline &360^\circ 0' 11'' \\ E &= 180^\circ 0' 11'' \\ &= 648,011''. \end{aligned}$$

$$\begin{aligned} \log r^2 &= \log r^2 \\ \log E &= 5.81159 \\ \log \pi &= 0.49715 \\ \text{colog } 648,000 &= 4.18842 - 10 \\ \log T &= 0.49716 + \log r^2 \\ T &= 3.1416 r^2. \end{aligned}$$

$$\begin{aligned} 13. \quad a &= 116^\circ 19' 45'', \\ A &= 160^\circ 42' 24'', \\ C &= 171^\circ 27' 15''. \end{aligned}$$

$$\begin{aligned} \cot x &= \cos a \tan C. \\ \sin(B - x) &= \cos A \sec C \sin x. \end{aligned}$$

$$\begin{aligned} \log \cos a &= 9.64692 \text{ (n)} \\ \log \tan C &= 9.17687 \text{ (n)} \\ \log \cot x &= 8.82379 \\ x &= 86^\circ 11' 13''. \end{aligned}$$

$$\begin{aligned} \log \cos A &= 9.97490 \text{ (n)} \\ \log \sec C &= 0.00485 \text{ (n)} \\ \log \sin x &= 9.99904 \\ \log \sin(B - x) &= 9.97879 \end{aligned}$$

$$\begin{aligned} B - x &= 72^\circ 14' 15''. \\ B &= 158^\circ 25' 28''. \end{aligned}$$

$$\begin{aligned} A &= 160^\circ 42' 24'' \\ B &= 158^\circ 25' 28'' \\ C &= 171^\circ 27' 15'' \\ \hline &490^\circ 35' 7'' \end{aligned}$$

$$\begin{aligned} E &= 310^\circ 35' 7'' \\ &= 1,118,107''. \end{aligned}$$

$$\begin{aligned} \log r^2 &= \log r^2 \\ \log E &= 6.04848 \\ \log \pi &= 0.49715 \\ \text{colog } 648,000 &= 4.18842 - 10 \\ \log T &= 0.73405 + \log r^2 \\ T &= 5.4206 r^2. \end{aligned}$$

14. Find the area of a triangle on the surface of the earth, regarded as a sphere, if each side of the triangle is equal to 1° , and the radius of the earth is taken as 3958 mi.

$$\begin{aligned} a &= 1^\circ \\ b &= 1^\circ \\ c &= 1^\circ \\ 2s &= 3^\circ \end{aligned}$$

$$\begin{aligned} \frac{1}{2}s &= 0^\circ 45'. \\ \frac{1}{2}(s - a) &= 0^\circ 15'. \\ \frac{1}{2}(s - b) &= 0^\circ 15'. \\ \frac{1}{2}(s - c) &= 0^\circ 15'. \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2}s &= 8.11696 \\ \log \tan \frac{1}{2}(s - a) &= 7.63982 \\ \log \tan \frac{1}{2}(s - b) &= 7.63982 \\ \log \tan \frac{1}{2}(s - c) &= 7.63982 \\ \log \tan^2 \frac{1}{4}E &= 1.03642 \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{4}E &= 5.51821 \\ \frac{1}{4}E &= 0^\circ 0' 6.8139'' \\ &= 27.2556''. \end{aligned}$$

$$\begin{aligned} \log r^2 &= 7.19496 \\ \log E &= 1.43546 \\ \log \pi &= 0.49715 \\ \text{colog } 648,000 &= 4.18842 - 10 \\ \log T &= 3.31599 \\ T &= 2070.1. \end{aligned}$$

Therefore the area is 2070.1 square miles.

15. In an equilateral triangle, given the side a , find the angle A .

$$a = b = c = a.$$

$$s = \frac{1}{2}(a + a + a) = 1\frac{1}{2}a.$$

$$\sin \frac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$$

$$= \sqrt{\frac{\sin \frac{1}{2}a \sin \frac{1}{2}a}{\sin a \sin a}}$$

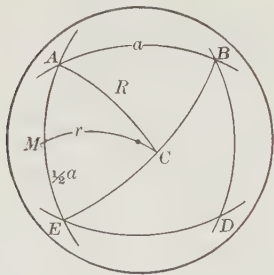
$$= \sqrt{\frac{\sin^2 \frac{1}{2}a}{\sin^2 a}}$$

$$= \frac{\sin \frac{1}{2}a}{\sin a}$$

$$= \frac{\sin \frac{1}{2}a}{2 \sin \frac{1}{2}a \cos \frac{1}{2}a}$$

$$= \frac{1}{2} \sec \frac{1}{2}a.$$

16. Given the side a of a regular spherical polygon of n sides, find



the angle A of the polygon, the distance R from the center of the

polygon to one of the vertices, and the distance r from the center to the middle point of one of the sides.

In the regular polygon $ABDE$ draw arcs of great circles from the vertices A, B , etc., through the center C , and from C to M , the middle of one side.

$$\text{Then } ACE = \frac{360^\circ}{n},$$

$$ACM = \frac{180^\circ}{n},$$

$$CAM = \frac{1}{2}A,$$

$$AM = \frac{1}{2}a,$$

$$AC = R,$$

$$MC = r.$$

By Napier's Rules,

$$\cos \frac{180^\circ}{n} = \cos \frac{1}{2}a \sin \frac{1}{2}A,$$

$$\sin \frac{1}{2}a = \sin R \sin \frac{180^\circ}{n},$$

$$\sin r = \tan \frac{1}{2}a \cot \frac{180^\circ}{n}.$$

Whence

$$\sin \frac{1}{2}A = \sec \frac{1}{2}a \cos \frac{180^\circ}{n},$$

$$\sin R = \sin \frac{1}{2}a \csc \frac{180^\circ}{n},$$

$$\sin r = \tan \frac{1}{2}a \cot \frac{180^\circ}{n}.$$

17. Compute the dihedral angles made by the faces of the five regular polyhedrons.

If a sphere is described about a vertex of the polyhedron as a center, with a radius equal to an edge of the polyhedron, the adjacent vertices of the polyhedron lie on the surface of the sphere and are the vertices of a regular spherical polygon, of which the angles are required.

If a is the length of a side of this polygon, that is, one of the angles of a face of the polyhedron, and n the number of sides,

that is, the number of faces of the polyhedron which meet at a vertex we have for the different cases:

POLYHEDRON	α	n
Tetrahedron	60°	3
Hexahedron	90°	3
Octahedron	60°	4
Dodecahedron	108°	3
Icosahedron	60°	5

But if A is an angle of the spherical polygon, we have, from Ex. 15,

$$\sin \frac{1}{2} A = \sec \frac{1}{2} \alpha \cos \frac{180^\circ}{n}.$$

Hence, for the different cases:

POLYHEDRON	$\sin \frac{1}{2} A$	$\log \sin \frac{1}{2} A$	A
Tetrahedron	$\frac{1}{3} \sqrt{3}$	9.76144	$70^\circ 31' 46''$
Hexahedron	$\frac{1}{2} \sqrt{2}$	9.84949	90°
Octahedron	$\frac{1}{3} \sqrt{6}$	9.91195	$109^\circ 28' 14''$
Dodecahedron	$\frac{1}{2} \sec 54^\circ$	9.92975	$116^\circ 33' 45''$
Icosahedron	$\frac{2}{3} \sqrt{3} \cos 36^\circ$	9.97043	$138^\circ 11' 36''$

18. The distance from Washington (W) to a certain place X , measured in degrees on a great-circle arc, is 9° , and of a place Y from Washington the distance is 12° . The angle XYW is 85° . What is the distance in degrees from X to Y ?

Given two sides and the included angle, to find the third side.

$$a = 12^\circ, b = 9^\circ, C = 85^\circ.$$

$$\tan m = \tan a \cos C.$$

$$\cos c = \cos a \sec m \cos (b - m)$$

$$\log \tan a = 9.32747$$

$$\log \cos a = 9.99040$$

$$\log \cos C = 9.94030$$

$$\log \sec m = 0.00008$$

$$\log \tan m = 8.26777$$

$$\log \cos (b - m) = 9.99582$$

$$m = 1^\circ 3' 41''.$$

$$\log \cos c = 9.98630$$

$$b - m = 7^\circ 56' 19''.$$

$$c = 14^\circ 19'.$$

$\therefore$ the distance from X to $Y = 14^\circ 19'.$

